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Friendship Among Triangle Centers

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Abstract. If we erect on the sides of a scalene triangle three squares, then at the vertices of the triangle we find new triangles, the *flanks*. We study pairs of triangle centers X and Y such that the triangle of Xs in the three flanks is perspective with ABC at Y, and vice versa. These centers X and Y we call *friends*. Some examples of friendship among triangle centers are given.

1. Flanks

Given a triangle ABC with side lengths $BC = a$, $CA = b$, and $AB = c$. By erecting squares AC_aC_bB, BA_bA_cC, and CB_cB_aA externally on the sides, we form new triangles AB_aC_a, BC_bA_b, and CA_cB_c, which we call the *flanks* of ABC. See Figure 1.

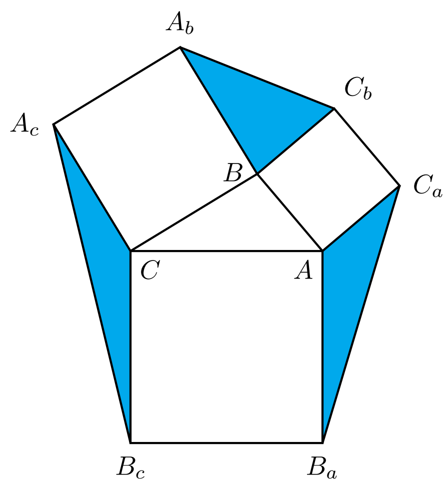

Figure 1

If we rotate the A-flank (triangle AB_aC_a) by $\frac{\pi}{2}$ about A, then the image of C_a is B, and that of B_a is on the line CA. Triangle ABC and the image of the A-flank form a larger triangle in which BA is a median. From this, ABC and the A-flank have equal areas. It is also clear that ABC is the A-flank triangle of the A-flank triangle. These observations suggest that there are a close relationship between ABC and its flanks.

2. Circumcenters of flanks

If P is a triangle center of ABC, we denote by P_A, P_B, and P_C the *same* center of the A-, B-, and C- flanks respectively.

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Let O be the circumcenter of triangle ABC. Consider the triangle $O_AO_BO_C$ formed by the circumcenters of the flanks. By the fact that the circumcenter is the intersection of the perpendicular bisectors of the sides, we see that $O_AO_BO_C$ is homothetic (parallel) to ABC, and that it bisects the squares on the sides of ABC. The distances between the corresponding sides of ABC and $O_AO_BO_C$ are therefore $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{c}{2}$.

3. Friendship of circumcenter and symmedian point

Now, homothetic triangles are perspective at their center of similitude. The distances from the center of similitude of ABC and $O_AO_BO_C$ to the sides of ABC are proportional to the distances between the corresponding sides of the two triangles, and therefore to the sides of ABC. This perspector must be the *symmedian point* K. [1]

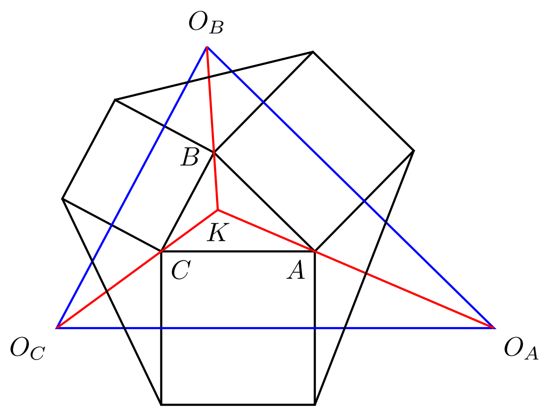

Figure 2

The triangle $O_AO_BO_C$ of *circumcenters* of the flanks is perspective with ABC at the *symmedian point* K of ABC. In particular, the A-Cevian of K in ABC (the line AK) is the same line as the A-Cevian of O_A in the A-flank. Since ABC is the A-flank of triangle AB_aC_a, the A-Cevian of K_A in the A-flank is the same line as the A-Cevian of O in ABC as well. Clearly, the same statement can be made for the B- and C-flanks. The triangle $K_AK_BK_C$ of *symmedian points* of the flanks is perspective with ABC at the *circumcenter* O.

For this relation we call the triangle centers O and K *friends*. See Figure 3. More generally, we say that P *befriends* Q if the triangle $P_AP_BP_C$ is perspective with ABC at Q. Such a friendship relation is always symmetric since, as we have remarked earlier, ABC is the A-, B-, C-flank respectively of its A-, B-, C-flanks.

[1]This is X_6 in [2, 3].

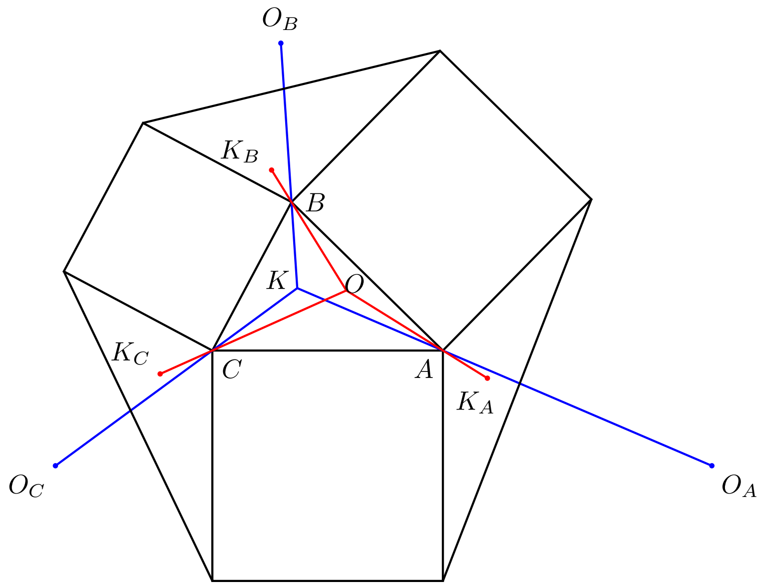

Figure 3

4. Isogonal conjugacy

It is easy to see that the bisector of an angle of ABC also bisects the corresponding angle of its flank. The incenter of a triangle, therefore, *befriends* itself.

Consider two friends P and Q. By reflection in the bisector of angle A, the line PAQ_A is mapped to the line joining the isogonal conjugates of P and Q_A.[2] We conclude:

Proposition. If two triangle centers are friends, then so are their isogonal conjugates.

Since the centroid G and the orthocenter H are respectively the isogonal conjugates of the symmedian point K and the circumcenter O, we conclude that G and H are friends.

5. The Vecten points

The centers of the three squares AC_aC_bB, BA_bA_cC and CB_cB_aA form a triangle perspective with ABC. The perspector is called the *Vecten point* of the triangle. [3] By the same token the centers of three squares constructed *inwardly* on the three sides also form a triangle perspective with ABC. The perspector is called the *second Vecten point*. [4] We show that each of the Vecten points befriends itself.

[2]For Q_A, this is the same line when isogonal conjugation is considered both in triangle ABC and in the A-flank.

[3]This is the point X_{485} of [3].

[4]This is the point X_{486} of [3], also called the *inner* Vecten point.

6. The Second Vecten points

O. Bottema [1] has noted that the position of the midpoint M of segment B_cC_b depends only on B, C, but not on A. More specifically, M is the apex of the isosceles right triangle on BC pointed towards A. [5]

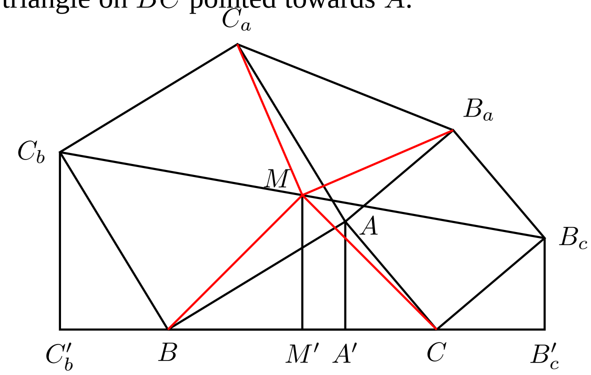

Figure 4

To see this, let A', M', B'_c and C'_b be the orthogonal projections of A, M, B_c and C_b respectively on the line BC. See Figure 4. Triangles $AA'C$ and CB'_cB_c are congruent by rotation through $\pm\frac{\pi}{2}$ about the center of the square CB_cB_aA. Triangles $AA'B$ and BC'_bC_b are congruent in a similar way. So we have $AA' = CB'_c = BC'_b$. It follows that M' is also the midpoint of BC. And we see that $C'_bC_b + B'_c + B_c = BA' + A'C = a$ so $MM' = \frac{a}{2}$. And M is as desired.

By symmetry M is also the apex of the isosceles right triangle on B_aC_a pointed towards A.

We recall that the triangle of apexes of similar isosceles triangles on the sides of ABC is perspective with ABC. The triangle of apexes is called a *Kiepert triangle*, and the *Kiepert perspector* $K(\phi)$ depends on the base angle ϕ (mod π) of the isosceles triangle. [6]

We conclude that AM is the A-Cevian of $K(-\frac{\pi}{4})$, also called the *second Vecten point* of both ABC and the A-flank. From similar observations on the B- and C-flanks, we conclude that the second Vecten point befriends itself.

7. Friendship of Kiepert perspectors

Given any real number t, Let X_t and Y_t be the points that divide CB_c and BC_b such that $CX_t : CB_c = BY_t : BC_b = t : 1$, and let M_t be their midpoint. Then BCM_t is an isosceles triangle, with base angle $\arctan t = \angle BAY_t$. See Figure 5.

Extend AX_t to X'_t on B_aB_c, and AY_t to Y'_t on C_aC_b and let M'_t be the midpoint of $X'_tY'_t$. Then $B_aC_aM'_t$ is an isosceles triangle, with base angle $\arctan\frac{1}{t} = \angle Y'_tAC_a = \frac{\pi}{2} - \angle BAY_t$. Also, by the similarity of triangles AX_tY_t and $AX'_tY'_t$

[5] Bottema introduced this result with the following story. Someone had found a treasure and hidden it in a complicated way to keep it secret. He found three marked trees, A, B and C, and thought of rotating BA through 90 degrees to BC_b, and CA through -90 degrees to CB_c. Then he chose the midpoint M of C_bB_c as the place to hide his treasure. But when he returned, he could not find tree A. He decided to guess its position and try. In a desperate mood he imagined numerous

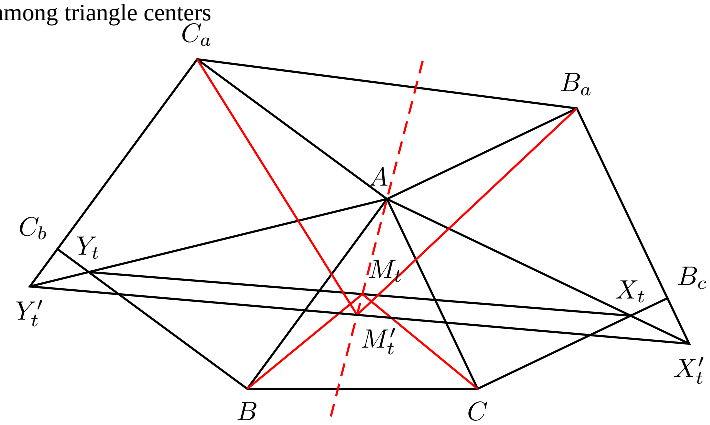

Figure 5

we see that A, M_t and M'_t are collinear. This shows that the Kiepert perspectors $K(\phi)$ and $K(\frac{\pi}{2}-\phi)$ are friends.

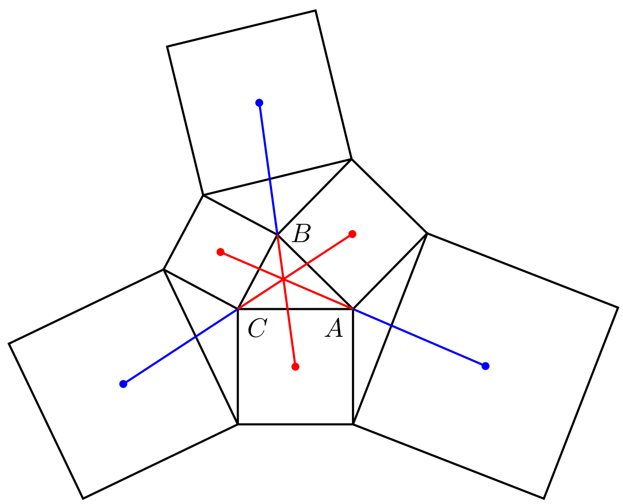

Figure 6

In particular, the first Vecten point $K(\frac{\pi}{4})$ also befriends itself. See Figure 6. The Fermat points $K(\pm\frac{\pi}{3})$ [7] are friends of the the Napoleon points $K(\frac{\pi}{6})$. [8]

Seen collectively, the *Kiepert hyperbola*, the locus of Kiepert perspectors, befriends itself; so does its isogonal transform, the Brocard axis OK.

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diggings without result. But, much to his surprise, he was able to recover his treasure on the very first try!

[6]By convention, ϕ is positive or negative according as the isosceles triangles are pointing outwardly or inwardly.

[7]These are the points X_{13} and X_{14} in [2, 3], also called the isogenic centers.

[8]These points are labelled X_{17} and X_{18} in [2, 3]. It is well known that the Kiepert triangles are equilateral.

References

[1] O. Bottema, Verscheidenheid XXXVIII, in *Verscheidenheden*, p.51, Nederlandse Vereniging van Wiskundeleraren / Wolters Noordhoff, Groningen (1978).
[2] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/˜ck6/encyclopedia/.

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