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Another Proof of the Erdős-Mordell Theorem

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Abstract. We give a proof of the famous Erdős-Mordell inequality using Ptolemy's theorem.

The following neat inequality is well-known:

Theorem. If from a point O inside a given triangle ABC perpendiculars OD, OE, OF are drawn to its sides, then $OA+OB+OC \geq 2(OD+OE+OF)$. Equality holds if and only if triangle ABC is equilateral.

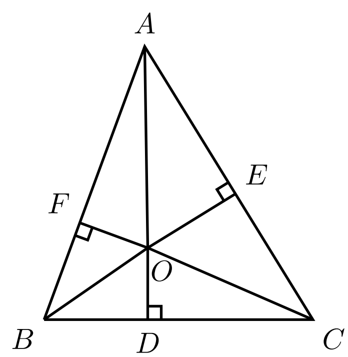

Figure 1

This was conjectured by Paul Erdős in 1935, and first proved by Louis Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's theorem by André Avez [5], angular computations with similar triangles by Leon Bankoff [2], area inequality by V. Komornik [6], or using trigonometry by Mordell and Barrow [1]. The purpose of this note is to give another elementary proof using Ptolemy's theorem.

Proof. Let HG denote the orthogonal projections of BC on the line FE. See Figure 2. Then, we have $BC \geq HG = HF + FE + EG$. It follows from $\angle BFH = \angle AFE = \angle AOE$ that the right triangles BFH and AOE are similar and $HF = \dfrac{OE}{OA}BF$. In a like manner we find that $EG = \dfrac{OF}{OA}CE$. Ptolemy's theorem applied to $AFOE$ gives

$$OA \cdot FE = AF \cdot OE + AE \cdot OF \quad \text{or} \quad FE = \frac{AF \cdot OE + AE \cdot OF}{OA}.$$

Combining these, we have

$$BC \geq \frac{OE}{OA}BF + \frac{AF \cdot OE + AE \cdot OF}{OA} + \frac{OF}{OA}CE,$$

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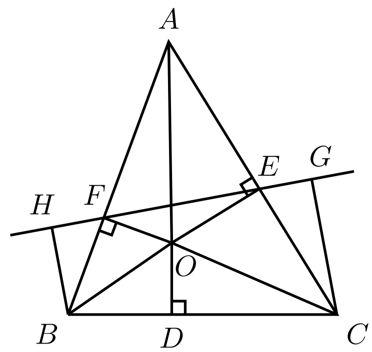

Figure 2

or

$$BC \cdot OA \geq OE \cdot BF + AF \cdot OE + AE \cdot OF + OF \cdot CE = OE \cdot AB + OF \cdot AC.$$

Dividing by BC, we have $OA \geq \frac{AB}{BC}OE + \frac{AC}{BC}OF$.

Applying the same reasoning to other projections, we have

$$OB \geq \frac{BC}{CA}OF + \frac{BA}{CA}OD \quad \text{and} \quad OC \geq \frac{CA}{AB}OD + \frac{CB}{AB}OE.$$

Adding these inequalities, we have

$$OA + OB + OC \geq \left(\frac{BA}{CA} + \frac{CA}{AB}\right)OD + \left(\frac{AB}{BC} + \frac{CB}{AB}\right)OE + \left(\frac{AC}{BC} + \frac{BC}{CA}\right)OF.$$

It follows from this and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ (for positive real numbers x, y) that

$$OA + OB + OC \geq 2(OD + OE + OF).$$

It is easy to check that equality holds if and only if $AB = BC = CA$ and O is the circumcenter of ABC. $\square$

References

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