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Heron Triangles:
A Gergonne-Cevian-and-Median Perspective

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Abstract. We give effective constructions of Heron triangles by considering the intersection of a median and a cevian through the Gergonne point.

1. Introduction

Heron gave the triangle area formula in terms of the sides a, b, c:

$$(*) \qquad \triangle = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{1}{2}(a+b+c).$$

He is further credited with the discovery of the integer sided and integer area triangle (13,14,15;84). Notice that this is a non-Pythagorean triangle, *i.e.*, it does not contain a right angle. We might as well say that with this discovery he challenged us to determine triangles having integer sides and area, *Heron triangles*. Dickson [2] sketches the early attempts to meet this challenge. The references [1, 3, 4, 5, 6, 7, 9, 10] describe recent attempts in that direction. The present discussion uses the intersection point of a Gergonne cevian (the line segment between a vertex and the point of contact of the incircle with the opposite side) and a median to generate Heron triangles. Why do we need yet another description? The answer is simple: Each new description provides new ways to solve, and hence to acquire new insights into, earlier Heron problems. More importantly, they pose new Heron challenges. We shall illustrate this. Dickson uses the name Heron triangle to describe one having rational sides and area. However, these rationals can always be rendered integers. Therefore for us a Heron triangle is one with integer sides and area except under special circumstances.

We use the standard notation: a, b, c for the sides BC, CA, AB of triangle ABC. We use the word side also in the sense of the length of a side. Furthermore, we assume $a \geq c$. No generality is lost in doing so because we may relabel the vertices if necessary.

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2. A preliminary result

We first solve this problem: Suppose three cevians of a triangle concur at a point. How does one determine the ratio in which the concurrence point sections one of them? The answer is given by

Theorem 1. *Let the cevians AD, BE, CF of triangle ABC concur at the point S. Then*
$$\frac{AS}{SD} = \frac{AE}{EC} + \frac{AF}{FB}.$$

Proof. Let $[T]$ denote the area of triangle T. We use the known result: if two triangles have a common altitude, then their areas are proportional to the corresponding bases. Hence, from Figure 1,

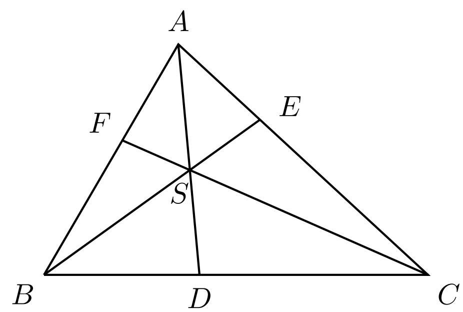

Figure 1

$$\frac{AS}{SD} = \frac{[ABS]}{[SBD]} = \frac{[ASC]}{[SDC]} = \frac{[ABS]+[ASC]}{[SBD]+[SDC]} = \frac{[ABS]}{[SBC]} + \frac{[ASC]}{[SBC]}. \quad (1)$$

But
$$\frac{AE}{EC} = \frac{[ABE]}{[EBC]} = \frac{[ASE]}{[ESC]} = \frac{[ABE]-[ASE]}{[EBC]-[ESC]} = \frac{[ABS]}{[SBC]}, \quad (2)$$

and likewise,
$$\frac{AF}{FB} = \frac{[ASC]}{[SBC]}. \quad (3)$$

Now, (1), (2), (3) complete the proof. □

In the above proof we used a property of equal ratios, namely, if $\frac{p}{q} = \frac{r}{s} = k$, then $k = \frac{p \pm q}{r \pm s}$. From Theorem 1 we deduce the following corollary that is important for our discussion.

Corollary 2. *In Figure 2, let AD denote the median, and BE the Gergonne cevian. Then* $\frac{AS}{SD} = \frac{2(s-a)}{s-c}$.

Proof. The present hypothesis implies $BD = DC$, and E is the point where the incircle is tangent with AC. It is well - known that $AE = s - a$, $EC = s - c$. Now, Ceva's theorem, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$, yields $\frac{AF}{FB} = \frac{s-a}{s-c}$. Then Theorem 1 upholds the claim of Corollary 2. $\square$

In the case of a Heron triangle, a, b, c and s are natural numbers. Therefore, $\frac{AS}{SD} = \frac{2(s-a)}{s-c} = \lambda$ is a rational ratio. Of course this will be true more generally even if $\triangle$ is not an integer; but that is beside the main point. Also, $a \geq c$ implies that $0 < \lambda \leq 2$. Next we show how each rational number λ generates an infinite family, a $\lambda-$family of Heron triangles.

3. Description of $\lambda-$family of Heron triangles

Theorem 3 gives expressions for the sides of the Heron triangle in terms of λ. At present we do not transform these rational sides integral. However, when we specify a rational number for λ then we do express a, b, c integral such that $\gcd(a,b,c) = 1$. An exception to this common practice may be made in the solution of a Heron problem that requires $\gcd(a,b,c) > 1$, in (*D1*) later, for example.

Theorem 3. *Let λ be a rational number such that $0 < \lambda \leq 2$. The $\lambda-$family of Heron triangles is described by*

$$(a,b,c) = (2(m^2 + \lambda^2 n^2),\ (2+\lambda)(m^2 - 2\lambda n^2),\ \lambda(m^2 + 4n^2)),$$

m, n *being relatively prime natural numbers such that* $m > \sqrt{2\lambda} \cdot n$.

Proof. From the definition we have

$$\frac{2(s-a)}{s-c} = \lambda \quad \text{or} \quad b = \frac{2+\lambda}{2-\lambda}(a-c).$$

If $\lambda \neq 2$, we assume $a - c = (2-\lambda)p$. This gives $b = (2+\lambda)p$. If $\lambda = 2$, then we define $b = 4p$. The rest of the description is common to either case. Next we calculate

$$\begin{aligned} a &= (2-\lambda)p + c, \quad s = c + 2p, \quad \text{and from } (*), \\ \triangle^2 &= 2\lambda p^2 (c+2p)(c-\lambda p). \qquad (4) \end{aligned}$$

To render (a,b,c) Heron we must have $(c+2p)(c-\lambda p) = 2\lambda q^2$. There is no need to distinguish two cases: 2λ itself a rational square or not. This fact becomes clearer later when we deduce Corollary 5. With the help of a rational number $\frac{m}{n}$ we may write down

$$c + 2p = \frac{m}{n} q, \quad \text{and} \quad c - \lambda p = \frac{n}{m}(2\lambda q).$$

We solve the above simultaneous equations for p and c:

$$p = \frac{m^2 - 2\lambda n^2}{(2+\lambda)mn} \cdot q, \qquad c = \frac{\lambda(m^2 + 4n^2)}{(2+\lambda)mn} \cdot q.$$

This yields

$$\frac{p}{m^2 - 2\lambda n^2} = \frac{q}{(2+\lambda)mn} = \frac{c}{\lambda(m^2+4n^2)}.$$

Since p, q, c, λ, m, n are positive we must have $m > \sqrt{2\lambda}\cdot n$. We may ignore the constant of proportionality so that

$$p = m^2 - 2\lambda n^2, \qquad q = (2+\lambda)mn \qquad c = \lambda(m^2+4n^2).$$

These values lead to the expressions for the sides a, b, c in the statement of Theorem 3. Also, $\triangle = 2\lambda(2+\lambda)mn(m^2 - 2\lambda n^2)$, see (4), indicates that the area is rational. $\square$

Here is a numerical illustration. Let $\lambda = 1$, $m = 4$, $n = 1$. Then Theorem 3 yields $(a,b,c) = (34,42,20)$. Here $\gcd(a,b,c) = 2$. In the study of Heron triangles often gcd$(a,b,c) > 1$. In such a case we divide the side length values by the gcd to list primitive values. Hence, $(a,b,c) = (17,21,10)$.

Now, suppose $\lambda = \frac{3}{2}$, $m = 5$, $n = 2$. Presently, Theorem 3 gives $(a,b,c) = (68, \frac{91}{2}, \frac{123}{2})$. As it is, the sides b and c are not integral. In this situation we render the sides integral (and divide by the gcd if it is greater than 1) so that $(a,b,c) = (136,91,123)$.

We should remember that Theorem 3 yields the same Heron triangle more than once if we ignore the order in which the sides appear. This depends on the number of ways in which the sides a, b, c may be permuted preserving the constraint $a \geq c$. For instance, the $(17,21,10)$ triangle above for $\lambda = 1$, $m = 4$, $n = 1$ may also be obtained when $\lambda = \frac{3}{7}$, $m = 12$, $n = 7$, or when $\lambda = \frac{6}{7}$, $m = 12$, $n = 7$. The verification is left to the reader. It is time to deduce a number of important corollaries from Theorem 3.

Corollary 4. *Theorem 3 yields the Pythagorean triangles* $(a,b,c) = (u^2+v^2, u^2-v^2, 2uv)$ *for* $\lambda = \dfrac{2v}{u}$, $m = 2$, $n = 1$.

Incidentally, we observe that the famous generators u, v of the Pythagorean triples/triangles readily tell us the ratio in which the Gergonne cevian BE intersects the median AD. Similar observation may be made throughout in an appropriate context.

Corollary 5. *Theorem 3 yields the isosceles Heron triangles* $(a,b,c) = (m^2+n^2, 2(m^2-n^2), m^2+n^2)$ *for* $\lambda = 2$.

Actually, $\lambda = 2$ yields $(a,b,c) = (m^2+4n^2, 2(m^2-4n^2), m^2+4n^2)$. However, the transformation $m \mapsto 2m$, $n \mapsto n$ results in the more familiar form displayed in Corollary 5.

Corollary 6. *Theorem 3 describes the complete set of Heron triangles.*

This is because the Gergonne cevian BE must intersect the median AD at a unique point. Therefore for all Heron triangles $0 < \lambda \leq 2$. Suppose first we fix λ at such a rational number. Then Theorem 3 gives the entire $\lambda-$family of Heron triangles each member of which has BE intersecting AD in the same ratio, that is

λ. Next we vary λ over rational numbers $0 < \lambda \leq 2$. By successive applications of the preceding remark the claim of Corollary 6 follows.

Corollary 7. *[Hoppe's Problem] Theorem 3 yields Heron triangles* $(a, b, c) = (m^2 + 9n^2, 2(m^2 + 3n^2), 3(m^2 + n^2))$ *having the sides in arithmetic progression for* $\lambda = \dfrac{m^2}{6n^2}$.

Here too a remark similar to the one following Corollary 5 applies. Corollaries 4 through 7 give us the key to the solution, often may be partial solutions of many Heron problems: Just consider appropriate $\lambda-$family of Heron triangles. We will continue to amplify on this theme in the sections to follow. To richly illustrate this we prepare a table of $\lambda-$families of Heron triangles. In Table 1, π denotes the perimeter of the triangle.

Table 1. $\lambda-$families of Heron triangles

λ	a	b	c	π	$\triangle$
1	$2(m^2+n^2)$	$3(m^2-2n^2)$	m^2+4n^2	$6m^2$	$6mn(m^2-2n^2)$
$1/2$	$4m^2+n^2$	$5(m^2-n^2)$	m^2+4n^2	$10m^2$	$10mn(m^2-n^2)$
$1/3$	$2(9m^2+n^2)$	$7(3m^2-2n^2)$	$3(m^2+4n^2)$	$42m^2$	$42mn(3m^2-2n^2)$
$2/3$	$9m^2+4n^2$	$4(3m^2-4n^2)$	$3(m^2+4n^2)$	$24m^2$	$24mn(3m^2-4n^2)$
$1/4$	$16m^2+n^2$	$9(2m^2-n^2)$	$2(m^2+4n^2)$	$36m^2$	$36mn(2m^2-n^2)$
$3/4$	$16m^2+9n^2$	$11(2m^2-3n^2)$	$6(m^2+4n^2)$	$44m^2$	$132mn(2m^2-3n^2)$
$1/5$	$2(25m^2+n^2)$	$11(5m^2-2n^2)$	$5(m^2+4n^2)$	$110m^2$	$110mn(5m^2-2n^2)$
$2/5$	$25m^2+4n^2$	$6(5m^2-4n^2)$	$5(m^2+4n^2)$	$60m^2$	$60mn(5m^2-4n^2)$
$3/5$	$2(25m^2+9n^2)$	$13(5m^2-6n^2)$	$15(m^2+4n^2)$	$130m^2$	$390mn(5m^2-6n^2)$
$4/5$	$25m^2+16n^2$	$7(5m^2-8n^2)$	$10(m^2+4n^2)$	$70m^2$	$140mn(5m^2-8n^2)$
$3/2$	$4m^2+9n^2$	$7(m^2-3n^2)$	$3(m^2+4n^2)$	$14m^2$	$42mn(m^2-3n^2)$
$4/3$	$9m^2+16n^2$	$5(3m^2-8n^2)$	$6(m^2+4n^2)$	$30m^2$	$60mn(3m^2-8n^2)$
$5/3$	$2(9m^2+25n^2)$	$11(3m^2-10n^2)$	$15(m^2+4n^2)$	$66m^2$	$330mn(3m^2-10n^2)$
$5/4$	$16m^2+25n^2$	$13(2m^2-5n^2)$	$10(m^2+4n^2)$	$52m^2$	$260mn(2m^2-5n^2)$
$7/4$	$16m^2+49n^2$	$15(2m^2-7n^2)$	$14(m^2+4n^2)$	$60m^2$	$420mn(2m^2-7n^2)$

4. Heron problems and solutions

In what follows we omit the word "determine" from each problem statement. "Heron triangles" will be contracted to HT, and we do *not* provide solutions in detail.

A. Involving sides. A1. HT in which two sides differ by a desired integer. In fact finding one such triangle is equivalent to finding an infinity! This is because they depend on the solution of the so-called Fermat-Pell equation $x^2 - dy^2 = e$, where e is an integer and d not an integer square. It is well-known that Fermat-Pell equations have an infinity of solutions (x, y) (i) when $e = 1$ and (ii) when $e \neq 1$ if there is one. The solution techniques are available in an introductory number theory text, or see [2].

HT in which the three sides are consecutive integers are completely given by Corollary 7. For example, $m = 3$, $n = 1$ gives the (3,4,5); $m = 2$, $n = 1$, the (13,14,15), and so on. Here two sides differ by 1 and incidentally, two sides by 2. However, there are other HT in which two sides differ by 1 (or 2). For another partial solution, consider $\lambda = 1$ family from Table 1. Here $a - c = 1 \Longleftrightarrow m^2 - 2n^2 = 1$. $m = 3$, $n = 2$ gives the $(26, 3, 25)$. $m = 17$, $n = 12$, the (866,3,865) triangle and so on. We observe that 3 is the common side of an infinity of HT. Actually, it is known that *every* integer greater than 2 is a common side of an infinity of HT [1, 2].

To determine a HT in which two sides differ by 3, take $\lambda = \frac{1}{2}$ family and set $b - a = 3$. This leads to the equation $m^2 - 6n^2 = 3$. The solution $(m, n) = (3, 1)$ gives $(a, b, c) = (37, 40, 13)$; $(m, n) = (27, 11)$ gives $(3037, 3040, 1213)$ and so on. This technique can be extended.

A2. A pair of HT having a common side. Consider the pairs $\lambda = 1$, $\lambda = \frac{1}{2}$; $\lambda = \frac{1}{3}$, $\lambda = \frac{2}{3}$; or some two distinct $\lambda-$families that give identical expressions for a particular side. For instance, $m = 3$, $n = 1$ in $\lambda = \frac{1}{3}$ and $\lambda = \frac{2}{3}$ families yields a pair $(164, 175, 39)$ and $(85, 92, 39)$. It is now easy to obtain as many pairs as one desires. This is a quicker solution than the one suggested by *A1*.

A3. A pair of HT in which a pair of corresponding sides are in the ratio $1:2$, $1:3$, $2:3$ etc. The solution lies in the column for side c.

A4. A HT in which two sides sum to a square. We consider $\lambda = \frac{1}{2}$ family where $a + c = 5(m^2 + n^2)$ is made square by $m = 11$, $n = 2$; $(488, 585, 137)$. It is now a simple matter to generate any number of them.

B. Involving perimeter. The perimeter column shows that it is a function of the single parameter m. This enables us to pose, and solve almost effortlessly, many perimeter related problems. To solve such problems by traditional methods would often at best be extremely difficult. Here we present a sample.

B1. A HT in which the perimeter is a square. A glance at Table 1 reveals that $\pi = 36m^2$ for $\lambda = \frac{1}{4}$ family. An infinity of primitive HT of this type is available.

B2. A pair of HT having equal perimeter. An infinity of solution is provided by the $\lambda = \frac{2}{5}$ and $\lambda = \frac{7}{4}$ families. All that is needed is to substitute identical values for m and suitable values to n to ensure the outcome of primitive HT.

B3. A finite number of HT all with equal perimeter. The solution is unbelievably simple! Take *any* λ family and put sufficiently large constant value for m and then vary the values of n only.

A pair of HT in which one perimeter is twice, thrice, ... another, or three or more HT whose perimeters are in arithmetic progression, or a set of four HT such that the sum or the product of two perimeters equals respectively the sum or the product of the other two perimeters are simple games to play. More extensive tables of $\lambda-$family HT coupled with a greater degree of observation ensures that ingenius problem posing solving activity runs wild.

C. Involving area. The $\lambda = \frac{1}{2}$ family has $\triangle = 10mn(m^2 - n^2)$. Now, $mn(m^2 - n^2)$ gives the area of the Pythagorean triangle $(m^2 - n^2, 2mn, m^2 + n^2)$. Because of this an obvious problem has posed and solved itself:

C1. Given a Pythagorean triangle there exists a non-Pythagorean Heron triangle such that the latter area is ten times the former.

It may happen that sometimes one of them may be primitive and the other not, or both may not be primitive. Also, for $m = 2$, $n = 1$, both are Pythagorean. However, there is the $(6, 25, 29)$ Heron triangle with $\triangle = 60$. This close relationship should enable us to put known vast literature on Pythagorean problems to good use, see the following problem for example.

C2. Two Heron triangles having equal area; two HT having areas in the ratio $r : s$.

In [2], pp. 172 – 175, this problem has been solved for right triangles. The primitive solutions are not guaranteed.

D. Miscellaneous problems. In this section we consider problems involving both perimeter and area.

D1. HT in which perimeter equals area. This is such a popular problem that it continues to resurface. It is known that there are just five such HT. The reader is invited to determine them. Hint: They are in $\lambda = \frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$, 1 and $\frac{4}{3}$ families. Possibly elsewhere too, see the remark preceding Corollary 4.

D2. HT in which π and $\triangle$ are squares. In $\lambda = \frac{1}{4}$ family we put $m = 169$, $n = 1$.

D3. Pairs of HT with equal perimeter and equal area in each pair. An infinity of such pairs may be obtained from $\lambda = \frac{1}{2}$ family. We put $m = u^2 + uv + v^2$, $n_1 = u^2 - v^2$ and $m = u^2 + uv + v^2$, $n_2 = 2uv + v^2$. For instance, $u = 3$, $v = 1$ *i.e.*, $m = 13$, $n_1 = 8$, $n_2 = 7$ produces a desired pair $(148, 105, 85)$ and $(145, 120, 73)$. They have $\pi_1 = \pi_2 = 338$ and $\triangle_1 = \triangle_2 = 4368$.

If we accept pairs of HT that may not be primitive then we may consider $\lambda = \frac{2}{3}$ family. Here, $m = p^2 + 3q^2$, $n_1 = p^2 - 3q^2$ and $m = p^2 + 3q^2$, $n_2 = \frac{1}{2}(-p^2 + 6pq + 3q^2)$.

E. Open problems. We may look upon the problem (D3) as follows: $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = 1$. This immediately leads to the following

Open problem 1. Suppose two HTs have perimeters π_1, π_2 and areas $\triangle_1$, $\triangle_2$ such that $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$, a rational number. Prove or disprove the existence of an infinity of HT such that for each pair $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$ holds.

For instance, $\lambda_1 = \frac{1}{5}$, (odd) $m_1 > 4k$, $n_1 = 4k$ and $\lambda_2 = \frac{4}{5}$, $m_2 > 4k$ (again odd), $m_2 = m_1$, $n_2 = 2k$ yield $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{11}{7}$ for $k = 1, 2, 3, \ldots$.

With some effort it is possible to find an infinity of pairs of HT such that for each pair, $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ for certain natural numbers e. This leads to

Open problem 2. Let e be a given natural number. Prove or disprove the existence of an infinity of pairs of HT such that for each pair $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ holds.

5. Conclusion

The present description of Heron triangles did provide simple solutions to certain Heron problems. Additionally it suggested new ones that arose naturally in our discussion. The reader is encouraged to try other $\lambda-$families for different solutions from the presented ones. There is much scope for problem posing and solving activity. Non-standard problems such as: find three Heron triangles whose perimeters (areas) are themselves the sides of a Heron triangle or a Pythagorean triangle. Equally important is to pose unsolved problems. A helpful step in this direction would be to consider Heron analogues of the large variety of existing Pythagorean problems.

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