A Morley Configuration

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Abstract. Given a triangle, the isogonal conjugates of the infinite points of the side lines of the Morley (equilateral) triangle is an equilateral triangle $PQR$ inscribed in the circumcircle. Their isotomic conjugates form another equilateral triangle $P'Q'R'$ inscribed in the Steiner circum-ellipse, homothetic to $PQR$ at the Steiner point. We show that under the one-to-one correspondence $P \leftrightarrow P'$ between the circumcircle and the Steiner circum-ellipse established by isogonal and then isotomic conjugations, this is the only case when both $PQR$ and $P'Q'R'$ are equilateral.

1. Introduction

Consider the Morley triangle $M_aM_bM_c$ of a triangle $ABC$, the equilateral triangle whose vertices are the intersections of pairs of angle trisectors adjacent to a side. Under isogonal conjugation, the infinite points of the Morley lines $M_bM_c$, $M_cM_a$, $M_aM_b$ correspond to three points $G_a$, $G_b$, $G_c$ on the circumcircle. These three points form the vertices of an equilateral triangle. This phenomenon is true for any three lines making $60^\circ$ angles with one another. ¹

Under isotomic conjugation, on the other hand, the infinite points of the same three Morley lines correspond to three points $T_a$, $T_b$, $T_c$ on the Steiner circum-ellipse. It is interesting to note that these three points also form the vertices of an equilateral triangle. Consider the mapping that sends a point $P$ to $P'$, the isotomic conjugate of the isogonal conjugate of $P$. This maps the circumcircle onto the Steiner circum-ellipse. The main result of this paper is that $G_aG_bG_c$ is the only equilateral triangle $PQR$ for which $P'Q'R'$ is also equilateral.

¹ In Figure 1, the isogonal conjugates of the infinite points of the three lines through $A$ are the intersections of the circumcircle with their reflections in the bisector of angle $A$. 

Figure 1
**Main Theorem.** Let \( PQR \) be an equilateral triangle inscribed in the circumcircle. The triangle \( P'Q'R' \) is equilateral if and only if \( P, Q, R \) are the isogonal conjugates of the infinite points of the Morley lines.

Before proving this theorem, we make some observations and interesting applications.

2. **Homothety of** \( G_aG_bG_c \) **and** \( T_aT_bT_c \)

The two equilateral triangles \( G_aG_bG_c \) and \( T_aT_bT_c \) are homothetic at the Steiner point \( S \), with ratio of homothety \( 1 : 4 \sin^2 \Omega \), where \( \Omega \) is the Brocard angle of triangle \( ABC \). The circumcircle of the equilateral triangle \( T_aT_bT_c \) has center at the third Brocard point \(^2\), the isotomic conjugate of the symmedian point, and is tangent to the circumcircle of \( ABC \) at the Steiner point \( S \). In other words, the circle centered at the third Brocard point and passing through the Steiner point intersects the Steiner circum-ellipse at three other points which are the vertices of an equilateral triangle homothetic to the Morley triangle. This circle has radius \( 4R \sin^2 \Omega \) and is smaller than the circumcircle, except when triangle \( ABC \) is equilateral.

The triangle \( G_aG_bG_c \) is the circum-tangential triangle in [3]. It is homothetic to the Morley triangle. From this it follows that the points \( G_a, G_b, G_c \) are the points of tangency with the circumcircle of the deltoid which is the envelope of the axes of inscribed parabolas. \(^3\)

\(^2\)This point is denoted by \( X_{76} \) in [3].

\(^3\)The axis of an inscribed parabola with focus \( F \) is the perpendicular from \( F \) to its Simson line, or equivalently, the homothetic image of the Simson line of the antipode of \( F \) on the circumcircle, with homothetic center \( G \) and ratio \( -2 \). In [5], van Lamoen has shown that the points of contact of Simson lines tangent to the nine-point circle also form an equilateral triangle homothetic to the Morley triangle.
3. Equilateral triangles inscribed in an ellipse

Let $\mathcal{E}$ be an ellipse centered at $O$, and $U$ a point on $\mathcal{E}$. With homothetic center $O$, ratio $-\frac{1}{2}$, maps $U$ to $u$. Construct the parallel through $u$ to its polar with respect to $\mathcal{E}$, to intersect the ellipse at $V$ and $W$. The circumcircle of $UVW$ intersects $\mathcal{E}$ at the Steiner point $S$ of triangle $UVW$. Let $M$ be the third Brocard point of $UVW$. The circle, center $M$, passing through $S$, intersects $\mathcal{E}$ at three other points which form the vertices of an equilateral triangle. See Figure 3.

![Figure 3](image)

From this it follows that the locus of the centers of equilateral triangles inscribed in the Steiner circum-ellipse of $ABC$ is the ellipse

$$
\sum_{\text{cyclic}} a^2 (a^2 + b^2 + c^2) x^2 + (a^2(b^2 + c^2) - (2b^4 + b^2c^2 + 2c^4))yz = 0
$$

with the same center and axes.

4. Some preliminary results

**Proposition 1.** If a circle through the focus of a parabola has its center on the directrix, there exists an equilateral triangle inscribed in the circle, whose side lines are tangent to the parabola.

**Proof.** Denote by $p$ the distance from the focus $F$ of the parabola to its directrix. In polar coordinates with the pole at $F$, let the center of the circle be the point $(\frac{p}{\cos \alpha}, \alpha)$. The radius of the circle is $R = \frac{p}{\cos \alpha}$. See Figure 4. If this center is at a distance $d$ to the line tangent to the parabola at the point $(\frac{p}{1+\cos \theta}, \theta)$, then

$$
d = \frac{d}{R} = \frac{\cos(\theta - \alpha)}{2 \cos \frac{\theta}{2}}.
$$

Thus, for $\theta = \frac{2}{3} \alpha$, $\frac{2}{3}(\alpha + \pi)$ and $\frac{2}{3}(\alpha - \pi)$, we have $d = \frac{2}{3}p$, and the lines tangent to the parabola at these three points form the required equilateral triangle. \qed

**Proposition 2.** If $P$ lies on the circumcircle, then the line $PP'$ passes through the Steiner point $S$.\footnote{More generally, if $u + v + w = 0$, the line joining $(\frac{u}{m-n} : \frac{v}{m-n} : \frac{w}{m-n})$ to $(\frac{u}{l-n} : \frac{v}{l-n} : \frac{w}{l-n})$ which is the fourth intersection of the two circumconics $\frac{u}{m-n} + \frac{v}{m-n} + \frac{w}{m-n} = 0$ and $\frac{u}{l-n} + \frac{v}{l-n} + \frac{w}{l-n} = 0.$}
It follows that a triangle $PQR$ inscribed in the circumcircle is always perspective with $P'Q'R'$ (inscribed in the Steiner circum-ellipse) at the Steiner point. The perspectrix is a line parallel to the tangent to the circumcircle at the focus of the Kiepert parabola.\(^5\)

We shall make use of the Kiepert parabola

\[
P : \sum (b^2 - c^2)^2 x^2 - 2(c^2 - a^2)(a^2 - b^2)yz = 0.
\]

This is the inscribed parabola with perspector the Steiner point $S$, focus $S' = \left( \frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2} \right)$,\(^6\) and the Euler line as directrix. For more on inscribed parabolas and inscribed conics in general, see [1].

**Proposition 3.** Let $PQ$ be a chord of the circumcircle. The following statements are equivalent:\(^7\)

(a) $PQ$ and $P'Q'$ are parallel.

(b) The line $PQ$ is tangent to the Kiepert parabola $P$.

(c) The Simson lines $s(P)$ and $s(Q)$ intersect on the Euler line.

**Proof.** If the line $PQ$ is $ux + vy + wz = 0$, then $P'Q'$ is $a^2ux + b^2vy + c^2wz = 0$. These two lines are parallel if and only if

\[
\frac{b^2 - c^2}{u} + \frac{c^2 - a^2}{v} + \frac{a^2 - b^2}{w} = 0,
\]

which means that $PQ$ is tangent to the Kiepert parabola.

The common point of the Simson lines $s(P)$ and $s(Q)$ is $(x : y : z)$, where

\[
x = \frac{(2b^2(c^2 + a^2 - b^2)v + 2c^2(a^2 + b^2 - c^2)w - (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)u)}{((a^2 + b^2 - c^2)v + (c^2 + a^2 - b^2)w - 2a^2u)},
\]

\(^5\)This line is also parallel to the trilinear polars of the two isodynamic points.

\(^6\)This is the point $X_{110}$ in [3].

\(^7\)These statements are also equivalent to (d): The orthopole of the line $PQ$ lies on the Euler line.
and $y$ and $z$ can be obtained by cyclically permuting $a, b, c,$ and $u, v, w$. This point lies on the Euler line if and only if (1) is satisfied.

In the following proposition, $(\ell_1, \ell_2)$ denotes the directed angle between two lines $\ell_1$ and $\ell_2$. This is the angle through which the line $\ell_1$ must be rotated in the positive direction in order to become parallel to, or to coincide with, the line $\ell_2$. See [2, §§16–19.].

**Proposition 4.** Let $P, Q, R$ be points on the circumcircle. The following statements are equivalent.

(a) The Simson lines $s(P), s(Q), s(R)$ are concurrent.

(b) $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$.

(c) $s(P)$ and $QR$ are perpendicular; so are $s(Q)$ and $RP$; $s(R)$ and $PQ$.

**Proof.** See [4, §§2.16–20].

**Proposition 5.** A line $\ell$ is parallel to a side of the Morley triangle if and only if

$$(AB, \ell) + (BC, \ell) + (CA, \ell) = 0 \pmod{\pi}.$$ 

**Proof.** Consider the Morley triangle $M_aM_bM_c$. The line $BM_c$ and $CM_b$ intersecting at $P$, the triangle $PM_bM_c$ is isosceles and $(M_cM_b, M_cP) = \frac{1}{3}(B + C)$. Thus, $(BC, M_bM_c) = \frac{1}{3}(B - C)$. Similarly, $(CA, M_bM_c) = \frac{1}{3}(C - A) + \frac{\pi}{3}$, and $(AB, M_bM_c) = \frac{1}{3}(A - B) - \frac{\pi}{3}$. Thus

$$(AB, M_bM_c) + (BC, M_bM_c) + (CA, M_bM_c) = 0 \pmod{\pi}.$$ 

There are only three directions of line $\ell$ for which $(AB, \ell) + (BC, \ell) + (CA, \ell) = 0$. These can only be the directions of the Morley lines.
5. Proof of Main Theorem

Let $\mathcal{P}$ be the Kiepert parabola of triangle $ABC$. By Proposition 1, there is an equilateral triangle $PQR$ inscribed in the circumcircle whose sides are tangent to $\mathcal{P}$. By Propositions 2 and 3, the triangle $P'Q'R'$ is equilateral and homothetic to $PQR$ at the Steiner point $S$. By Proposition 3 again, the Simson lines $s(P)$, $s(Q)$, $s(R)$ concur. It follows from Proposition 4 that $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$. Since the lines $PQ$, $QR$, and $RP$ make $60^\circ$ angles with each other, we have

$$(AB, PQ) + (BC, PQ) + (CA, PQ) = 0 \pmod{\pi},$$

and $PQ$ is parallel to a side of the Morley triangle by Proposition 5. Clearly, this is the same for $QR$ and $RP$. By Proposition 4, the vertices $P$, $Q$, $R$ are the isogonal conjugates of the infinite points of the Morley sides.
**Uniqueness:** For \( M(x : y : z) \), let

\[ f(M) = \frac{x + y + z}{a^2 + b^2 + c^2}. \]

The determinant of the affine mapping \( P \mapsto P', Q \mapsto Q', R \mapsto R' \) is

\[ \frac{f(P)f(Q)f(R)}{a^2b^2c^2}. \]

This determinant is positive for \( P, Q, R \) on the circumcircle, which does not intersect the Lemoine axis \( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 0 \). Thus, if both triangles are equilateral, the similitude \( P \mapsto P', Q \mapsto Q', R \mapsto R' \) is a direct one. Hence,

\[ (SP', SQ') = (SP, SQ) = (RP, RQ) = (RP', R'Q'), \]

and the circle \( P'Q'R' \) passes through \( S \). Now, through any point on an ellipse, there is a unique circle intersecting the ellipse again at the vertices of an equilateral triangle. This establishes the uniqueness, and completes the proof of the theorem.

**6. Concluding remarks**

We conclude with a remark and a generalization.

(1) The reflection of \( G_aG_bG_c \) in the circumcenter is another equilateral triangle \( PQR \) (inscribed in the circumcircle) whose sides are parallel to the Morley lines.\(^8\) This, however, does not lead to an equilateral triangle inscribed in the Steiner circum-ellipse.

(2) Consider the circum-hyperbola \( C \) through the centroid \( G \) and the symmedian point \( K \).\(^9\) For any point \( P \) on \( C \), let \( C_P \) be the circumconic with perspector \( P \), intersecting the circumcircle again at a point \( S_P \).\(^{10}\) For every point \( M \) on the

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\(^8\)This is called the circumnormal triangle in [3].

\(^9\)The center of this hyperbola is the point \( (a^4(b^2 - c^2)^2 : b^4(c^2 - a^2)^2 : c^4(a^2 - b^2)^2) \).

\(^{10}\)The perspector of a circumconic is the perspector of the triangle bounded by the tangents to the conic at the vertices of \( ABC \). If \( P = (u : v : w) \), the circumconic \( C_P \) has center \( (u(v + w - u) : v(w + u - v) : w(u + v - w)) \), and \( S_P \) is the point \( (\frac{1}{b^4w - c^2v} : \frac{1}{c^4u - a^2w} : \frac{1}{a^4v - b^2u}) \). See Footnote 4.
circumcircle, denote by $M'$ the second common point of $C_U$ and the line $MS_P$. Then, if $G_a, G_b, G_c$ are the isogonal conjugates of the infinite points of the Morley lines, $G'_a G'_b G'_c$ is homothetic to $G_a G_b G_c$ at $S_U$. The reason is simple: Proposition 3 remains true. For $U = G$, this gives the equilateral triangle $T_a T_b T_c$ inscribed in the case of the Steiner circum-ellipse. Here is an example. For $U = (a(b + c) : b(c + a) : c(a + b))$, \(^{11}\) we have the circumellipse with center the Spieker center $(b + c : c + a : a + b)$. The triangles $G_a G_b G_c$ and $G'_a G'_b G'_c$ are homothetic at $X_{100} = (\frac{a}{b+c} : \frac{b}{c+a} : \frac{c}{a+b})$, and the circumcircle of $G'_a G'_b G'_c$ is the incircle of the anticomplementary triangle, center the Nagel point, and ratio of homothety $R : 2r$.

![Figure 9](image_url)

References


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\(^{11}\)This is the point $X_{37}$ in [3].