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Concurrency of Four Euler Lines

Antreas P. Hatzipolakis, Floor van Lamoen, Barry Wolk, and Paul Yiu

Abstract. Using tripolar coordinates, we prove that if P is a point in the plane of triangle ABC such that the Euler lines of triangles PBC, APC and ABP are concurrent, then their intersection lies on the Euler line of triangle ABC. The same is true for the Brocard axes and the lines joining the circumcenters to the respective incenters. We also prove that the locus of P for which the four Euler lines concur is the same as that for which the four Brocard axes concur. These results are extended to a family $\mathcal{L}_n$ of lines through the circumcenter. The locus of P for which the four $\mathcal{L}_n$ lines of ABC, PBC, APC and ABP concur is always a curve through 15 finite real points, which we identify.

1. Four line concurrency

Consider a triangle ABC with incenter I. It is well known [13] that the Euler lines of the triangles IBC, AIC and ABI concur at a point on the Euler line of ABC, the Schiffler point with homogeneous barycentric coordinates [1]

$$\left(\frac{a(s-a)}{b+c} : \frac{b(s-b)}{c+a} : \frac{c(s-c)}{a+b}\right).$$

There are other notable points which we can substitute for the incenter, so that a similar statement can be proven relatively easily. Specifically, we have the following interesting theorem.

Theorem 1. *Let P be a point in the plane of triangle ABC such that the Euler lines of the component triangles PBC, APC and ABP are concurrent. Then the point of concurrency also lies on the Euler line of triangle ABC.*

When one tries to prove this theorem with homogeneous coordinates, calculations turn out to be rather tedious, as one of us has noted [14]. We present an easy analytic proof, making use of tripolar coordinates. The same method applies if we replace the Euler lines by the Brocard axes or the OI-lines joining the circumcenters to the corresponding incenters.

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[1] This appears as X_{21} in Kimberling's list [7]. In the expressions of the coordinates, s stands for the semiperimeter of the triangle.

2. Tripolar coordinates

Given triangle ABC with $BC = a$, $CA = b$, and $AB = c$, consider a point P whose distances from the vertices are $PA = \lambda$, $PB = \mu$ and $PC = \nu$. The precise relationship among λ, μ, and ν dates back to Euler [4]:

$$(\mu^2 + \nu^2 - a^2)^2\lambda^2 + (\nu^2 + \lambda^2 - b^2)^2\mu^2 + (\lambda^2 + \mu^2 - c^2)^2\nu^2$$
$$-(\mu^2 + \nu^2 - a^2)(\nu^2 + \lambda^2 - b^2)(\lambda^2 + \mu^2 - c^2) - 4\lambda^2\mu^2\nu^2 = 0.$$

See also [1, 2]. Geometers in the 19th century referred to the triple (λ, μ, ν) as the *tripolar* coordinates of P. A comprehensive introduction can be found in [12]. [2] This series begins with the following easy theorem.

Proposition 2. *An equation of the form $\ell\lambda^2 + m\mu^2 + n\nu^2 + q = 0$ represents a circle or a line according as $\ell + m + n$ is nonzero or otherwise.*

The center of the circle has homogeneous barycentric coordinates $(\ell : m : n)$. If $\ell + m + n = 0$, the line is orthogonal to the direction $(\ell : m : n)$. Among the applications one finds the equation of the Euler line in tripolar coordinates [op. cit. §26]. [3]

Proposition 3. *The tripolar equation of the Euler line is*

$$(b^2 - c^2)\lambda^2 + (c^2 - a^2)\mu^2 + (a^2 - b^2)\nu^2 = 0. \tag{1}$$

We defer the proof of this proposition to §5 below. Meanwhile, note how this applies to give a simple proof of Theorem 1.

3. Proof of Theorem 1

Let P be a point with tripolar coordinates (λ, μ, ν) such that the Euler lines of triangles PBC, APC and ABP intersect at a point Q with tripolar coordinates (λ', μ', ν'). We denote the distance PQ by ρ.

Applying Proposition 3 to the triangles PBC, APC and ABP, we have

$$\begin{aligned}
(\nu^2 - \mu^2)\rho^2 &+ (\mu^2 - a^2)\mu'^2 + (a^2 - \nu^2)\nu'^2 = 0,\\
(b^2 - \lambda^2)\lambda'^2 &+ (\lambda^2 - \nu^2)\rho^2 + (\nu^2 - b^2)\nu'^2 = 0,\\
(\lambda^2 - c^2)\lambda'^2 &+ (c^2 - \mu^2)\mu'^2 + (\mu^2 - \lambda^2)\rho^2 = 0.
\end{aligned}$$

Adding up these equations, we obtain (1) with λ', μ', ν' in lieu of λ, μ, ν. This shows that Q lies on the Euler line of ABC.

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[2] [5] and [8] are good references on tripolar coordinates.

[3] The tripolar equations of the lines in §§5 – 7 below can be written down from the barycentric equations of these lines. The calculations in these sections, however, do not make use of these barycentric equations.

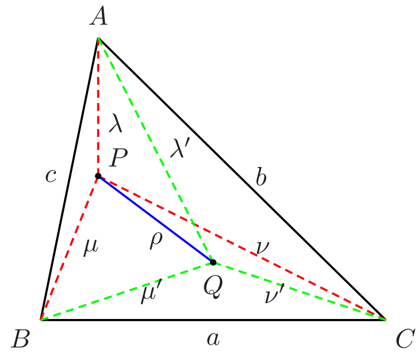

Figure 1

4. Tripolar equations of lines through the circumcenter

O. Bottema [2, pp.37–38] has given a simple derivation of the equation of the Euler line in tripolar coordinates. He began with the observation that since the point-circles

$$\lambda^2 = 0, \qquad \mu^2 = 0, \qquad \nu^2 = 0,$$

are all orthogonal to the circumcircle,[4] for arbitrary t_1, t_2, t_3, the equation

$$t_1\lambda^2 + t_2\mu^2 + t_3\nu^2 = 0 \tag{2}$$

represents a circle orthogonal to the circumcircle. By Proposition 2, this represents a line through the circumcenter if and only if $t_1 + t_2 + t_3 = 0$.

5. Tripolar equation of the Euler line

Consider the centroid G of triangle ABC. By the Apollonius theorem, and the fact that G divides each median in the ratio $2 : 1$, it is easy to see that the tripolar coordinates of G satisfy

$$\lambda^2 : \mu^2 : \nu^2 = 2b^2 + 2c^2 - a^2 : 2c^2 + 2a^2 - b^2 : 2a^2 + 2b^2 - c^2.$$

It follows that the Euler line OG is defined by (2) with t_1, t_2, t_3 satisfying

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & + & t_3 & = & 0, \\ (2b^2+2c^2-a^2)t_1 & + & (2c^2+2a^2-b^2)t_2 & + & (2a^2+2b^2-c^2)t_3 & = & 0, \end{array}$$

or
$$t_1 : t_2 : t_3 = b^2 - c^2 : c^2 - a^2 : a^2 - b^2.$$

This completes the proof of Proposition 3.

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[4]These point-circles are evidently the vertices of triangle ABC.

6. Tripolar equation of the OI-line

For the incenter I, we have

$$\lambda^2 : \mu^2 : \nu^2 = \csc^2\frac{A}{2} : \csc^2\frac{B}{2} : \csc^2\frac{C}{2} = \frac{s-a}{a} : \frac{s-b}{b} : \frac{s-c}{c},$$

where $s = \frac{a+b+c}{2}$. The tripolar equation of the OI-line is given by (2) with t_1, t_2, t_3 satisfying

$$t_1 + t_2 + t_3 = 0, \qquad \frac{s-a}{a}t_1 + \frac{s-b}{b}t_2 + \frac{s-c}{c}t_3 = 0.$$

From these, $t_1 : t_2 : t_3 = \frac{1}{b} - \frac{1}{c} : \frac{1}{c} - \frac{1}{a} : \frac{1}{a} - \frac{1}{b}$, and the tripolar equation of the OI-line is

$$\left(\frac{1}{b} - \frac{1}{c}\right)\lambda^2 + \left(\frac{1}{c} - \frac{1}{a}\right)\mu^2 + \left(\frac{1}{a} - \frac{1}{b}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the OI-lines.

7. Tripolar equation of the Brocard axis

The Brocard axis is the line joining the circumcenter to the symmedian point. Since this line contains the two isodynamic points, whose tripolar coordinates, by definition, satisfy

$$\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c},$$

it is easy to see that the tripolar equation of the Brocard axis is[5]

$$\left(\frac{1}{b^2} - \frac{1}{c^2}\right)\lambda^2 + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\mu^2 + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the Brocard axes.

8. The lines $\mathcal{L}_n$

The resemblance of the tripolar equations in §§5 – 7 suggests consideration of the family of lines through the circumcenter:

$$\mathcal{L}_n : \qquad (b^n - c^n)\lambda^2 + (c^n - a^n)\mu^2 + (a^n - b^n)\nu^2 = 0,$$

for nonzero integers n. The Euler line, the Brocard axis, and the OI-line are respectively $\mathcal{L}_n$ for $n = 2, -2$, and -1. In homogeneous barycentric coordinates,

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[5]The same equation can be derived directly from the tripolar distances of the symmedian point: $AK^2 = \frac{b^2c^2(2b^2+2c^2-a^2)}{(a^2+b^2+c^2)^2}$ etc. This can be found, for example, in [11, p.118].

the equation of $\mathcal{L}_n$ is[6]

$$\sum_{\text{cyclic}} (a^n(b^2-c^2)-(b^{n+2}-c^{n+2}))x = 0.$$

The line $\mathcal{L}_1$ contains the points [7]

$$(2a+b+c : a+2b+c : a+b+2c)$$

and

$$(a(b+c)-(b-c)^2 : b(c+a)-(c-a)^2 : c(a+b)-(a-b)^2).$$

Theorem 1 obviously applies when the Euler lines are replaced by $\mathcal{L}_n$ lines for a fixed nonzero integer n.

9. Intersection of the $\mathcal{L}_n$ lines

It is known that the locus of P for which the Euler lines ($\mathcal{L}_2$) of triangles PBC, APC and ABP are concurrent is the union of the circumcircle and the Neuberg cubic.[8] See [10, p.200]. Fred Lang [9] has computed the locus for the Brocard axes ($\mathcal{L}_{-2}$) case, and found exactly the same result. The coincidence of these two loci is a special case of the following theorem.

Theorem 4. *Let n be a nonzero integer. The $\mathcal{L}_n$ lines of triangles PBC, APC and ABP concur (at a point on $\mathcal{L}_n$) if and only if the $\mathcal{L}_{-n}$ lines of the same triangles concur (at a point on $\mathcal{L}_{-n}$).*

Proof. Consider the component triangles PBC, APC and ABP of a point P. If P has tripolar coordinates (L, M, N), then the $\mathcal{L}_n$ lines of these triangles have tripolar equations

$$\mathcal{L}_n(PBC): \quad (N^n-M^n)\rho^2+(M^n-a^n)\mu^2+(a^n-N^n)\nu^2=0,$$

$$\mathcal{L}_n(APC): \quad (b^n-L^n)\lambda^2+(L^n-N^n)\rho^2+(N^n-b^n)\nu^2=0,$$

$$\mathcal{L}_n(ABP): \quad (L^n-c^n)\lambda^2+(c^n-M^n)\mu^2+(M^n-L^n)\rho^2=0,$$

where ρ is the distance between P and a variable point (λ, μ, ν).[9] These equations can be rewritten as

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[6]This can be obtained from the tripolar equation by putting

$$\lambda^2 = \frac{1}{(x+y+z)^2}(c^2y^2+(b^2+c^2-a^2)yz+b^2z^2)$$

and analogous expressions for μ^2 and ν^2 obtained by cyclic permutations of a, b, c and x, y, z.

[7]These are respectively the midpoint between the incenters of ABC and its medial triangle, and the symmedian point of the excentral triangle of the medial triangle.

[8]The Neuberg cubic is defined as the locus of points P such that the line joining P to its isogonal conjugate is parallel to the Euler line.

[9]See Figure 1, with λ, μ, ν replaced by L, M, N, and λ', μ', ν' by λ, μ, ν respectively.

$$\begin{array}{rcrcrcl} & - & (M^n-a^n)(\rho^2-\mu^2) & + & (N^n-a^n)(\rho^2-\nu^2) & = & 0, \\ (L^n-b^n)(\rho^2-\lambda^2) & & & - & (N^n-b^n)(\rho^2-\nu^2) & = & 0, \\ -(L^n-c^n)(\rho^2-\lambda^2) & + & (M^n-c^n)(\rho^2-\mu^2) & & & = & 0. \end{array} \tag{3}$$

One trivial solution to these equations is $\rho = \lambda = \mu = \nu$, which occurs only when the variable point is the circumcenter O, with P on the circumcircle. In this case the $\mathcal{L}_n$ lines all concur at the point O, for all n. Otherwise, we have a solution to (3) with at least one of the values $\rho^2 - \lambda^2$, $\rho^2 - \mu^2$, and $\rho^2 - \nu^2$ being non-zero. And the condition for a solution of this kind is

$$(L^n - b^n)(M^n - c^n)(N^n - a^n) = (L^n - c^n)(M^n - a^n)(N^n - b^n). \tag{4}$$

This condition is clearly necessary. Conversely, take P satisfying (4). This says that (3), as linear homogeneous equations in $\rho^2 - \lambda^2$, $\rho^2 - \mu^2$, and $\rho^2 - \nu^2$, have a nontrivial solution (u, v, w), which is determined up to a scalar multiple. Then the equations of the $\mathcal{L}_n$ lines of triangles ABP and PBC can be rewritten as $(\frac{1}{u} - \frac{1}{v})XP^2 - \frac{1}{u}XA^2 + \frac{1}{v}XB^2 = 0$ and $(\frac{1}{v} - \frac{1}{w})XP^2 - \frac{1}{v}XB^2 + \frac{1}{w}XC^2 = 0$. If X is a point common to these two lines, then it satisfies

$$\frac{XP^2 - XA^2}{u} = \frac{XP^2 - XB^2}{v} = \frac{XP^2 - XC^2}{w}$$

and also lies on the $\mathcal{L}_n$ line of triangle APC.

Note that (4) is clearly equivalent to

$$\left(\frac{1}{L^n}-\frac{1}{b^n}\right)\left(\frac{1}{M^n}-\frac{1}{c^n}\right)\left(\frac{1}{N^n}-\frac{1}{a^n}\right) = \left(\frac{1}{L^n}-\frac{1}{c^n}\right)\left(\frac{1}{M^n}-\frac{1}{a^n}\right)\left(\frac{1}{N^n}-\frac{1}{b^n}\right),$$

which, by exactly the same reasoning, is the concurrency condition for the $\mathcal{L}_{-n}$ lines of the same triangles. □

Corollary 5. *The locus of P for which the Brocard axes of triangles PBC, APC and ABP are concurrent (at a point on the Brocard axis of triangle ABC) is the union of the circumcircle and the Neuberg cubic.*

Let $\mathcal{C}_n$ be the curve with tripolar equation

$$(\lambda^n - b^n)(\mu^n - c^n)(\nu^n - a^n) = (\lambda^n - c^n)(\mu^n - a^n)(\nu^n - b^n),$$

so that together with the circumcircle, it constitutes the locus of points P for which the four $\mathcal{L}_n$ lines of triangles PBC, APC, ABP and ABC concur.[10] The symmetry of equation (4) leads to the following interesting fact.

Corollary 6. *If P lies on the $\mathcal{C}_n$ curve of triangle ABC, then A (respectively B, C) lies on the $\mathcal{C}_n$ curve of triangle PBC (respectively APC, ABP).*

Remark. The equation of $\mathcal{C}_n$ can also be written in one of the following forms:

$$\sum_{\text{cyclic}} (b^n - c^n)(a^n\lambda^n + \mu^n\nu^n) = 0$$

[10]By Theorem 4, it is enough to consider n positive.

or

$$\begin{vmatrix} \lambda^n + a^n & \mu^n + b^n & \nu^n + c^n \\ a^n\lambda^n & b^n\mu^n & c^n\nu^n \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

10. Points common to $\mathcal{C}_n$ curves

Proposition 7. *A complete list of finite real points common to all $\mathcal{C}_n$ curves is as follows:*

(1) the vertices A, B, C and their reflections on the respective opposite side,
(2) the apexes of the six equilateral triangles erected on the sides of ABC,
(3) the circumcenter, and
(4) the two isodynamic points.

Proof. It is easy to see that each of these points lies on $\mathcal{C}_n$ for every positive integer n. For the isodynamic points, recall that $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. We show that $\mathcal{C}_1$ and $\mathcal{C}_2$ meet precisely in these 15 points. From their equations

$$(\lambda - b)(\mu - c)(\nu - a) = (\lambda - c)(\mu - a)(\nu - b) \tag{5}$$

and

$$(\lambda^2 - b^2)(\mu^2 - c^2)(\nu^2 - a^2) = (\lambda^2 - c^2)(\mu^2 - a^2)(\nu^2 - b^2). \tag{6}$$

If both sides of (5) are zero, it is easy to list the various cases. For example, solutions like $\lambda = b, \mu = a$ lead to a vertex and its reflection through the opposite side (in this case C and its reflection in AB); solutions like $\lambda = b$, $\nu = b$ lead to the apexes of equilateral triangles erected on the sides of ABC (in this case on AC). Otherwise we can factor and divide, getting

$$(\lambda + b)(\mu + c)(\nu + a) = (\lambda + c)(\mu + a)(\nu + b).$$

Together with (5), this is easy to solve. The only solutions in this case are $\lambda = \mu = \nu$ and $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, giving respectively $P = O$ and the isodynamic points. $\square$

Remarks. (1) If P is any of the points listed above, then this result says that the triangles ABC, PBC, APC, and ABP have concurrent $\mathcal{L}_n$ lines, for all non-zero integers n. There is no degeneracy in the case where P is an isodynamic point, and we then get an infinite sequence of four-fold concurrences.

(2) The curve $\mathcal{C}_4$ has degree 7, and contains the two circular points at infinity, each of multiplicity 3. These, together with the 15 finite real points above, account for all 21 intersections of $\mathcal{C}_2$ and $\mathcal{C}_4$.

11. Intersections of Euler lines and of Brocard axes

For $n = \pm 2$, the curve $\mathcal{C}_n$ is the Neuberg cubic

$$\sum_{\text{cyclic}} ((b^2 - c^2)^2 + a^2(b^2 + c^2) - 2a^4)x(c^2y^2 - b^2z^2) = 0$$

in homogeneous barycentric coordinates. Apart from the points listed in Proposition 7, this cubic contains the following notable points: the orthocenter, incenter and excenters, the Fermat points, and the Parry reflection point.[11] A summary of interesting properties of the Neuberg cubic can be found in [3]. Below we list the corresponding points of concurrency, giving their coordinates. For points like the Fermat points and Napoleon points resulting from erecting equilateral triangles on the sides, we label the points by $\epsilon = +1$ or -1 according as the equilateral triangles are constructed exterior to ABC or otherwise. Also, Δ stands for the area of triangle ABC. For functions like F_a, F_b, F_c indexed by a, b, c, we obtain F_b and F_c from F_a by cyclic permutations of a, b, c.

P	Intersection of Euler lines	Intersection of Brocard axes
Circumcenter	Circumcenter	Circumcenter
Reflection of vertex on opposite side	Intercept of Euler line on the side line	Intercept of Brocard axis on the side line
Orthocenter	Nine-point center	Orthocenter of orthic triangle
Incenter	Schiffler point	Isogonal conjugate of Spieker center
Excenters $I_a = (-a : b : c)$ $I_b = (a : -b : c)$ $I_c = (a : b : -c)$	$(\frac{as}{b+c} : \frac{b(s-c)}{c-a} : \frac{c(s-b)}{-a+b})$ $(\frac{a(s-c)}{-b+c} : \frac{bs}{c+a} : \frac{c(s-a)}{a-b})$ $(\frac{a(s-b)}{b-c} : \frac{b(s-a)}{-c+a} : \frac{cs}{a+b})$	$(\frac{a^2}{b+c} : \frac{b^2}{c-a} : \frac{c^2}{-a+b})$ $(\frac{a^2}{-b+c} : \frac{b^2}{c+a} : \frac{c^2}{a-b})$ $(\frac{a^2}{b-c} : \frac{b^2}{-c+a} : \frac{c^2}{a+b})$
ϵ-Fermat point	centroid	Isogonal conjugate of $(-\epsilon)$-Napoleon point
ϵ-isodynamic point		Isogonal conjugate of ϵ-Napoleon point

Apexes of ϵ-equilateral triangles erected on the sides of ABC. Let P be the apex of an equilateral triangle erected the side BC. This has coordinates

$$\left(-2a^2 : a^2+b^2-c^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta : c^2+a^2-b^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta\right).$$

The intersection of the Euler lines has coordinates

$$\Big(-a^2(a^2-b^2)(a^2-c^2) \;:\; (a^2-b^2)(a^2b^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta(a^2+b^2-c^2)) \;:\; (a^2-c^2)(a^2c^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta(c^2+a^2-b^2))\Big),$$

and the Brocard axis intersection is the point

$$\Big(a^2(a^2-b^2)(a^2-c^2)(-\epsilon(b^2+c^2-a^2)+4\sqrt{3}\Delta)$$
$$: b^2(a^2-b^2)(-\epsilon(a^4+2b^4+3c^4-5b^2c^2-4c^2a^2-3a^2b^2)+4\sqrt{3}\Delta(c^2+a^2))$$
$$: c^2(a^2-c^2)(-\epsilon(a^4+3b^4+2c^4-5b^2c^2-3c^2a^2-4a^2b^2)+4\sqrt{3}\Delta(a^2+b^2))\Big).$$

[11]Bernard Gibert has found that the Fermat points of the anticomplementary triangle of ABC also lie on the Neuberg cubic. These are the points X_{616} and X_{617} in [7]. Their isogonal conjugates (in triangle ABC) clearly lie on the Neuberg cubic too. Ed.

Isodynamic points. For the ϵ-isodynamic point, the Euler line intersections are

$$\begin{aligned}&(a^2(\sqrt{3}b^2c^2 + \epsilon\cdot 4\Delta(b^2+c^2-a^2))\\ &: b^2(\sqrt{3}c^2a^2 + \epsilon\cdot 4\Delta(c^2+a^2-b^2))\\ &: c^2(\sqrt{3}a^2b^2 + \epsilon\cdot 4\Delta(a^2+b^2-c^2))).\end{aligned}$$

These points divide the segment GO harmonically in the ratio $8\sin A\sin B\sin C : 3\sqrt{3}$.[12] The Brocard axis intersections for the Fermat points and the isodynamic points are illustrated in Figure 2.

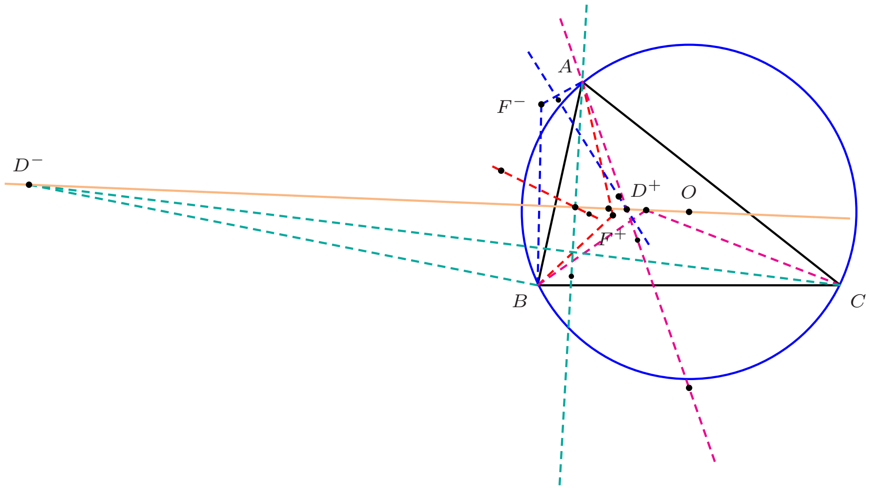

Figure 2

The Parry reflection point. This is the reflection of the circumcenter in the focus of the Kiepert parabola. [13] Its coordinates, and those of the Euler line and Brocard axis intersections, can be described with the aids of three functions.

(1) Parry reflection point: $(a^2P_a : b^2P_b : c^2P_c)$,
(2) Euler line intersection: $(a^2P_af_a : b^2P_bf_b : c^2P_cf_c)$,
(3) Brocard axis intersection: $(a^2f_ag_a : b^2f_bg_b : c^2f_cg_c)$, where

$$\begin{aligned}P_a &= a^8 - 4a^6(b^2+c^2) + a^4(6b^4+b^2c^2+6c^4)\\ &\quad -a^2(b^2+c^2)(4b^4-5b^2c^2+4c^4) + (b^2-c^2)^2(b^4+4b^2c^2+c^4),\\ f_a &= a^6 - 3a^4(b^2+c^2) + a^2(3b^4-b^2c^2+3c^4) - (b^2-c^2)^2(b^2+c^2),\\ g_a &= 5a^8 - 14a^6(b^2+c^2) + a^4(12b^4+17b^2c^2+12c^4)\\ &\quad -a^2(b^2+c^2)(2b^2+c^2)(b^2+2c^2) - (b^2-c^2)^4.\end{aligned}$$

[12] These coordinates, and those of the Brocard axis intersections, can be calculated by using the fact that triangle PBC has $(-\epsilon)$-isodynamic point at the vertex A and circumcenter at the point

$$(a^2((b^2+c^2-a^2)-\epsilon\cdot 4\sqrt{3}\Delta) : b^2((c^2+a^2-b^2)+\epsilon\cdot 4\sqrt{3}\Delta) : c^2((a^2+b^2-c^2)+\epsilon\cdot 4\sqrt{3}\Delta)).$$

[13] The Parry reflection point is the point X_{399} in [6]. The focus of the Kiepert parabola is the point on the circumcircle with coordinates $\left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$.

This completes the identification of the Euler line and Brocard axis intersections for points on the Neuberg cubic. The identification of the locus for the $\mathcal{L}_{\pm 1}$ problems is significantly harder. Indeed, we do not know of any interesting points on this locus, except those listed in Proposition 7.

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Antreas P. Hatzipolakis: 81 Patmou Street, Athens 11144, Greece
E-mail address: xpolakis@otenet.gr

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Barry Wolk: Department of Mathematics, University of Manitoba, Wennipeg, Manitoba, R3T 2N2, Canada
E-mail address: wolkb@cc.umanitoba.ca

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: yiu@fau.edu