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The Simson cubic

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Abstract. The Simson cubic is the locus of the trilinear poles of the Simson lines. There exists a conic such that a point M lies on the Simson cubic if and only if the line joining M to its isotomic conjugate is tangent to this conic. We also characterize cubics which admit pivotal conics for a given isoconjugation.

1. Introduction

Antreas P. Hatzipolakis [2] has raised the question of the locus of points for which the triangle bounded by the pedal cevians is perspective. More precisely, given triangle ABC, let $A_{[P]}B_{[P]}C_{[P]}$ be the pedal triangle of a point P, and consider the intersection points

$$Q_a := BB_{[P]} \cap CC_{[P]}, \qquad Q_b := CC_{[P]} \cap AA_{[P]}, \qquad Q_c := AA_{[P]} \cap BB_{[P]}.$$

We seek the locus of P for which the triangle $Q_aQ_bQ_c$ is perspective with ABC. See Figure 1. This is the union of

(1a) the Darboux cubic consisting of points whose pedal triangles are cevian,[1]
(1b) the circumcircle together with the line at infinity.

The loci of the perspector in these cases are respectively

(2a) the Lucas cubic consisting of points whose cevian triangles are pedal,[2]
(2b) a cubic related to the Simson lines.

We give an illustration of the Darboux and Lucas cubics in the Appendix. Our main interest is in the singular case (2b) related to the Simson lines of points on the circumcircle. The curve in (2b) above is indeed the locus of the tripoles[3] of the Simson lines. Let P be a point on the circumcircle, and $\mathsf{t}(P) = (u : v : w)$ the tripole of its Simson line $\mathsf{s}(P)$. This means that the perpendicular to the sidelines at the points

$$U = (0 : v : -w), \quad V = (-u : 0 : w), \quad W = (u : -v : 0) \tag{1}$$

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[1]This is the isogonal cubic with pivot the de Longchamps point, the reflection of the orthocenter in the circumcenter. A point P lies on this cubic if and only if its the line joining P to its isogonal conjugate contains the de Longchamps point.

[2]This is the isotomic cubic with pivot $\mathsf{i}(H)$, the isotomic conjugate of the orthocenter. A point P lies on this cubic if and only if its the line joining P to its isotomic conjugate contains the point $\mathsf{i}(H)$.

[3]We use the term tripole as a short form of trilinear pole.

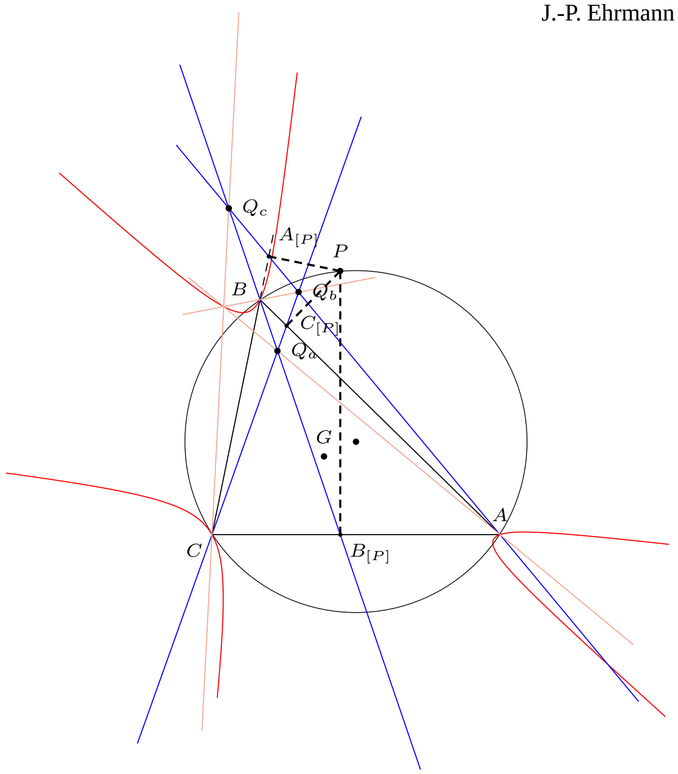

Figure 1

are concurrent (at P on the circumcircle). In the notations of John H. Conway,[4] the equations of these perpendiculars are

$$\begin{array}{ccccccc} (S_Bv+S_Cw)x & + & a^2vy & + & a^2wz & = & 0, \\ b^2ux & + & (S_Cw+S_Au)y & + & b^2wz & = & 0, \\ c^2ux & + & c^2vy & + & (S_Au+S_Bv)z & = & 0. \end{array}$$

Elimination of x, y, z leads to the cubic

$$\mathcal{E}:\ S_Au(v^2+w^2)+S_Bv(w^2+u^2)+S_Cw(u^2+v^2)-(a^2+b^2+c^2)uvw=0.$$

This is clearly a self-isotomic cubic, *i.e.*, a point P lies on the cubic if and only if its isotomic conjugate does. We shall call $\mathcal{E}$ the *Simson cubic* of triangle ABC.

2. A parametrization of the Simson cubic

It is easy to find a rational parametrization of the Simson cubic. Let P be a point on the circumcircle. Regarded as the isogonal conjugate of the infinite point of a

[4]For a triangle ABC with side lengths a, b, c, denote

$$S_A=\frac{1}{2}(b^2+c^2-a^2),\quad S_B=\frac{1}{2}(c^2+a^2-b^2),\quad S_C=\frac{1}{2}(a^2+b^2-c^2).$$

These satisfy a number of basic relations. We shall, however, only need the obvious relations

$$S_B+S_C=a^2,\qquad S_C+S_A=b^2,\quad S_A+S_B=c^2.$$

line $px + qy + rz = 0$, the point P has homogeneous barycentric coordinates

$$\left(\frac{a^2}{q-r} : \frac{b^2}{r-p} : \frac{c^2}{p-q}\right).$$

The pedals of P on the side lines are the points U, V, W in (1) with

$$\begin{aligned} u &= \frac{1}{q-r}(-a^2p + S_Cq + S_Br), \\ v &= \frac{1}{r-p}(S_Cp - b^2q + S_Ar), \\ w &= \frac{1}{p-q}(S_Bp + S_Aq - c^2r). \end{aligned} \qquad (2)$$

This means that the tripole of the Simson line $\mathsf{s}(P)$ of P is the point $\mathsf{t}(P) = (u : v : w)$. The system (2) therefore gives a rational parametrization of the Simson cubic. It also shows that $\mathcal{E}$ has a singularity, which is easily seen to be an isolated singularity at the centroid. [5]

3. Pivotal conic of the Simson cubic

We have already noted that $\mathcal{E}$ is a self-isotomic cubic. In fact, the isotomic conjugate of $\mathsf{t}(P)$ is the point $\mathsf{t}(P')$, where P' is the antipode of P (with respect to the circumcircle). [6]

It is well known that the Simson lines of antipodal points intersect (orthogonally) on the nine-point circle. As this intersection moves on the nine-point circle, the line joining the tripoles $\mathsf{t}(P)$, $\mathsf{t}(P')$ of the orthogonal Simson lines $\mathsf{s}(P)$, $\mathsf{s}(P')$ envelopes the conic $\mathcal{C}$ dual to the nine-point circle. This conic has equation[7]

$$\sum_{\text{cyclic}} (b^2 - c^2)^2x^2 - 2(b^2c^2 + c^2a^2 + a^2b^2 - a^4)yz = 0,$$

and is the inscribed ellipse in the anticomplementary triangle with center the symmedian point of triangle ABC, $K = (a^2 : b^2 : c^2)$. The Simson cubic $\mathcal{E}$ can therefore be regarded as an isotomic cubic with the ellipse $\mathcal{C}$ as pivot. See Figure 2.

Proposition 1. *The pivotal conic $\mathcal{C}$ is tritangent to the Simson cubic $\mathcal{E}$ at the tripoles of the Simson lines of the isogonal conjugates of the infinite points of the Morley sides.*

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[5] If P is an infinite point, its pedals are the infinite points of the side lines. The triangle $Q_aQ_bQ_c$ in question is the anticomplementary triangle, and has perspector at the centroid G.

[6] The antipode of P has coordinates

$$\left(\frac{a^2}{-a^2p + S_Cq + S_Br} : \frac{b^2}{S_Cp - b^2q + S_Ar} : \frac{c^2}{S_Bp + S_Aq - c^2r}\right).$$

[7] The equation of the nine-point circle is $\sum_{\text{cyclic}} S_Ax^2 - a^2yz = 0$. We represent this by a symmetric matrix A. The dual conic is then represented by the adjoint matrix of A.

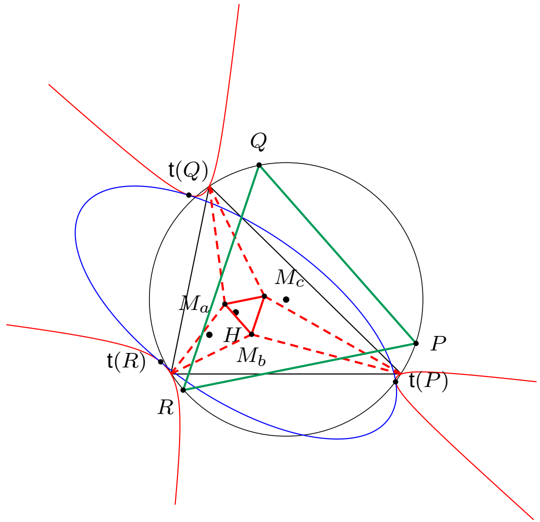

Figure 2

Proof. Since $\mathcal{C}$ is the dual of the nine-point circle, the following statements are equivalent:

(1) $\mathsf{t}(P)$ lies on $\mathcal{C} \cap \mathcal{E}$.

(2) $\mathsf{s}(P')$ is tangent to the nine-point circle.

(3) $\mathsf{s}(P)$ passes through the nine-point center.

Thus, $\mathcal{C}$ and $\mathcal{E}$ are tangent at the three points $\mathsf{t}(P)$ for which the Simson lines $S(P)$ pass through the nine-point center. If P, Q, R are the isogonal conjugates of the infinite points of the side lines of the Morley triangle, then PQR is an equilateral triangle and the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ are perpendicular to QR, RP, PQ respectively. See [1]. Let H be the orthocenter of triangle ABC, and consider the midpoints P_1, Q_1, R_1 of HP, HQ, HR. Since $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through P_1, Q_1, R_1 respectively, these Simson lines are the altitudes of the triangle $P_1Q_1R_1$. As this triangle is equilateral and inscribed in the nine-point circle, the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through the nine-point center. □

Remarks. (1) The triangle PQR is called the circum-tangential triangle of ABC in [3].

(2) The ellipse $\mathcal{C}$ intersects the Steiner circum - ellipse at the four points

$$\left(\tfrac{1}{b-c} : \tfrac{1}{c-a} : \tfrac{1}{a-b}\right), \qquad \left(\tfrac{1}{b-c} : \tfrac{1}{c+a} : \tfrac{-1}{a+b}\right),$$
$$\left(\tfrac{-1}{b+c} : \tfrac{1}{c-a} : \tfrac{1}{a+b}\right), \qquad \left(\tfrac{1}{b+c} : \tfrac{-1}{c+a} : \tfrac{1}{a-b}\right).$$

These points are the perspectors of the four inscribed parabolas tangent respectively to the tripolars of the incenter and of the excenters. In Figure 3, we illustrate the parabolas for the incenter and the B-excenter. The foci are the isogonal conjugates of the infinite points of the lines $\pm ax \pm by \pm cz = 0$, and the directrices are the corresponding lines of reflections of the foci in the side lines of triangle ABC.

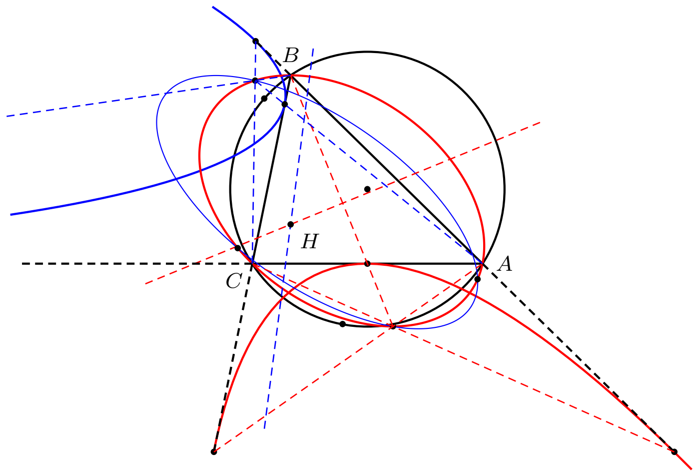

Figure 3

4. Intersection of $\mathcal{E}$ with a line tangent to $\mathcal{C}$

Consider again the Simson line of P and P' intersecting orthogonally at a point N on the nine-point circle. There is a third point Q on the circumcircle whose Simson line $\mathsf{s}(Q)$ passes through N.

- Q is the intersection of the line HN with the circumcircle, H being the orthocenter.
- The line $\mathsf{t}(P)\mathsf{t}(P')$ intersects again the cubic at $\mathsf{t}(Q')$.
- The tangent lines at $\mathsf{t}(P)$ and $\mathsf{t}(P')$ to the cubic intersect at $\mathsf{t}(Q)$ on the cubic.

If the line $\mathsf{t}(P)\mathsf{t}(P')$ touches $\mathcal{C}$ at S, then

(i) S and $\mathsf{t}(Q')$ are harmonic conjugates with respect to $\mathsf{t}(P)$ and $\mathsf{t}(P')$;

(ii) the isotomic conjugate of S is the tripole of the line tangent at N to the nine-point circle.

See Figure 4.

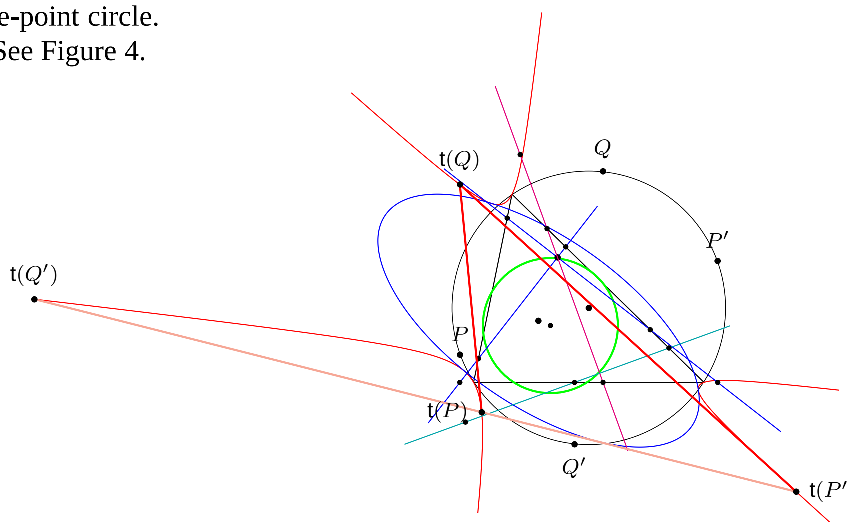

Figure 4

5. Circumcubics invariant under a quadratic transformation

Let $\mathcal{E}$ be a circumcubic invariant under a quadratic transformation τ defined by

$$\tau(x:y:z) = \left(\frac{f^2}{x} : \frac{g^2}{y} : \frac{h^2}{z}\right).$$

The fixed points of τ are the points $(\pm f : \pm g : \pm h)$, which form a harmonic quadruple.

Consider a circumcubic $\mathcal{E}$ invariant under τ. Denote by U, V, W the "third" intersections of $\mathcal{E}$ with the side lines. Then, either U, V, W lie on same line or UVW is perspective with ABC.

The latter case is easier to describe. If UVW is perspective with ABC at P, then $\mathcal{E}$ is the $\tau-$cubic with pivot P, *i.e.*, a point Q lies on $\mathcal{E}$ if and only if the line joining Q and $\tau(Q)$ passes through P.

On the other hand, if U, V, W are collinear, their coordinates can be written as in (1) for appropriate choice of u, v, w, so that the line containing them is the tripolar of the point $(u : v : w)$. In this case, then the equation of $\mathcal{E}$ is

$$\sum_{\text{cyclic}} f^2 yz(wy + vz) + txyz = 0$$

for some t.

(a) If $\mathcal{E}$ contains exactly one of the fixed points $F = (f : g : h)$, then

$$t = -2(ghu + hfv + fgw).$$

In this case, $\mathcal{E}$ has a singularity at F. If $M = (x : y : z)$ in barycentric coordinates with respect to ABC, then with respect to the precevian triangle of F (the three other invariant points), the tangential coordinates of the line joining M to $\tau(M)$ are

$$(p:q:r) = \left(\frac{gz - hy}{(g+h-f)(gz+hy)} : \frac{hx - fz}{(h+f-g)(hx+fz)} : \frac{fy - gx}{(f+g-h)(fy+gx)}\right).$$

As the equation of $\mathcal{E}$ can be rewritten as

$$\frac{p_0}{p} + \frac{q_0}{q} + \frac{r_0}{r} = 0,$$

where

$$p_0 = \frac{f(hv + gw)}{g+h-f}, \quad q_0 = \frac{g(fw + hu)}{h+f-g}, \quad r_0 = \frac{h(gu + fv)}{f+g-h},$$

it follows that the line $M\tau(M)$ envelopes a conic inscribed in the precevian triangle of F.

Conversely, if $\mathcal{C}$ is a conic inscribed in the precevian triangle $A^F B^F C^F$, the locus of M such as the line $M\tau(M)$ touches $\mathcal{C}$ is a $\tau-$cubic with a singularity at F. The tangent lines to $\mathcal{E}$ at F are the tangent lines to $\mathcal{C}$ passing through P.

Note that if F lies on $\mathcal{C}$, and T the tangent to $\mathcal{C}$ at P, then $\mathcal{E}$ degenerates into the union of T and T^*.

(b) If $\mathcal{E}$ passes through two fixed points F and, A^F, then it degenerates into the union of FA^F and a conic.

(c) If the cubic $\mathcal{E}$ contains none of the fixed points, each of the six lines joining two of these fixed points contains, apart from a vertex of triangle ABC, a pair of points of $\mathcal{E}$ conjugate under τ. In this case, the lines $M\tau(M)$ cannot envelope a conic, because this conic should be tangent to the six lines, which is clearly impossible.

We close with a summary of the results above.

Proposition 2. *Let $\mathcal{E}$ be a circumcubic and τ a quadratic transformation of the form*

$$\tau(x : y : z) = (f^2yz : g^2zx : h^2xy).$$

The following statements are equivalent.

(1) $\mathcal{E}$ is $\tau-$invariant with pivot a conic.

(2) $\mathcal{E}$ passes through one and only one fixed points of τ, has a singularity at this point, and the third intersections of $\mathcal{E}$ with the side lines lie on a line ℓ.

In this case, if $\mathcal{E}$ contains the fixed point $F = (f : g : h)$, and if ℓ is the tripolar of the point $(u : v : w)$, then the equation of $\mathcal{E}$ is

$$-2(ghu + hfv + fgw)xyz + \sum_{\text{cyclic}} ux(h^2y^2 + g^2z^2) = 0. \qquad (3)$$

The pivotal conic is inscribed in the precevian triangle of F and has equation[8]

$$\sum_{\text{cyclic}} (gw - hv)^2x^2 - 2(ghu^2 + 3fu(hv + gw) + f^2vw)yz = 0.$$

Appendix

Proposition 3. *Let ℓ be the tripolar of the point $(u : v : w)$, intersecting the sidelines of triangle ABC at U, V, W with coordinates given by (1), and $F = (f : g : h)$ a point not on ℓ nor the side lines of the reference triangle. The locus of M for which the three intersections $AM \cap FU$, $BM \cap FV$ and $CM \cap FW$ are collinear is the cubic $\mathcal{E}$ defined by (3) above.*

Proof. These intersections are the points

$$\begin{aligned} AM \cap FU &= (f(wy + vz) : (hv + gw)y : (hv + gw)z), \\ BM \cap FV &= ((fw + hu)x : g(uz + wx) : (fw + hu)z), \\ CM \cap FW &= ((gu + fv)x : (gu + fv)y : h(vx + uy)). \end{aligned}$$

The corresponding determinant is $(fvw+gwu+huv)R$ where R is the expression on the left hand side of (3). □

The Simson cubic is the particular case $F = G$, the centroid, and ℓ the line

$$\frac{x}{S_A} + \frac{y}{S_B} + \frac{z}{S_C} = 0,$$

which is the tripolar of the isotomic conjugate of the orthocenter H.

[8] The center of this conic is the point $(f(v+w-u)+u(g+h-f) : g(w+u-v)+v(h+f-g) : h(u+v-w)+w(f+g-h)$.

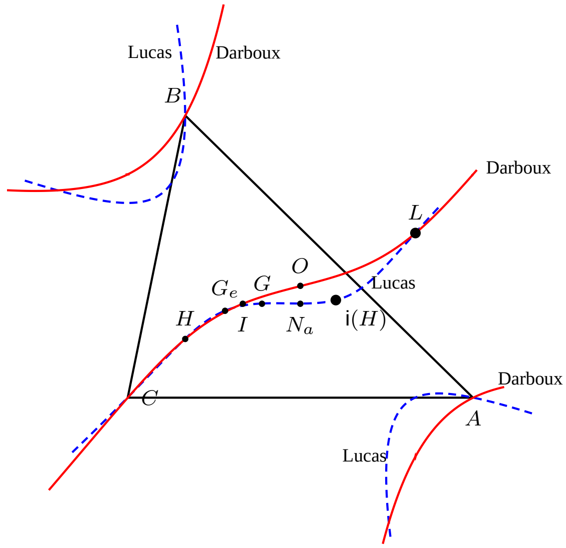

Figure 5. The Darboux and Lucas cubics

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