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The Kiepert Pencil of Kiepert Hyperbolas

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Abstract. We study Kiepert triangles $\mathcal{K}(\phi)$ and their iterations $\mathcal{K}(\phi,\psi)$, the Kiepert triangles $\mathcal{K}(\psi)$ relative to Kiepert triangles $\mathcal{K}(\phi)$. For arbitrary ϕ and ψ, we show that $\mathcal{K}(\phi,\psi)=\mathcal{K}(\psi,\phi)$. This iterated Kiepert triangle is perspective to each of ABC, $\mathcal{K}(\phi)$, and $\mathcal{K}(\psi)$. The Kiepert hyperbolas of $\mathcal{K}(\phi)$ form a pencil of conics (rectangular hyperbolas) through the centroid, and the two infinite points of the Kiepert hyperbola of the reference triangle. The centers of the hyperbolas in this Kiepert pencils are on the line joining the Fermat points of the medial triangle of ABC. Finally we give a construction of the degenerate Kiepert triangles. The vertices of these triangles fall on the parallels through the centroid to the asymptotes of the Kiepert hyperbola.

1. Preliminaries

Given triangle ABC with side lengths a, b, c, we adopt the notation of John H. Conway. Let S denote *twice* the area of the triangle, and for every θ, write $S_\theta = S\cdot\cot\theta$. In particular, from the law of cosines,

$$S_A=\frac{b^2+c^2-a^2}{2},\quad S_B=\frac{c^2+a^2-b^2}{2},\quad S_C=\frac{a^2+b^2-c^2}{2}.$$

The sum $S_A+S_B+S_C=\frac{1}{2}(a^2+b^2+c^2)=S_\omega$ for the Brocard angle ω of the triangle. See, for example, [2, p.266] or [3, p.47]. For convenience, a product $S_\phi\cdot S_\psi\cdots$ is simply written as $S_{\phi\psi\cdots}$. We shall make use of the following fundamental formulae.

Lemma 1 (Conway)**.** *The following relations hold:*
(a) $a^2=S_B+S_C$, $b^2=S_C+S_A$, *and* $c^2=S_A+S_B$;
(b) $S_A+S_B+S_C=S_\omega$;
(c) $S_{AB}+S_{BC}+S_{CA}=S^2$;
(d) $S_{ABC}=S^2\cdot S_\omega-a^2b^2c^2$.

Proposition 2 (Distance formula)**.** *The square distance between two points with* absolute *barycentric coordinates $P=x_1A+y_1B+z_1C$ and $Q=x_2A+y_2B+z_2C$ is given by*

$$|PQ|^2=S_A(x_1-x_2)^2+S_B(y_1-y_2)^2+S_C(z_1-z_2)^2. \tag{1}$$

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Proposition 3 (Conway). *Let P be a point such that the directed angles PBC and PCB are respectively ϕ and ψ. The homogeneous barycentric coordinates of P are*

$$(-a^2 \;:\; S_C+S_\psi \;:\; S_B+S_\phi).$$

Since the cotangent function has period π, we may always choose ϕ and ψ in the range $-\frac{\pi}{2} < \phi,\ \psi \leq \frac{\pi}{2}$. See Figure 1.

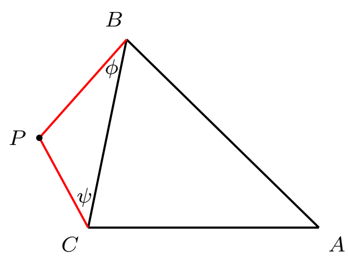

Figure 1

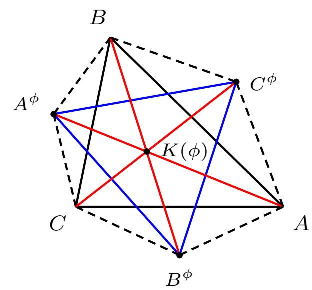

Figure 2

2. The Kiepert triangle $\mathcal{K}(\phi)$

Given an angle ϕ, let A^ϕ, B^ϕ, C^ϕ be the apexes of isosceles triangles on the sides of ABC with base angle ϕ. These are the points

$$\begin{aligned} A^\phi &= (-a^2 : S_C+S_\phi : S_B+S_\phi),\\ B^\phi &= (S_C+S_\phi : -b^2 : S_A+S_\phi),\\ C^\phi &= (S_B+S_\phi : S_A+S_\phi : -c^2). \end{aligned} \tag{2}$$

They form the *Kiepert triangle* $\mathcal{K}(\phi)$, which is perspective to ABC at the *Kiepert perspector*

$$K(\phi) = \left(\frac{1}{S_A+S_\phi} : \frac{1}{S_B+S_\phi} : \frac{1}{S_C+S_\phi}\right). \tag{3}$$

See Figure 2. If $\phi = \frac{\pi}{2}$, this perspector is the orthocenter H. The Kiepert triangle $\mathcal{K}(\frac{\pi}{2})$ is one of three degenerate Kiepert triangles. Its vertices are the infinite points in the directions of the altitudes. The other two are identified in §2.3 below.

The Kiepert triangle $\mathcal{K}(\phi)$ has the same centroid $G = (1:1:1)$ as the reference triangle ABC. This is clear from the coordinates given in (2) above.

2.1. *Side lengths.* We denote by a_ϕ, b_ϕ, and c_ϕ the lengths of the sides $B^\phi C^\phi$, $C^\phi A^\phi$, and $A^\phi B^\phi$ of the Kiepert triangle $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$, these side lengths are given by

$$\begin{aligned} 4S_\phi^2\cdot a_\phi^2 &= a^2S_\phi^2 + S^2(4S_\phi+S_\omega+3S_A),\\ 4S_\phi^2\cdot b_\phi^2 &= b^2S_\phi^2 + S^2(4S_\phi+S_\omega+3S_B),\\ 4S_\phi^2\cdot c_\phi^2 &= c^2S_\phi^2 + S^2(4S_\phi+S_\omega+3S_C). \end{aligned} \tag{4}$$

Here is a simple relation among these side lengths.

Proposition 4. *If $\phi \neq \frac{\pi}{2}$,*

$$b_\phi^2 - c_\phi^2 = \frac{1-3\tan^2\phi}{4} \cdot (b^2 - c^2);$$

similarly for $c_\phi^2 - a_\phi^2$ and $a_\phi^2 - b_\phi^2$.

If $\phi = \pm\frac{\pi}{6}$, we have $b_\phi^2 = c_\phi^2 = a_\phi^2$, and the triangle is equilateral. This is Napoleon's theorem.

2.2. *Area.* Denote by S' twice the area of $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$,

$$S' = \frac{S}{(2S_\phi)^3}\begin{vmatrix} -a^2 & S_C+S_\phi & S_B+S_\phi \\ S_C+S_\phi & -b^2 & S_A+S_\phi \\ S_B+S_\phi & S_A+S_\phi & -c^2 \end{vmatrix} = \frac{S}{4S_\phi^2}(S_\phi^2 + 2S_\omega S_\phi + 3S^2). \tag{5}$$

2.3. *Degenerate Kiepert triangles.* The Kiepert triangle $\mathcal{K}(\phi)$ degenerates into a line when $\phi = \frac{\pi}{2}$ as we have seen above, or $S' = 0$. From (5), this latter is the case if and only if $\phi = \omega_\pm$ for

$$\cot\omega_\pm = -\cot\omega \pm \sqrt{\cot^2\omega - 3}. \tag{6}$$

See §5.1 and Figures 8A,B for the construction of the two finite degenerate Kiepert triangles.

2.4. *The Kiepert hyperbola.* It is well known that the locus of the Kiepert perspectors is the Kiepert hyperbola

$$\mathcal{K}: \qquad (b^2-c^2)yz + (c^2-a^2)zx + (a^2-b^2)xy = 0.$$

See, for example, [1]. In this paper, we are dealing with the Kiepert hyperbolas of various triangles. This particular one (of the reference triangle) will be referred to as the *standard* Kiepert hyperbola. It is the rectangular hyperbola with asymptotes the Simson lines of the intersections of the circumcircle with the Brocard axis OK (joining the circumcenter and the symmedian point). Its center is the point $((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$ on the nine-point circle. The asymptotes, regarded as infinite points, are the points $K(\phi)$ for which

$$\frac{1}{S_A+S_\phi} + \frac{1}{S_B+S_\phi} + \frac{1}{S_C+S_\phi} = 0.$$

These are $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ for $\omega_\pm$ given by (6) above.

3. Iterated Kiepert triangles

Denote by A', B', C' the magnitudes of the angles A^ϕ, B^ϕ, C^ϕ of the Kiepert triangle $\mathcal{K}(\phi)$. From the expressions of the side lengths in (4), we have

$$S'_{A'} = \frac{1}{4S_\phi^2}(S_A S_\phi^2 + 2S^2 S_\phi + S^2(2S_\omega - 3S_A)) \tag{7}$$

together with two analogous expressions for $S'_{B'}$ and $S'_{C'}$.

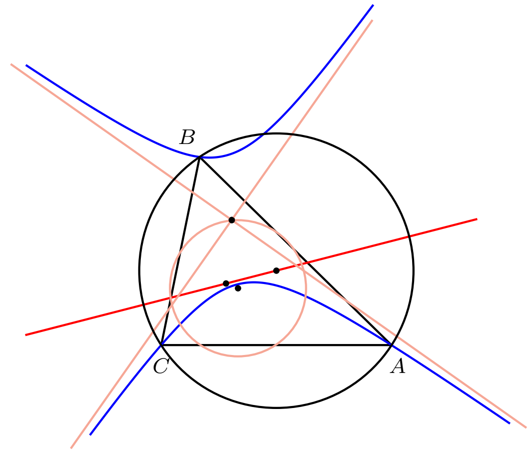

Figure 3

Consider the Kiepert triangle $\mathcal{K}(\psi)$ of $\mathcal{K}(\phi)$. The coordinates of the apex $A^{\phi,\psi}$ with respect to $\mathcal{K}(\phi)$ are $(-a_\phi^2 : S'_{C'} + S'_\psi : S'_{B'} + S'_\psi)$. Making use of (5) and (7), we find the coordinates of the vertices of $\mathcal{K}(\phi,\psi)$ with reference to ABC, as follows.

$$A^{\phi,\psi} = (-(2S^2 + a^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi)),$$
$$B^{\phi,\psi} = (S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : -(2S^2 + b^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi)),$$
$$C^{\phi,\psi} = (S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi) : -(2S^2 + c^2(S_\phi + S_\psi) + 2S_{\phi\psi})).$$

From these expressions we deduce a number of interesting properties of the iterated Kiepert triangles.

1. The symmetry in ϕ and ψ of these coordinates shows that the triangles $\mathcal{K}(\phi,\psi)$ and $\mathcal{K}(\psi,\phi)$ coincide.

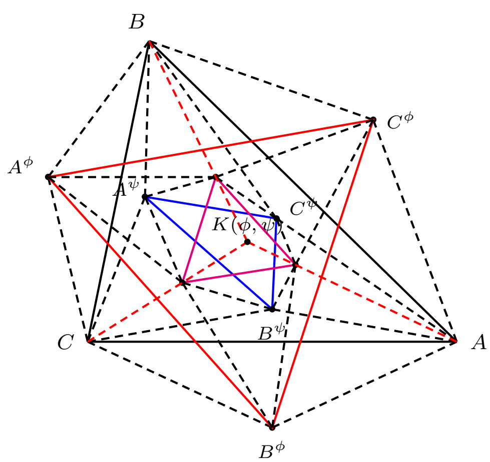

Figure 4

2. It is clear that the iterated Kiepert triangle $\mathcal{K}(\phi,\psi)$ is perspective with each of $\mathcal{K}(\phi)$ and $\mathcal{K}(\psi)$, though the coordinates of the perspectors $K_\phi(\psi)$ and $K_\psi(\phi)$ are very tedious. It is interesting, however, to note that $\mathcal{K}(\phi,\psi)$ is also perspective with ABC. See Figure 4. The perspector has relatively simple coordinates:

$$K(\phi,\psi)=\left(\frac{1}{S^2+S_A(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_B(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_C(S_\phi+S_\psi)-S_{\phi\psi}}\right).$$

3. This perspector indeed lies on the Kiepert hyperbola of ABC; it is the Kiepert perspector $K(\theta)$, where

$$\cot\theta=\frac{1-\cot\phi\cot\psi}{\cot\phi+\cot\psi}=-\cot(\phi+\psi).$$

In other words,

$$K(\phi,\psi)=K(-(\phi+\psi)). \tag{8}$$

From this we conclude that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has the same infinite points of the standard Kiepert hyperbola, *i.e.*, their asymptotes are parallel.

4. The triangle $\mathcal{K}(\phi,-\phi)$ is homothetic to ABC at G, with ratio of homothety $\frac{1}{4}(1-3\tan^2\phi)$. Its vertices are

$$\begin{aligned}
A^{\phi,-\phi} &= (-2(S^2-S_\phi^2):S^2+S_\phi^2:S^2+S_\phi^2),\\
B^{\phi,-\phi} &= (S^2+S_\phi^2:-2(S^2-S_\phi^2):S^2+S_\phi^2),\\
C^{\phi,-\phi} &= (S^2+S_\phi^2:S^2+S_\phi^2:-2(S^2-S_\phi^2)).
\end{aligned}$$

See also [4].

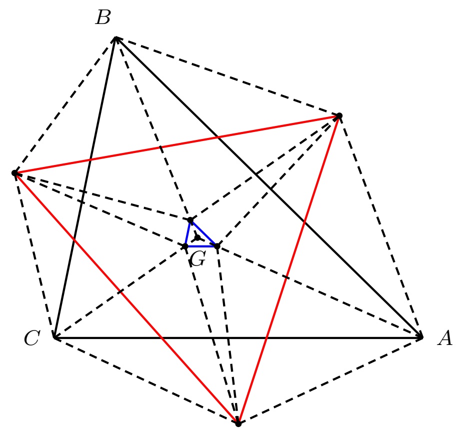

Figure 5

4. The Kiepert hyperbola of $\mathcal{K}(\phi)$

Since the Kiepert triangle $\mathcal{K}(\phi)$ has centroid G, its Kiepert hyperbola $\mathcal{K}_\phi$ contains G. We show that it also contains the circumcenter O.

Proposition 5. *If* $\phi\neq\frac{\pi}{2},\ \pm\frac{\pi}{6}$, $O=K_\phi(-(\frac{\pi}{2}-\phi))$.

Proof. Let $\psi = -(\frac{\pi}{2} - \phi)$, so that $S_\psi = -\frac{S^2}{S_\phi}$. Note that

$$A^{\phi,\psi} = \left(-a^2 : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_C : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_B\right),$$

while

$$A^\phi = (-a^2 : S_C + S_\phi : S_B + S_\phi).$$

These two points are distinct unless $\phi = \frac{\pi}{2}, \pm\frac{\pi}{6}$. Subtracting these two coordinates we see that the line $\ell_a := A^\phi A^{\phi,\psi}$ passes through $(0 : 1 : 1)$, the midpoint of BC. This means, by the construction of A^ϕ, that ℓ_a is indeed the perpendicular bisector of BC, and thus passes through O. By symmetry this proves the proposition. □

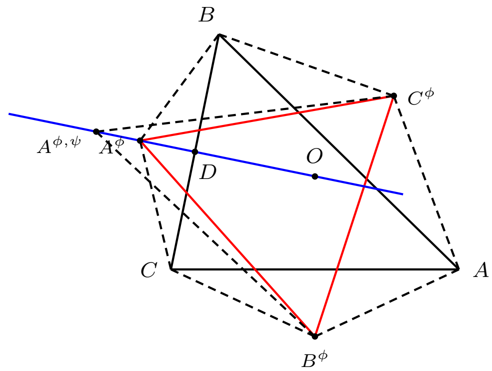

Figure 6

The Kiepert hyperbolas of the Kiepert triangles therefore form the pencil of conics through the centroid G, the circumcenter O, and the two infinite points of the standard Kiepert hyperbola. The Kiepert hyperbola $\mathcal{K}_\phi$ is the one in the pencil that contains the Kiepert perspector $K(\phi)$, since $K(\phi) = K_\phi(-2\phi)$ according to (8). Now, the line containing $K(\phi)$ and the centroid has equation

$$(b^2 - c^2)(S_A + S_\phi)x + (c^2 - a^2)(S_B + S_\phi)y + (a^2 - b^2)(S_C + S_\phi)z = 0.$$

It follows that the equation of $\mathcal{K}_\phi$ is of the form

$$\sum_{\text{cyclic}} (b^2 - c^2)yz + \lambda(x + y + z)(\sum_{\text{cyclic}} (b^2 - c^2)(S_A + S_\phi)x) = 0,$$

where λ is determined by requiring that the conic passes through the circumcenter $O = (a^2 S_A : b^2 S_B : c^2 S_C)$. This gives $\lambda = \frac{1}{2S_\phi}$, and the equation of the conic can be rewritten as

$$2S_\phi(\sum_{\text{cyclic}} (b^2 - c^2)yz) + (x + y + z)(\sum_{\text{cyclic}} (b^2 - c^2)(S_A + S_\phi)x) = 0.$$

Several of the hyperbolas in the pencil are illustrated in Figure 7.

The locus of the centers of the conics in a pencil is in general a conic. In the case of the Kiepert pencil, however, this locus is a line. This is clear from Proposition 4 that the center of $\mathcal{K}_\phi$ has coordinates

$$((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2)$$

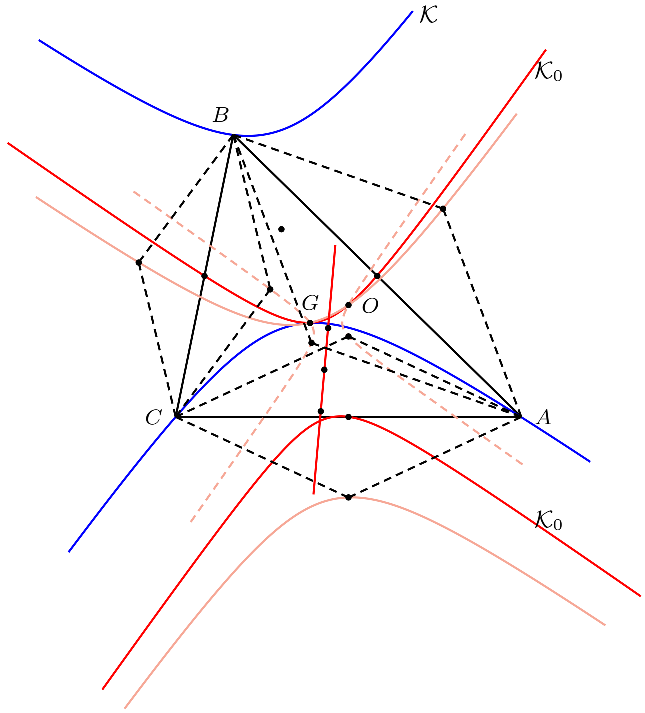

Figure 7

relative to $A^\phi B^\phi C^\phi$, and from (2) that the coordinates of A^ϕ, B^ϕ, C^ϕ are linear functions of S_ϕ. This is the line joining the Fermat points of the medial triangle.

5. Concluding remarks

5.1. *Degenerate Kiepert conics.* There are three degenerate Kiepert triangles corresponding to the three degenerate members of the Kiepert pencil, which are the three pairs of lines connecting the four points G, O, $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ defining the pencil. The Kiepert triangles $\mathcal{K}(\omega_\pm)$ degenerate into the straight lines $GI_\mp$. The vertices are found by intersecting the line with the perpendicular bisectors of the sides of ABC. The centers of these degenerate Kiepert conics are also on the circle with OG as diameter.

5.2. *The Kiepert hyperbolas of the Napoleon triangles.* The Napoleon triangles $\mathcal{K}(\pm\frac{\pi}{6})$ being equilateral do not posses Kiepert hyperbolas, the centroid being the only finite Kiepert perspector. The rectangular hyperbolas $\mathcal{K}_{\pm\pi/6}$ in the pencil are the circumconics through this common perspector G and O. The centers of these rectangular hyperbolas are the Fermat points of the medial triangle.

5.3. *Kiepert coordinates.* Every point outside the standard Kiepert hyperbola $\mathcal{C}$, and other than the circumcenter O, lies on a unique member of the Kiepert pencil, *i.e.*, it can be *uniquely* written as $K_\phi(\psi)$. As an example, the symmedian point

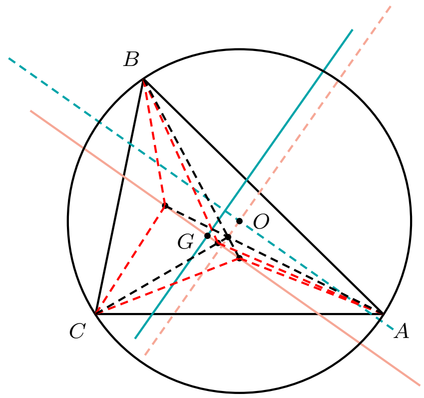

Figure 8A

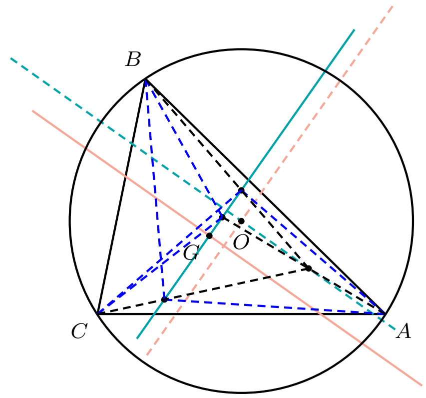

Figure 8B

$K = K_\phi(\psi)$ for $\phi = \omega$ (the Brocard angle) and $\psi = \text{arccot}(\frac{1}{3}\cot\omega)$. We leave the details to the readers.

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