Simple Constructions of the Incircle of an Arbelos

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Abstract. We give several simple constructions of the incircle of an arbelos, also known as a shoemaker’s knife.

Archimedes, in his Book of Lemmas, studied the arbelos bounded by three semicircles with diameters $AB$, $AC$, and $CB$, all on the same side of the diameters. ¹ See Figure 1. Among other things, he determined the radius of the incircle of the arbelos. In Figure 2, $GH$ is the diameter of the incircle parallel to the base $AB$, and $G'$, $H'$ are the (orthogonal) projections of $G$, $H$ on $AB$. Archimedes showed that $GHH'G'$ is a square, and that $AG'$, $G'H'$, $H'B$ are in geometric progression. See [1, pp. 307–308].

In this note we give several simple constructions of the incircle of the arbelos. The elegant Construction 1 below was given by Leon Bankoff [2]. The points of tangency are constructed by drawing circles with centers at the midpoints of two of the semicircles of the arbelos. In validating Bankoff’s construction, we obtain Constructions 2 and 3, which are easier in the sense that one is a ruler-only construction, and the other makes use only of the midpoint of one semicircle.

¹The arbelos is also known as the shoemaker’s knife. See [3].
Theorem 1 (Bankoff [2]). Let $P$ and $Q$ be the midpoints of the semicircles $(BC)$ and $(AC)$ respectively. If the incircle of the arbelos is tangent to the semicircles $(BC)$, $(AC)$, and $(AB)$ at $X$, $Y$, $Z$ respectively, then

(i) $A$, $C$, $X$, $Z$ lie on a circle, center $Q$;
(ii) $B$, $C$, $Y$, $Z$ lie on a circle, center $P$.

**Proof.** Let $D$ be the intersection of the semicircle $(AB)$ with the line perpendicular to $AB$ at $C$. See Figure 3. Note that $AB \cdot AC = AD^2$ by Euclid’s proof of the Pythagorean theorem. Consider the inversion with respect to the circle $A(D)$. This interchanges the points $B$ and $C$, and leaves the line $AB$ invariant. The inver- sive images of the semicircles $(AB)$ and $(AC)$ are the lines $\ell$ and $\ell'$ perpendicular to $AB$ at $C$ and $B$ respectively. The semicircle $(BC)$, being orthogonal to the invariant line $AB$, is also invariant under the inversion. The incircle $XYZ$ of the arbelos is inverted into a circle tangent to the semicircle $(BC)$, and the lines $\ell$, $\ell'$, at $P$, $Y'$, $Z'$ respectively. Since the semicircle $(BC)$ is invariant, the points $A$, $X$, and $P$ are collinear. The points $Y'$ and $Z'$ are such that $BPZ'$ and $CPY'$ are lines making $45^\circ$ angles with the line $AB$. Now, the line $BPZ'$ also passes through the midpoint $L$ of the semicircle $(AB)$. The inver- sive image of this line is a circle passing through $A$, $C$, $X$, $Z$. Since inversion is conformal, this circle also makes a $45^\circ$ angle with the line $AB$. Its center is therefore the midpoint $Q$ of the semicircle $(AC)$. This proves that the points $X$ and $Z$ lies on the circle $P(A)$.

The same reasoning applied to the inversion in the circle $B(D)$ shows that $Y$ and $Z$ lie on the circle $P(B)$. □

Theorem 1 justifies Construction 1. The above proof actually gives another construction of the incircle $XYZ$ of the arbelos. It is, first of all, easy to construct the circle $PY'Z'$. The points $X$, $Y$, $Z$ are then the intersections of the lines $AP$, $AY'$, and $AZ'$ with the semicircles $(BC)$, $(CA)$, and $(AB)$ respectively. The following two interesting corollaries justify Constructions 2 and 3.

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2Euclid’s Elements, Book I, Proposition 47.
**Corollary 2.** The lines $AX$, $BY$, and $CZ$ intersect at a point $S$ on the incircle $XYZ$ of the arbelos.

**Proof.** We have already proved that $A$, $X$, $P$ are collinear, as are $B$, $Y$, $Q$. In Figure 4, let $S$ be the intersection of the line $AP$ with the circle $XYZ$. The inversive image $S'$ (in the circle $A(D)$) is the intersection of the same line with the circle $PY'Z'$. Note that

$$\angle AS'Z' = \angle PS'Z' = \angle PY'Z' = 45^\circ = \angle ABZ'$$

so that $A$, $B$, $S'$, $Z'$ are concyclic. Considering the inversive image of this circle, we conclude that the line $CZ$ contains $S$. In other words, the lines $AP$ and $CZ$ intersect at the point $S$ on the circle $XYZ$. Likewise, $BQ$ and $CZ$ intersect at the same point.

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[Figure 4]

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**Corollary 3.** Let $M$ be the midpoint of the semicircle $(AB)$ on the opposite side of the arbelos.

(i) The points $A$, $B$, $X$, $Y$ lie on a circle, center $M$.

(ii) The line $CZ$ passes through $M$.

**Proof.** Consider Figure 5 which is a modification of Figure 3. Since $C$, $P$, $Y'$ are on a line making a $45^\circ$ angle with $AB$, its inversive image (in the circle $A(D)$) is a circle through $A$, $B$, $X$, $Y$, also making a $45^\circ$ angle with $AB$. The center of this circle is necessarily the midpoint $M$ of the semicircle $AB$ on the opposite side of the arbelos.

Join $A$, $M$ to intersect the line $\ell$ at $M'$. Since $\angle BAM' = 45^\circ = \angle BZ'M'$, the four points $A$, $Z'$, $B$, $M'$ are concyclic. Considering the inversive image of the circle, we conclude that the line $CZ$ passes through $M$.

\[\square\]

The center of the incircle can now be constructed as the intersection of the lines joining $X$, $Y$, $Z$ to the centers of the corresponding semicircles of the arbelos.
However, a closer look into Figure 4 reveals a simpler way of locating the center of the incircle $XYZ$. The circles $XYZ$ and $PY'Z'$, being inversive images, have the center of inversion $A$ as a center of similitude. This means that the center of the incircle $XYZ$ lies on the line joining $A$ to the midpoint of $Y'Z'$, which is the opposite side of the square erected on $BC$, on the same side of the arbelos. The same is true for the square erected on $AC$. This leads to the following Construction 4 of the incircle of the arbelos:

![Construction 4](image)

References


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