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$P\ell$-Perpendicularity

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Abstract. It is well known that perpendicularity yields an involution on the line at infinity $\mathcal{L}^\infty$ mapping perpendicular directions to each other. Many notions of triangle geometry depend on this involution. Since in projective geometry the perpendicular involution is not different from other involutions, theorems using standard perpendicularity in fact are valid more generally.

In this paper we will classify alternative perpendicularities by replacing the orthocenter H by a point P and $\mathcal{L}^\infty$ by a line ℓ. We show what coordinates undergo with these changes and give some applications.

1. Introduction

In the Euclidean plane we consider a reference triangle ABC. We shall perform calculations using homogeneous barycentric coordinates. In these calculations $(f : g : h)$ denotes the barycentrics of a point, while $[l : m : n]$ denotes the line with equation $lx + my + nz = 0$. The line at infinity $\mathcal{L}^\infty$, for example, has coordinates $[1 : 1 : 1]$.

Perpendicularity yields an involution on the line at infinity, mapping perpendicular directions to each other. We call this involution *the standard perpendicularity*, and generalize it by replacing the orthocenter H by another point P with coordinates $(f : g : h)$, stipulating that the cevians of P be "perpendicular" to the corresponding sidelines of ABC. To ensure that P is outside the sidelines of ABC, we assume $fgh \neq 0$.

Further we let the role of $\mathcal{L}^\infty$ be taken over by another line $\ell = [l : m : n]$ not containing P. To ensure that ℓ does not pass through any of the vertices of ABC, we assume $lmn \neq 0$ as well. We denote by $[L]^\ell$ the intersection of a line L with ℓ. When we replace H by P and $\mathcal{L}^\infty$ by ℓ, we speak of *$P\ell$-perpendicularity*.

Many notions of triangle geometry, like rectangular hyperbolas, circles, and isogonal conjugacy, depend on the standard perpendicularity. Replacing the standard perpendicularity by $P\ell$-perpendicularity has its effects on these notions. Also, with the replacement of the line of infinity $\mathcal{L}^\infty$, we have to replace affine notions like midpoint and the center of a conic by their projective generalizations. So it may seem that there is a lot of triangle geometry to be redone, having to prove many generalizations. Nevertheless, there are at least two advantages in making

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calculations in generalized perpendicularities. (1) Calculations using coordinates in P-perpendicularity are in general easier and more transparent than when we use specific expressions for the orthocenter H. (2) We give ourselves the opportunity to work with different perpendicularities simultaneously. Here, we may find new interesting views to the triangle in the Euclidean context.

2. $P\ell$-Perpendicularity

In the following we assume some basic results on involutions. These can be found in standard textbooks on projective geometry, such as [2, 3, 8].

2.1. *$P\ell$-rectangular conics.* We generalize the fact that all hyperbolas from the pencil through A, B, C, H are rectangular hyperbolas. Let $\mathcal{P}$ be the pencil of circumconics through P. The elements of $\mathcal{P}$ we call $P\ell$-rectangular conics. According to Desargues' extended Involution Theorem (see, for example, [2, §16.5.4], [8, p.153], or [3, §6.72]) each member of $\mathcal{P}$ must intersect a line ℓ in two points, which are images under an involution $\tau_{P\ell}$. This involution we call the $P\ell$-perpendicularity.

Since an involution is determined by two pairs of images, $\tau_{P\ell}$ can be defined by the degenerate members of the pencil, the pairs of lines (BC, PA), (AC, PB), and (AB, PC). Two of these pairs are sufficient.

If two lines L and M intersect ℓ in a pair of images of $\tau_{P\ell}$, then we say that they are $P\ell$-perpendicular, and write $L \perp_{P\ell} M$. Note that for any ℓ, this perpendicularity replaces the altitudes of a triangle by the cevians of P as lines $P\ell$-perpendicular to the corresponding sides.

The involution $\tau_{P\ell}$ has two fixed points J_1 and J_2, real if $\tau_{P\ell}$ is hyperbolic, and complex if $\tau_{P\ell}$ is elliptic.

Again, by Desargues' Involution Theorem, every nondegenerate triangle $P_1P_2P_3$ has the property that the lines through the vertices $P\ell$-perpendicular to the opposite sides are concurrent at a point. We call this point of concurrence the $P\ell$-orthocenter of the triangle.

Remark. In order to be able to make use of the notions of parallelism and midpoints, and to perform calculations with simpler coordinates, it may be convenient to only replace H by P, but not $\mathcal{L}^\infty$ by another line. In this case we speak of P-perpendicularity. Each $P\ell$-perpendicularity corresponds to the Q-perpendicularity for an appropriate Q by the mappings $(x : y : z) \leftrightarrow (lx : my : nz)$.[1]

2.2. *Representation of $\tau_{P\ell}$ in coordinates.*

Theorem 1. *For $P = (f : g : h)$ and $\ell = [l : m : n]$, the $P\ell$-perpendicularity is given by*

$$\tau_{P\ell} : (f_L : g_L : h_L) \mapsto \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n} \right). \tag{1}$$

[1] These mappings can be constructed by the $(l : m : n)$-reciprocal conjugacy followed by isotomic conjugacy and conversely, as explained in [4].

Proof. Let L be a line passing through C with $[L]^\ell = (f_L : g_L : h_L)$, and let $B_L = L \cap AB = (f_L : g_L : 0)$. We will consider triangle AB_LC. We have noted above that the $P\ell$-altitudes of triangle AB_LC are concurrent. Two of them are very easy to identify. The $P\ell$-altitude from C simply is $CP = [-g : f : 0]$. On the other hand, since $[BP]^\ell = (fm : -lf - hn : hm)$, the $P\ell$-altitude from B_L is $[-hmg_L : hmf_L : fmg_L + flf_L + hnf_L]$. These two $P\ell$-altitudes intersect in the point: [2]

$$X = (f(fmg_L + flf_L + hnf_L) : gn(hf_L - fh_L) : hm(fg_L - gf_Lg)).$$

Finally, we find that the third $P\ell$-altitude meets ℓ in

$$[AX]^\ell = \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n}\right),$$

which indeed satisfies (1). □

3. $P\ell$-circles

Generalizing the fact that in the standard perpendicularity, all circles pass through the two circular points at infinity, we define a $P\ell$-circle to be any conic through the fixed points J_1 and J_2 of the involution $\tau_{P\ell}$. This viewpoint leads to another way of determining the involution, based on the following well known fact, which can be found, for example, in [2, §5.3]:

> Let a conic $\mathcal{C}$ intersect a line L in two points I and J. The involution τ on L with fixed points I and J can be found as follows: Let X be a point on L, then $\tau(X)$ is the point of intersection of L and the polar of X with respect to $\mathcal{C}$.

It is clear that applying this to a $P\ell$-circle with line ℓ we get the involution $\tau_{P\ell}$. In particular this shows us that in any $P\ell$-circle $\mathcal{C}$ a radius and the tangent to $\mathcal{C}$ through its endpoint are P-perpendicular. Knowing this, and restricting ourselves to P-circles, i.e. $\ell = \mathcal{L}^\infty$, we can conclude that all P-circles are homothetic in the sense that parallel radii of two P-circles have parallel tangents, or equivalently, that two parallel radii of two P-circles have a ratio that is independent of its direction.[3]

We now identify the most important $P\ell$-circle.

Theorem 2. *The conic $\mathcal{O}_{P\ell}$:*

$$f(gm + hn)yz + g(fl + hn)xz + h(fl + gm)xy = 0 \tag{2}$$

is the $P\ell$-circumcircle.

Proof. Clearly A, B and C are on the conic given by the equation. Let $J = (f_1 : g_1 : h_1)$, then with (1) the condition that J is a fixed point of $\tau_{P\ell}$ gives

$$\left(\frac{fgh_1 - fg_1h}{l} : \frac{f_1gh - fgh_1}{m} : \frac{fg_1h - f_1gh}{n}\right) = (f_1 : g_1 : h_1)$$

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[2]In computing the coordinates of X, we have have used of the fact that $lf_L + mg_L + nh_L = 0$.

[3]Note here that the ratio might involve a real radius and a complex radius. This happens for instance when we have in the real plane two hyperbolas sharing asymptotes, but on alternative sides of these asymptotes.

which, under the condition $f_1 l + g_1 m + h_1 n = 0$, is equivalent to (2). This shows that the fixed points J_1 and J_2 of τ_P lie on $\mathcal{C}_P$ and proves the theorem. □

As the 'center' of $\mathcal{O}_{P\ell}$ we use the pole of ℓ with respect to $\mathcal{O}_{P\ell}$. This is the point

$$O_{P\ell} = \left(\frac{mg+nh}{l} : \frac{lf+nh}{m} : \frac{lf+mg}{n}\right).$$

3.1. *$P\ell$-Nine Point Circle.* The 'centers' of $P\ell$-rectangular conics, ***i.e.***, elements of the pencil $\mathcal{P}$ of conics through A, B, C, P, form a conic through the traces of P, [4] the 'midpoints' [5] of the triangle sides, and also the 'midpoints' of AP, BP and CP. This conic $\mathcal{N}_{P\ell}$ is an analogue of the nine-point conics, its center is the 'midpoint' of P and $O_{P\ell}$.

The conic through A, B, C, P, and J_1 (or J_2) clearly must be tangent to ℓ, so that J_1 (J_2) is the 'center' of this conic. So both J_1 and J_2 lie on $\mathcal{N}_{P\ell}$, which makes it a $P\ell$-circle.

4. $P\ell$-conjugacy

In standard perpendicularity we have the isogonal conjugacy τ_H as the natural (reciprocal) conjugacy. It can be defined by combining involutions on the pencils of lines through the vertices of ABC. The involution that goes with the pencil through A is defined by two pairs of lines. The first pair is AB and AC, the second pair is formed by the lines through A perpendicular to AB and to AC. Of course this involution maps to each other lines through A making opposite angles to AB and AC respectively. Similarly we have involutions on the pencil through B and C. The isogonal conjugacy is found by taking the images of the cevians of a point P under the three involutions. These images concur in the isogonal conjugate of P.

This isogonal conjugacy finds its P-perpendicular cognate in the following reciprocal conjugacy:

$$\tau_{P\ell c} : (x:y:z) \mapsto \left(\frac{f(mg+nh)}{lx} : \frac{g(lf+nh)}{my} : \frac{h(lf+mg)}{nz}\right), \qquad (3)$$

which we will call the $P\ell$-conjugacy. This naming is not unique, since for each line ℓ' there is a point Q so that the $P\ell$- and $Q\ell'$-conjugacies are equal. In particular, if $\ell = \mathcal{L}^\infty$, this reciprocal conjugacy is

$$(x:y:z) \mapsto \left(\frac{f(g+h)}{x} : \frac{g(h+f)}{y} : \frac{h(f+g)}{z}\right).$$

[4] These are the 'centers' of the degenerate elements of $\mathcal{P}$.

[5] The 'midpoint' of XY is the harmonic conjugate of $[XY]^\ell$ with respect to X and Y. The 'midpoints' of the triangle sides are also the traces of the trilinear pole of ℓ.

Clearly the $P\ell$-conjugacy maps P to $O_{P\ell}$. This provides us with a construction of the conjugacy. See [4]. [6] From (2) it is also clear that this conjugacy transforms $\mathcal{C}_{P\ell}$ into ℓ and back.

Now we note that any reciprocal conjugacy maps any line to a circumconic of ABC, and conversely. In particular, any line through $O_{P\ell}$ is mapped to a conic from the pencil $\mathcal{P}$, a $P\ell$-rectangular conic. This shows that $\tau_{P\ell c}$ maps the $P\ell$-perpendicularity to the involution on $\mathcal{O}_{P\ell}$ mapping each point X to the second point of intersection of $O_{P\ell}X$ with $\mathcal{C}_{P\ell}$.

The four points

$$\left(\pm\sqrt{\frac{f(mg+nh)}{l}} : \pm\sqrt{\frac{g(lf+nh)}{m}} : \pm\sqrt{\frac{h(lf+mg)}{n}}\right)$$

are the fixed points of the $P\ell$-conjugacy. They are the centers of the $P\ell$-circles tritangent to the sidelines of ABC.

5. Applications of P-perpendicularity

As mentioned before, it is convenient not to change the line at infinity $\mathcal{L}^\infty$ into ℓ and speak only of P-perpendicularity. This notion is certainly less general. Nevertheless, it works with simpler coordinates and it allows one to make use of parallelism and ratios in the usual way. For instance, the Euler line is generalized quite easily, because the coordinates of O_P are $(g+h : f+h : f+g)$, so that it is easy to see that $PG : GO_P = 2 : 1$.

We give a couple of examples illustrating the convenience of the notion of P-perpendicularity in computations and understanding.

5.1. *Construction of ellipses.* Note that the equation (1) does not change when we exchange $(f : g : h)$ and $(x : y : z)$. So we have:

Proposition 3. *P lies on $\mathcal{O}_{Q\ell}$ if and only if Q lies on $\mathcal{O}_{P\ell}$.*

When we restrict ourselves to P-perpendicularity, Proposition 3 is helpful in finding the axes of a circumellipse of a triangle. Let's say that the ellipse is $\mathcal{O}_P$. [7] If we find the fourth intersection X of a circumellipse and the circumcircle, then the X-circumcircle $\mathcal{O}_X$ passes through H as well as P, and thus it is a rectangular hyperbola as well as a P-rectangular conic. This means that the asymptotes of $\mathcal{O}_X$ must correspond to the directions of the axes of the ellipse. These yield indeed the only diameters of the ellipse to which the tangents at the endpoints are (standard) perpendicular. Note also that this shows that all conics through A, B, C, X, apart from the circumcircle have parallel axes. Figure 1 illustrates the case when $P = G$, the centroid, and X = Steiner point. Here, $\mathcal{O}_X =$ is the Kiepert hyperbola.

The knowledge of P-perpendicularity can be helpful when we try to draw conics in dynamic geometry software. This can be done without using foci.

[6] In [4] we can find more ways to construct the $P\ell$-conjugacy, for instance, by using the degenerate triangle where AP, BP and CP meet ℓ.

[7] When we know the center O_P of $\mathcal{O}_P$, we can find P by the ratio $O_PG : GP = 1 : 2$.

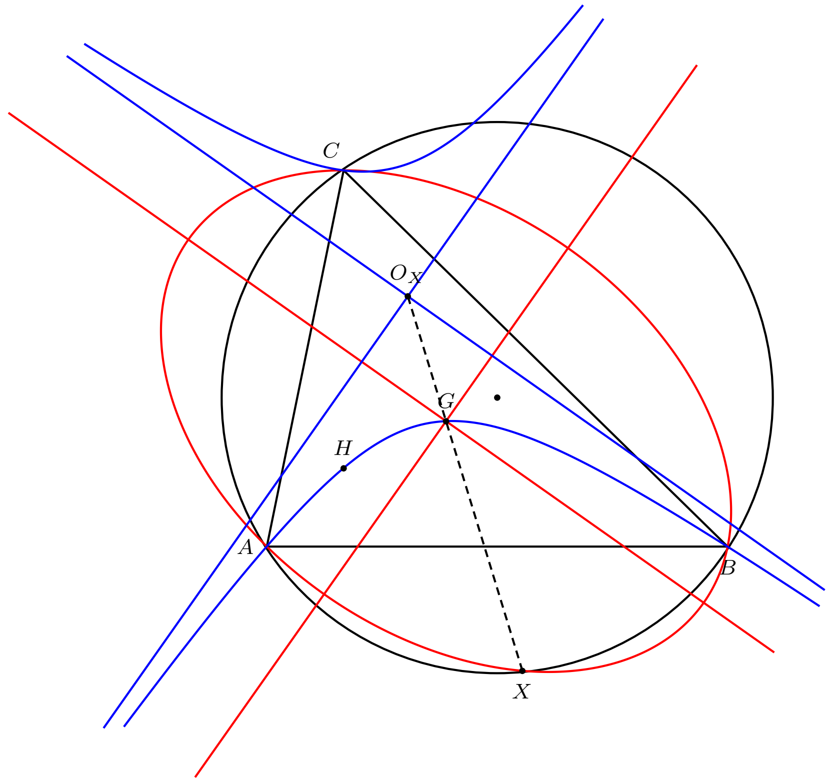

Figure 1

If we have the center O_P of a conic through three given points, say ABC, we easily find P as well. Also by reflecting one of the vertices, say A, through O_P we have the endpoints of a diameter, say AA_r. Then if we let a line m go through A, and a line n which is P-perpendicular to m through A_r. Their point of intersection lies on the P-circle through ABC. See Figure 2.

5.2. *Simson-Wallace lines.* Given a generic finite point $X = (x : y : z)$, let $A' \in BC$ be the point such that $A'X \parallel AP$, and let B' and C' be defined likewise, then we call $A'B'C'$ the *triangle of P-traces* of X. This triangle is represented by the following matrix:

$$M = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & gx + (g+h)y & hx + (g+h)z \\ fy + (f+h)x & 0 & hy + (f+h)z \\ fz + (f+g)x & gz + (f+g)y & 0 \end{pmatrix} \quad (4)$$

We are interested in the conic that plays a role similar to the circumcircle in the occurrence of Simson-Wallace lines.[8] To do so, we find that $A'B'C'$ is degenerate

[8] In [5] Miguel de Guzmán generalizes the Simson-Wallace line more drastically. He allows three arbitrary directions of projection, with the only restriction that these directions are not all equal, each not parallel to the side to which it projects.

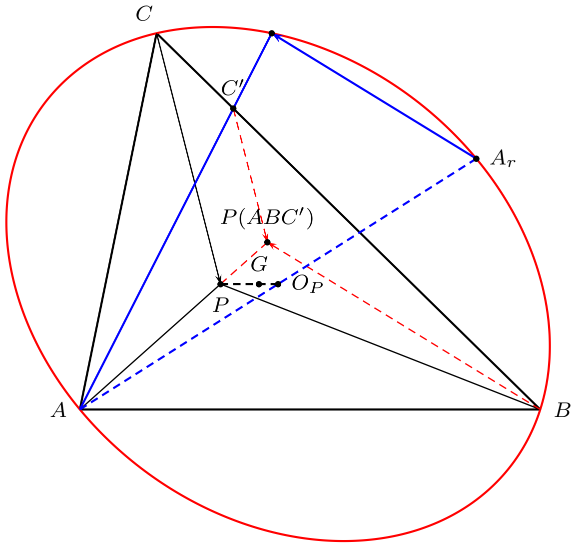

Figure 2

iff determinant $|M| = 0$, which can be rewritten as

$$(f+g+h)(x+y+z)(\tilde{f}yz+\tilde{g}xz+\tilde{h}xy)=0, \tag{5}$$

where

$$\tilde{f}=f(g+h), \qquad \tilde{g}=g(f+h), \qquad \tilde{h}=h(f+g).$$

Using that X and P are finite points, (5) can be rewritten into (2), so that the locus is the P-circle $\mathcal{C}_P$. See Figure 3.

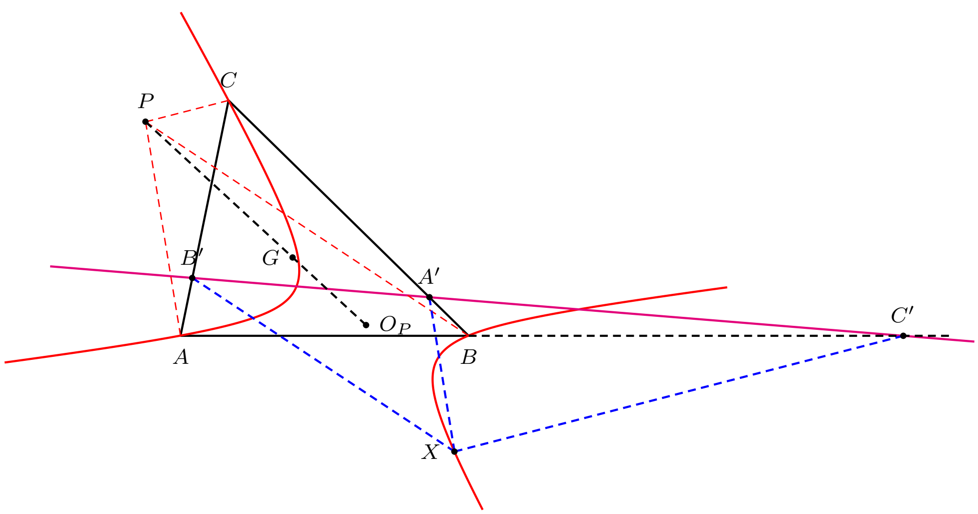

Figure 3

We further remark that since the rows of matrix M in (4) add up to $(f+g+h)X+(x+y+z)P$, the P-Simson-Wallace line $A'B'C'$ bisects the segment XP when $X \in \mathcal{C}_P$. Thus, the point of intersection of $A'B'C'$ and XP lies on $\mathcal{N}_P$.

5.3. *The Isogonal Theorem.* The following theorem generalizes the Isogonal Theorem. [9] We shall make use of the involutions τ_{PA}, τ_{PB} and τ_{PC} that the P-conjugacy causes on the pencil of lines through A, B and C respectively.

Theorem 4. *For $I \in \{A,B,C\}$, consider lines l_I and l'_I unequal to sidelines of ABC that are images under τ_{PI}. Let $A_1 = l_B \cap l'_C$, $B_1 = l_C \cap l'_A$ and $C_1 = l_A \cap l'_B$. We call $A_1B_1C_1$ a P-*conjugate triangle*. Then triangles ABC and $A_1B_1C_1$ are perspective.*

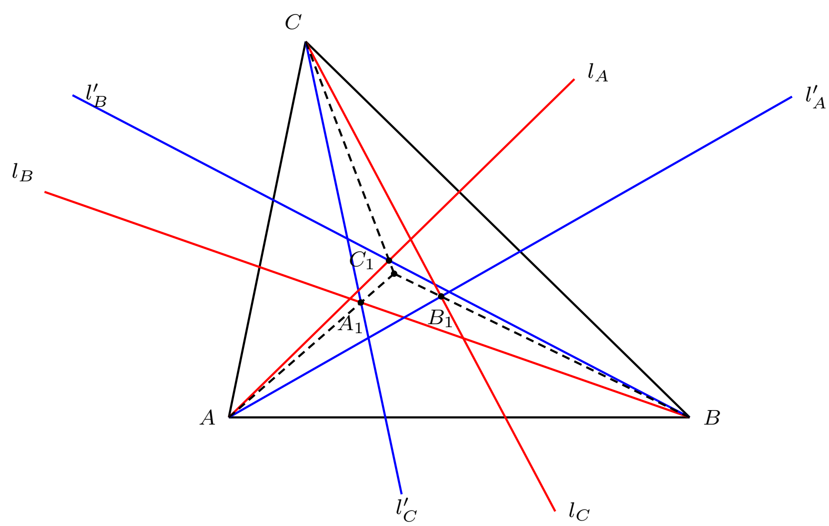

Figure 4

Proof. For $I \in \{A,B,C\}$, let $P_I = (x_I : y_I : z_I) \in l_I$ be a point different from I. We find, for instance, $l_A = [0 : z_A : -y_A]$ and $l'_C = [\tilde{g}/y_C : -\tilde{f}/x_C : 0]$. Consequently $B_1 = (\tilde{f}y_A/x_C : \tilde{g}y_A/y_C : \tilde{g}z_A/y_C)$. In the same way we find coordinates for A_1 and C_1 so that the P-conjugate triangle $A_1B_1C_1$ is given by

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \tilde{f}x_C/x_B & \tilde{f}y_C/x_B & \tilde{h}x_C/z_B \\ \tilde{f}y_A/x_C & \tilde{g}y_A/y_C & \tilde{g}z_A/y_C \\ \tilde{h}x_B/z_A & \tilde{g}z_B/y_A & \tilde{h}z_B/z_A \end{pmatrix}.$$

With these coordinates it is not difficult to verify that $A_1B_1C_1$ is perspective to ABC. This we leave to the reader. □

[9] This theorem states that a triangle $A_1B_1C_1$ with $\angle BAC_1 = \angle CAB_1$, $\angle CBA_1 = \angle ABC_1$ and $\angle ACB_1 = \angle BCA_1$ is perspective to ABC. See [1, p.55], also [6, 9], and [7, Theorem 6D].

Interchanging the lines l_I and l'_I in Theorem 4 above, we see that the P-conjugates of $A_1B_1C_1$ form a triangle $A_2B_2C_2$ perspective to ABC as well. This is its desmic mate. [10] Now, each triangle perspective to ABC is mapped to its desmic mate by a reciprocal conjugacy. From this and Theorem 4 we see that the conditions 'perspective to ABC' and 'desmic mate is also an image under a reciprocal conjugacy' are equivalent.

5.3.1. Each P-conjugate triangle can be written in coordinates as

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = M_1 = \begin{pmatrix} \tilde{f} & w & v \\ w & \tilde{g} & u \\ v & u & \tilde{h} \end{pmatrix}.$$

Let a second P-conjugate triangle be given by

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} = M_2 = \begin{pmatrix} \tilde{f} & W & V \\ W & \tilde{g} & U \\ V & U & \tilde{h} \end{pmatrix}.$$

Considering linear combinations $tM_1 + uM_2$ it is clear that the following proposition holds.

Proposition 5. *Let $A_1B_1C_1$ and $A_2B_2C_2$ be two distinct P-conjugate triangles. Define $A' = A_1A_2 \cap BC$ and B', C' analogously. Then $A'B'C'$ is a cevian triangle. In fact, if $A''B''C''$ is such that the cross ratios $(A_1A_2A'A'')$, $(B_1B_2B'B'')$ and $(C_1C_2C'C'')$ are equal, then $A''B''C''$ is perspective to ABC as well.*

The following corollary uses that the points where the cevians of P meet $\mathcal{L}^\infty$ is a P-conjugate triangle.

Corollary 6. *Let $A_1B_1C_1$ be a P-conjugate triangle. Let A' be the P-perpendicular projections of A_1 on BC, B_1 on CA, and C_1 on AB respectively. Let $A''B''C''$ be such that $A'A_1 : A''A_1 = B'B_1 : B''B_1 = C'C_1 : C''C_1 = t$, then $A''B''C''$ is perspective to ABC. As t varies, the perspector traverses the P-rectangular circumconic through the perspector of $A_1B_1C_1$.*

5.4. *The Darboux cubic.* We conclude with an observation on the analogues of the Darboux cubic. It is well known that the locus of points X whose pedal triangles are perspective to ABC is a cubic curve, the Darboux cubic. We generalize this to triangles of P-traces.

First, let us consider the lines connecting the vertices of ABC and the triangle of P-traces of X given in (4). Let μ_{ij} denote the entry in row i and column j of (4), then we find as matrix of coefficients of these lines

$$N = \begin{pmatrix} 0 & -\mu_{13} & \mu_{12} \\ \mu_{23} & 0 & -\mu_{21} \\ -\mu_{32} & \mu_{31} & 0 \end{pmatrix}. \qquad (6)$$

[10] See for instance [4].

These lines concur iff $\det N = 0$. This leads to the cubic equation

$$(-f+g+h)x(\tilde{h}y^2-\tilde{g}z^2)+(f-g+h)y(\tilde{f}z^2-\tilde{h}x^2)+(f+g-h)z(\tilde{g}x^2-\tilde{f}y^2) = 0. \tag{7}$$

We will refer to this cubic as the *P-Darboux cubic*. The cubic consists of the points Q such that Q and its P-conjugate are collinear with the point $(-f+g+h : f-g+h : f+g-h)$, which is the reflection of P in O_P.

It is seen easily from (4) and (6) that if we interchange $(f : g : h)$ and $(x : y : z)$, then (7) remains unchanged. From this we can conclude:

Proposition 7. *For two points P and Q be two points not on the sidelines of triangle ABC, P lies on the Q-Darboux cubic if and only if Q lies on the P-Darboux cubic.*

This example, and others in §5.1, demonstrate the fruitfulness of considering different perpendicularities simultaneously.

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