Forum Geometricorum
Volume 1 (2001) 173–175.

FORUM GEOM
ISSN 1534-1178

# A Feuerbach Type Theorem on Six Circles

Lev Emelyanov

According to the famous Feuerbach theorem there exists a circle which is tangent internally to the incircle and externally to each of the excircles of a triangle. This is the nine-point circle of the triangle. We obtain a similar result by replacing the excircles with circles each tangent internally to the circumcircle and to the sides at the traces of a point. We make use of Casey's theorem. See, for example, [1, 2].

**Theorem** (Casey)**.** *Given four circles $\mathcal{C}_i$, $i = 1, 2, 3, 4$, let $t_{ij}$ be the length of a common tangent between $\mathcal{C}_i$ and $\mathcal{C}_j$. The four circles are tangent to a fifth circle (or line) if and only if for appropriate choice of signs,*

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

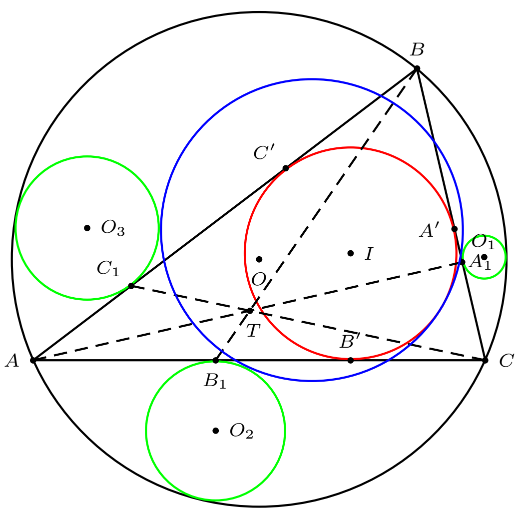


Figure 1

In this note we establish the following theorem. Let $ABC$ be a triangle of side lengths $BC = a$, $CA = b$, and $AB = c$.

**Theorem.** *Let points $A_1$, $B_1$ and $C_1$ be on the sides $BC$, $CA$ and $AB$ respectively of triangle $ABC$. Construct three circles $(O_1)$, $(O_2)$ and $(O_3)$ outside the triangle which is tangent to the sides of $ABC$ at $A_1$, $B_1$ and $C_1$ respectively and also tangent to the circumcircle of $ABC$. The circle tangent externally to these three circles is also tangent to the incircle of triangle $ABC$ if and only if the lines $AA_1$, $BB_1$ and $CC_1$ are concurrent.*

Publication Date: December 13, 2001. Communicating Editor: Paul Yiu.

*Proof.* Let in our case $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ be the circles $(O_1)$, $(O_2)$, $(O_3)$ and the incircle respectively. With reference to Figure 1, we show that

$$t_{12}t_{34} - t_{13}t_{42} - t_{14}t_{23} = 0, \tag{1}$$

where $t_{12}$, $t_{13}$ and $t_{23}$ are the lengths of the common extangents, $t_{34}$, $t_{24}$ and $t_{14}$ are the lengths of the common intangents.

Let $(A)$ be the degenerate circle $A(0)$ (zero radius) and $t_i(A)$ be the length of the tangent from $A$ to $\mathcal{C}_i$. Similar notations apply to vertices $B$ and $C$. Applying Casey's theorem to circles $(A)$, $(B)$, $(O_1)$ and $(C)$, which are all tangent to the circumcircle, we have

$$t_1(A) \cdot a = c \cdot CA_1 + b \cdot A_1B.$$

From this we obtain $t_1(A)$, and similarly $t_2(B)$ and $t_3(C)$:

$$t_1(A) = \frac{c \cdot CA_1 + b \cdot A_1B}{a}, \tag{2}$$

$$t_2(B) = \frac{a \cdot AB_1 + c \cdot B_1C}{b}, \tag{3}$$

$$t_3(C) = \frac{b \cdot BC_1 + a \cdot C_1A}{c}. \tag{4}$$

Applying Casey's theorem to circles $(B)$, $(C)$, $(O_2)$ and $(O_3)$, we have

$$t_2(B)t_3(C) = a \cdot t_{23} + CB_1 \cdot C_1B.$$

Using (3) and (4), we obtain $t_{23}$, and similarly, $t_{13}$ and $t_{12}$:

$$t_{23} = \frac{a \cdot C_1A \cdot AB_1 + b \cdot AB_1 \cdot BC_1 + c \cdot AC_1 \cdot CB_1}{bc}, \tag{5}$$

$$t_{13} = \frac{b \cdot A_1B \cdot BC_1 + c \cdot BC_1 \cdot CA_1 + a \cdot BA_1 \cdot AC_1}{ca}, \tag{6}$$

$$t_{12} = \frac{c \cdot B_1C \cdot CA_1 + a \cdot CA_1 \cdot AB_1 + b \cdot CB_1 \cdot BA_1}{ab}. \tag{7}$$

In the layout of Figure 1, with $A'$, $B'$, $C'$ the touch points of the incircle with the sides, the lengths of the common tangents of the circles $(O_1)$, $(O_2)$, $(O_3)$ with the incircle are

$$t_{14} = A_1A' = -CA_1 + CA' = -CA_1 + \frac{a+b-c}{2}, \tag{8}$$

$$t_{24} = B_1B' = -AB_1 + AB' = -AB_1 + \frac{b+c-a}{2}, \tag{9}$$

$$t_{34} = C_1C' = BC_1 - BC' = BC_1 - \frac{c+a-b}{2}. \tag{10}$$

Substituting (5)-(10) into (1) and simplifying, we obtain

$$t_{12}t_{34} - t_{13}t_{24} - t_{14}t_{23} = \frac{F(a,b,c)}{abc} \cdot (AB_1 \cdot BC_1 \cdot CA_1 - A_1B \cdot B_1C \cdot C_1A),$$

where

$$F(a,b,c) = 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$$

Since $F(a,b,c)$ can be rewritten as

$$(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b),$$

it is clearly nonzero. It follows that $t_{12}t_{34}-t_{13}t_{24}-t_{14}t_{23}=0$ if and only if

$$AB_1\cdot BC_1\cdot CA_1-A_1B\cdot B_1C\cdot C_1A=0. \tag{11}$$

By the Ceva theorem, (11) is the condition for the concurrency of $AA_1$, $BB_1$ and $CC_1$. It is clear that for different positions of the touch points of circles $(O_1)$, $(O_2)$ and $(O_3)$ relative to those of the incircle, the proofs are analogous. □

## References

[1] J. L. Coolidge, *A Treatise on Circles and Spheres*, 1917, Chelsea reprint.

[2] I. M. Yaglom, *Geometric Transformations*, 3 volumes, Mathematical Association of America, 1968.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
*E-mail address*: emelyanov@kaluga.ru