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A Pair of Kiepert Hyperbolas

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Abstract. The solution of a locus problem of Hatzipolakis can be expressed in terms of a simple relationship concerning points on a pair of Kiepert hyperbolas associated with a triangle. We study a generalization.

Let P be a finite point in the plane of triangle ABC. Denote by a, b, c the lengths of the sides BC, CA, AB respectively, and by A_H, B_H, C_H the feet of the altitudes. We consider rays through P in the directions of the altitudes AA_H, BB_H, CC_H, and, for a nonzero constant k, choose points A', B', C' on these rays such that

$$PA' = ka, \qquad PB' = kb, \qquad PC' = kc. \tag{1}$$

Antreas P. Hatzipolakis [1] has asked, for $k = 1$, for the locus of P for which triangle $A'B'C'$ is perspective with ABC.

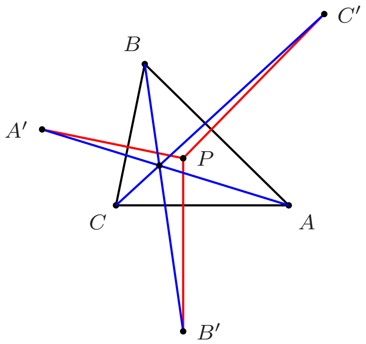

Figure 1

We tackle the general case by making use of homogeneous barycentric coordinates with respect to ABC. Thus, write $P = (u : v : w)$. In the notations introduced by John H. Conway, [1]

$$\begin{aligned}
A' =& (uS - k(u+v+w)a^2 : vS + k(u+v+w)S_C : wS + k(u+v+w)S_B),\\
B' =& (uS + k(u+v+w)S_C : vS - k(u+v+w)b^2 : wS + k(u+v+w)S_A),\\
C' =& (uS + k(u+v+w)S_B : vS + k(u+v+w)S_A : wS - k(u+v+w)c^2).
\end{aligned}$$

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[1] Let ABC be a triangle of side lengths a, b, c, and area $\frac{1}{2}S$. For each ϕ, $S_\phi := S \cdot \cot\phi$. Thus, $S_A = \frac{1}{2}(b^2+c^2-a^2)$, $S_B = \frac{1}{2}(c^2+a^2-b^2)$, and $S_C = \frac{1}{2}(a^2+b^2-c^2)$. These satisfy $S_AS_B + S_BS_C + S_CS_A = S^2$ and other simple relations. For a brief summary, see [3, §1].

The equations of the lines AA', BB', CC' are

$$\begin{aligned}
&& (wS+k(u+v+w)S_B)y - (vS+k(u+v+w)S_C)z &= 0, & (2)\\
-(wS+k(u+v+w)S_A)x && +(uS+k(u+v+w)S_C)z &= 0, & (3)\\
(vS+k(u+v+w)S_A)x - (uS+k(u+v+w)S_B)y && &= 0. & (4)
\end{aligned}$$

These three lines are concurrent if and only if

$$\begin{vmatrix} 0 & wS+k(u+v+w)S_B & -(vS+k(u+v+w)S_C) \\ -(wS+k(u+v+w)S_A) & 0 & uS+k(u+v+w)S_C \\ vS+k(u+v+w)S_A & -(uS+k(u+v+w)S_B) & 0 \end{vmatrix} = 0.$$

This condition can be rewritten as

$$kS(u+v+w)(S\cdot K(u,v,w) - k(u+v+w)L(u,v,w)) = 0,$$

where

$$\begin{aligned}
K(u,v,w) &= (b^2-c^2)vw + (c^2-a^2)wu + (a^2-b^2)uv, & (5)\\
L(u,v,w) &= (b^2-c^2)S_Au + (c^2-a^2)S_Bv + (a^2-b^2)S_Cw. & (6)
\end{aligned}$$

Note that $K(u,v,w)=0$ and $L(u,v,w)=0$ are respectively the equations of the Kiepert hyperbola and the Euler line of triangle ABC. Since P is a finite point and k is nonzero, we conclude, by writing $k=\tan\phi$, that the locus of P for which $A'B'C'$ is perspective with ABC is the rectangular hyperbola

$$S_\phi K(u,v,w) - (u+v+w)L(u,v,w) = 0 \qquad (7)$$

in the pencil generated by the Kiepert hyperbola and the Euler line.

Floor van Lamoen [2] has pointed out that this hyperbola (7) is the Kiepert hyperbola of a Kiepert triangle of the dilated (anticomplementary) triangle of ABC. Specifically, let $\mathcal{K}(\phi)$ be the Kiepert triangle whose vertices are the apexes of similar isosceles triangles of base angles ϕ constructed on the sides of ABC. It is shown in [3] that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has equation

$$2S_\phi(\sum_{\text{cyclic}}(b^2-c^2)yz) + (x+y+z)(\sum_{\text{cyclic}}(b^2-c^2)(S_A+S_\phi)x) = 0.$$

If we replace x, y, z respectively by $v+w$, $w+u$, $u+v$, this equation becomes (7) above. This means that the hyperbola (7) is the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC. [2]

The orthocenter H and the centroid G are always on the locus. Trivially, if $P=H$, the perspector is the same point H. For $P=G$, the perspector is the point [3]

$$\left(\frac{1}{3kS_A+S} : \frac{1}{3kS_B+S} : \frac{1}{3kS_C+S}\right),$$

[2] The Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC is also the dilated triangle of the Kiepert triangle $\mathcal{K}(\phi)$ of triangle ABC.

[3] In the notations of [3], this is the Kiepert perspector $K(\arctan 3k)$.

the second common point of Kiepert hyperbola and the tangent at P to the locus of P, the Kiepert hyperbola of the dilated triangle of $\mathcal{K}(\phi)$.

Now we identify the perspector when P is different from G. Addition of the equations (2,3,4) of the lines AA', BB', CC' gives

$$(v-w)x+(w-u)y+(u-v)z=0.$$

This is the equation of the line joining P to the centroid G, showing that the perspector lies on the line GP.

We can be more precise. Reorganize the equations (2,3,4) as

$$k(S_By-S_Cz)u+(k(S_By-S_Cz)-Sz)v+(k(S_By-S_Cz)+Sy)w=0, \quad (8)$$
$$(k(S_Cz-S_Ax)+Sz)u+\qquad (k(S_Cz-S_Ax)v+(k(S_Cz-S_Ax)-Sx)w=0, \quad (9)$$
$$(k(S_Ax-S_By)-Sy)u+(k(S_Ax-S_By)+Sx)v+\qquad k(S_Ax-S_By)w=0. \quad (10)$$

Note that the combination x·(8) + y·(9) + z·(10) gives

$$k(u+v+w)(x(S_By-S_Cz)+y(S_Cz-S_Ax)+z(S_Ax-S_By)=0.$$

Since k and $u+v+w$ are nonzero, we have

$$(S_C-S_B)yz+(S_A-S_C)zx+(S_B-S_A)xy=0,$$

or equivalently, $(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0$. It follows that the perspector is also on the Kiepert hyperbola.

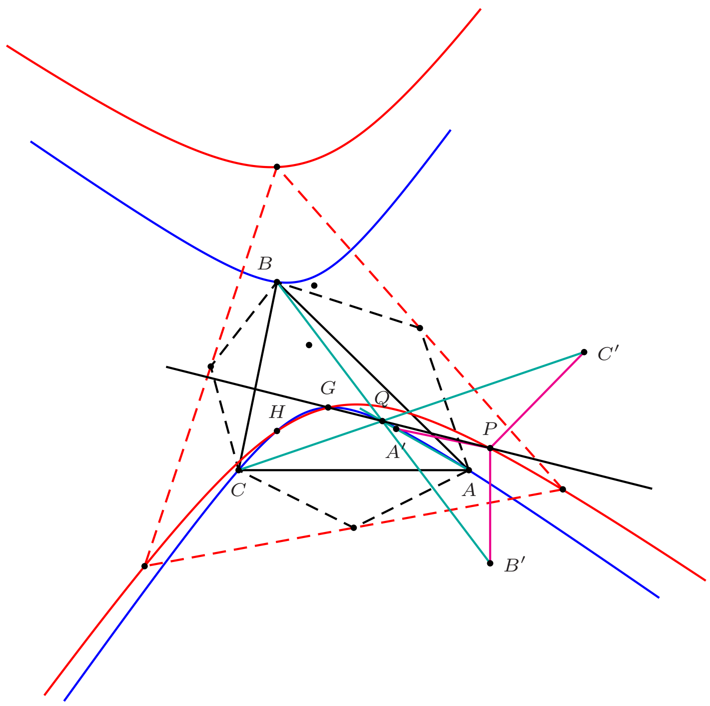

Figure 2

We summarize these results in the following theorem.

Theorem. *Let $k = \tan\phi$ be nonzero, and points A', B', C' be given by (1) along the rays through P parallel to the altitudes. The lines AA', BB', CC' are concurrent if and only if P lies on the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle. The intersection of these lines is the second intersection of the line GP and the Kiepert hyperbola of triangle ABC.*

If we change, for example, the orientation of PA', the locus of P is the rectangular hyperbola with center at the apex of the isosceles triangle on BC of base angle ϕ,[4] asymptotes parallel to the A-bisectors, and passing through the orthocenter H (and also the A-vertex $A^G = (-1 : 1 : 1)$ of the dilated triangle). For $P = A^G$, the perspector is the point $\left(\dfrac{1}{kS_A + S} : \dfrac{1}{kS_B - S} : \dfrac{1}{kS_C - S}\right)$, and for $P \neq A^G$, the second common point of the line PA^G and the rectangular circum-hyperbola with center the midpoint of BC.

We conclude by noting that for a positive k, the locus of P for which we can choose points A', B', C' on the perpendiculars through P to BC, CA, AB such that the lines AA', BB', CC' concur and the distances from P to A', B', C' are respectively k times the lengths of the corresponding side is the union of 8 rectangular hyperbolas.

References

[1] A. P. Hatzipolakis, Hyacinthos message 2510, March 1, 2001.
[2] F. M. van Lamoen, Hyacinthos message 2541, March 6, 2001.
[3] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

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[4] This point has coordinates $(-a^2 : S_C + S_\phi : S_B + S_\phi)$.