Forum Geometricorum
Volume 2 (2002) 5–13.

FORUM GEOM
ISSN 1534-1178

Some Concurrencies from Tucker Hexagons

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Abstract. We present some concurrencies in the figure of Tucker hexagons together with the centers of their Tucker circles. To find the concurrencies we make use of extensions of the sides of the Tucker hexagons, isosceles triangles erected on segments, and special points defined in some triangles.

1. The Tucker hexagon $\mathcal{T}_\phi$ and the Tucker circle $\mathcal{C}_\phi$

Consider a scalene (nondegenerate) reference triangle ABC in the Euclidean plane, with sides $a = BC$, $b = CA$ and $c = AB$. Let B_a be a point on the sideline CA. Let C_a be the point where the line through B_a antiparallel to BC meets AB. Then let A_c be the point where the line through C_a parallel to CA meets BC. Continue successively the construction of parallels and antiparallels to complete a hexagon $B_aC_aA_cB_cC_bA_b$ of which B_aC_a, A_cB_c and C_bA_b are antiparallel to sides BC, CA and AB respectively, while B_cC_b, A_cC_a and A_bB_a are parallel to these respective sides.

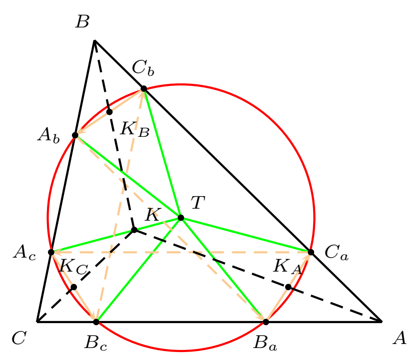

Figure 1

This is the well known way to construct a *Tucker hexagon*. Each Tucker hexagon is circumscribed by a circle, the *Tucker circle*. The three antiparallel sides are congruent; their midpoints K_A, K_B and K_C lie on the symmedians of ABC in such a way that $AK_A : AK = BK_B : BK = CK_C : CK$, where K denotes the symmedian point. See [1, 2, 3].

1.1. *Identification by central angles.* We label by $\mathcal{T}_\phi$ the specific Tucker hexagon in which the congruent central angles on the chords B_aC_a, C_bA_b and A_cB_c have measure 2ϕ. The circumcircle of the Tucker hexagon is denoted by $\mathcal{C}_\phi$, and its radius by r_ϕ. In this paper, the points B_a, C_a, A_b, C_b, A_c and B_c are the vertices of $\mathcal{T}_\phi$, and T denotes the center of the Tucker circle $\mathcal{C}_\phi$.

Publication Date: January 25, 2002. Communicating Editor: Paul Yiu.
The author thanks the Communicating Editor for his assistance with §7.

Let M_a, M_b and M_c be the midpoints of A_bA_c, B_aB_c and C_aC_b respectively. Since

$$\angle M_bTM_c = B + C, \quad \angle M_cTM_a = C + A, \quad \angle M_aTM_b = A + B,$$

the top angles of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b have measures $2(A - \phi)$, $2(B - \phi)$, and $2(C - \phi)$ respectively. [1]

From these top angles, we see that the distances from T to the sidelines of triangle ABC are $r_\phi \cos(A - \phi)$, $r_\phi \cos(B - \phi)$ and $r_\phi \cos(C - \phi)$ respectively, so that in homogeneous barycentric coordinates,

$$T = (a\cos(A - \phi) : b\cos(B - \phi) : c\cos(C - \phi)).$$

For convenience we write $\overline{\phi} := \frac{\pi}{2} - \phi$. In the notations introduced by John H. Conway, [2]

$$T = (a^2(S_A + S_{\overline{\phi}}) : b^2(S_B + S_{\overline{\phi}}) : c^2(S_C + S_{\overline{\phi}})). \tag{1}$$

This shows that T is the isogonal conjugate of the Kiepert perspector $K(\overline{\phi})$.[3] We shall, therefore, write $K^*(\overline{\phi})$ for T. It is clear that $K^*(\overline{\phi})$ lies on the Brocard axis, the line through the circumcenter O and symmedian point K.

Some of the most important $K^*(\overline{\phi})$ are listed in the following table, together with the corresponding number in Kimberling's notation of [4, 5]. We write ω for the Brocard angle.

ϕ	$K^*(\overline{\phi})$	Kimberling's Notation
0	Circumcenter	X_3
ω	Brocard midpoint	X_{39}
$\pm\frac{\pi}{4}$	Kenmotu points	X_{371}, X_{372}
$\pm\frac{\pi}{3}$	Isodynamic centers	X_{15}, X_{16}
$\frac{\pi}{2}$	Symmedian point	X_6

1.2. *Coordinates.* Let K' and C_b' be the feet of the perpendiculars from $K^*(\overline{\phi})$ and C_b to BC. By considering the measures of sides and angles in $C_bC_b'K'K^*(\overline{\phi})$ we find that the (directed) distances α from C_b to BC as

$$\begin{aligned} \alpha &= r_\phi(\cos(A - \phi) - \cos(A + \phi)) \\ &= 2r_\phi \sin A \sin\phi. \end{aligned} \tag{2}$$

In a similar fashion we find the (directed) distance β from C_b to AC as

$$\begin{aligned} \beta &= r_\phi(\cos(B - \phi) + \cos(A - C + \phi)) \\ &= 2r_\phi \sin C \sin(A + \phi). \end{aligned} \tag{3}$$

[1]Here, a negative measure implies a negative orientation for the isosceles triangle.

[2]For an explanation of the notation and a brief summary, see [7, §1].

[3]This is the perspector of the triangle formed by the apexes of isosceles triangles on the sides of ABC with base angles $\overline{\phi}$. See, for instance, [7].

Combining (2) and (3) we obtain the barycentric coordinates of C_b:

$$\begin{aligned}C_b =&(a^2 \sin\phi : bc(\sin(A+\phi) : 0)\\ =&(a^2 : S_A + S_\phi : 0).\end{aligned}$$

In this way we find the coordinates for the vertices of the Tucker hexagon as

$$\begin{aligned}B_a &= (S_C + S_\phi : 0 : c^2), & C_a &= (S_B + S_\phi : b^2 : 0),\\ A_c &= (0 : b^2 : S_B + S_\phi), & B_c &= (a^2 : 0 : S_A + S_\phi),\\ C_b &= (a^2 : S_A + S_\phi : 0), & A_b &= (0 : S_C + S_\phi : c^2).\end{aligned} \tag{4}$$

Remark. The radius of the Tucker circle is $r_\phi = \frac{R\sin\omega}{\sin(\phi+\omega)}$.

2. Triangles of parallels and antiparallels

With the help of (4) we find that the three antiparallels from the Tucker hexagons bound a triangle $A_1B_1C_1$ with coordinates:

$$\begin{aligned}A_1 =&\left(\frac{a^2(S_A - S_\phi)}{S_A + S_\phi} : b^2 : c^2\right),\\ B_1 =&\left(a^2 : \frac{b^2(S_B - S_\phi)}{S_B + S_\phi} : c^2\right),\\ C_1 =&\left(a^2 : b^2 : \frac{c^2(S_C - S_\phi)}{S_C + S_\phi}\right).\end{aligned} \tag{5}$$

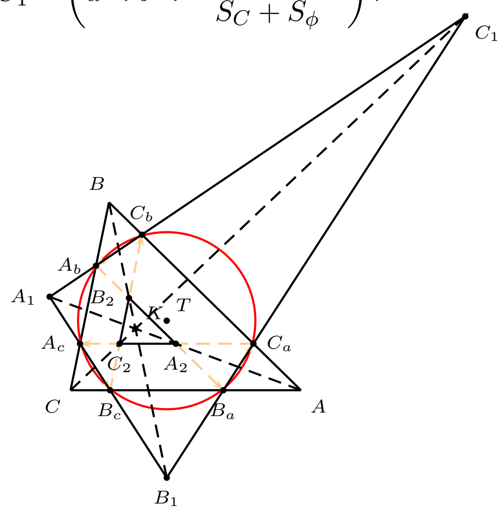

Figure 2

In the same way the parallels bound a triangle $A_2B_2C_2$ with coordinates:

$$\begin{aligned}A_2 =&(-(S_A - S_\phi) : b^2 : c^2),\\ B_2 =&(a^2 : -(S_B - S_\phi) : c^2),\\ C_2 =&(a^2 : b^2 : -(S_C - S_\phi)).\end{aligned} \tag{6}$$

It is clear that the three triangles are perspective at the symmedian point K. See Figure 2. Since ABC and $A_2B_2C_2$ are homothetic, we have a very easy construction of Tucker hexagons without invoking antiparallels: construct a triangle homothetic to ABC through K, and extend the sides of this triangles to meet the sides of ABC in six points. These six points form a Tucker hexagon.

3. Congruent rhombi

Fix ϕ. Recall that K_A, K_B and K_C are the midpoints of the antiparallels B_aC_a, A_bC_b and A_cB_c respectively. With the help of (4) we find

$$\begin{aligned} K_A =&(a^2 + 2S_\phi : b^2 : c^2),\\ K_B =&(a^2 : b^2 + 2S_\phi : c^2),\\ K_C =&(a^2 : b^2 : c^2 + 2S_\phi). \end{aligned} \tag{7}$$

Reflect the point $K^*(\overline{\phi})$ through K_A, K_B and K_C to A_ϕ, B_ϕ and C_ϕ respectively. These three points are the opposite vertices of three congruent rhombi from the point $T = K^*(\overline{\phi})$. Inspired by the figure of the *Kenmotu point* X_{371} in [4, p.268], which goes back to a collection of *Sangaku problems* from 1840, the author studied these rhombi in [6] without mentioning their connection to Tucker hexagons.

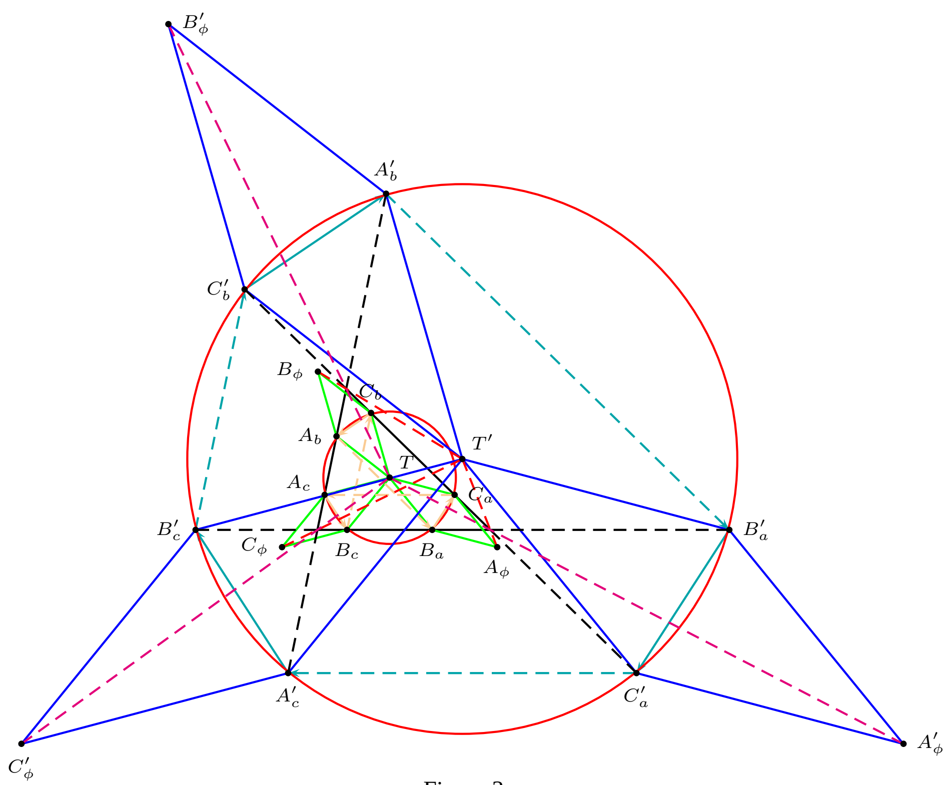

Figure 3

With the help of the coordinates for $K^*(\overline{\phi})$ and K_A found in (1) and (7) we find after some calculations,

$$\begin{aligned}
A_\phi =&(a^2(S_A - S_{\overline{\phi}}) - 4S^2 : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}})),\\
B_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) - 4S^2 : c^2(S_C - S_{\overline{\phi}})),\\
C_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}}) - 4S^2).
\end{aligned} \tag{8}$$

From these, it is clear that ABC and $A_\phi B_\phi C_\phi$ are perspective at $K^*(-\overline{\phi})$.

The perspectivity gives spectacular figures, because the rhombi formed from $\mathcal{T}_\phi$ and $\mathcal{T}_{-\phi}$ are parallel. See Figure 3. In addition, it is interesting to note that $K^*(\overline{\phi})$ and $K^*(-\overline{\phi})$ are *harmonic conjugates* with respect to the circumcenter O and the symmedian point K.

4. Isosceles triangles on the sides of $A_bA_cB_cB_aC_aC_b$

Consider the hexagon $A_bA_cB_cB_aC_aC_b$. Define the points A_3, B_3, C_3, A_4, B_4 and C_4 as the apexes of isosceles triangles $A_cA_bA_3$, $B_aB_cB_3$, $C_bC_aC_3$, $B_aC_aA_4$, $C_bA_bB_4$ and $A_cB_cC_4$ of base angle ψ, where all six triangles have positive orientation when $\psi > 0$ and negative orientation when $\psi < 0$. See Figure 4.

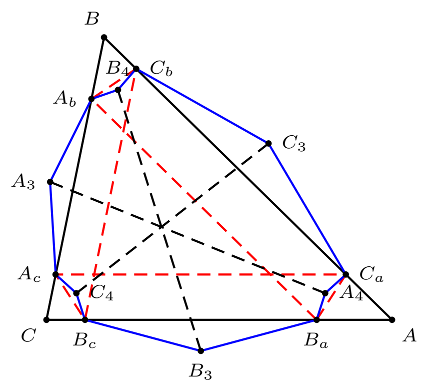

Figure 4

Proposition 1. *The lines A_3A_4, B_3B_4 and C_3C_4 are concurrent.*

Proof. Let $B_aC_a = C_bA_b = A_cB_c = 2t$, where t is given positive sign when C_aB_a and BC have equal directions, and positive sign when these directions are opposite. Note that $K_AK_BK_C$ is homothetic to ABC and that $K^*(\overline{\phi})$ is the circumcenter of $K_AK_BK_C$. Denote the circumradius of $K_AK_BK_C$ by ρ. Then we find the following:

- the signed distance from K_AK_c to AC is $t\sin B = t\frac{|K_AK_C|}{2\rho}$;
- the signed distance from AC to B_3 is $\frac{1}{2}\tan\psi|K_AK_C| - t\tan\psi\cos B$;
- the signed distance from A_4C_4 to K_AK_C is $t\tan\psi\cos B$.

Adding these signed distances we find that the signed distance from A_4C_4 to B_3 is equal to $(\frac{t}{2\rho} + \frac{\tan\psi}{2})|K_AK_C|$. By symmetry we see the signed distances from the sides B_4C_4 and A_4B_4 to A_3 and C_3 respectively are $|K_BK_C|$ and $|K_AK_B|$ multiplied by the same factor. Since triangles $K_AK_BK_C$ and $A_4B_4C_4$ are similar, the three distances are proportional to the sidelengths of $A_4B_4C_4$. Thus, $A_3B_3C_3$ is a Kiepert triangle of $A_4B_4C_4$. From this, we conclude that A_3A_4, B_3B_4 and C_3C_4 are concurrent. □

5. Points defined in *pap* triangles

Let ϕ vary and consider the triangle $A_2C_aB_a$ formed by the lines B_aA_b, B_aC_a and C_aA_c. We call this the A-*pap* triangle, because it consists of a **p**arallel, an **a**ntiparallel and again a **p**arallel. Let the parallels B_aA_b and C_aA_c intersect in A_2. Then, A_2 is the reflection of A through K_A. It clearly lies on the $A-$symmedian. See also §2. The A-*pap* triangle $A_2C_aB_a$ is oppositely similar to ABC. Its vertices are

$$\begin{aligned} A_2 =&(-(S_A - S_\phi) : b^2 : c^2), \\ C_a =&(S_B + S_\phi : b^2 : 0), \\ B_a =&(S_C + S_\phi : 0 : c^2). \end{aligned} \tag{9}$$

Now let $P = (u : v : w)$ be some point given in homogeneous barycentric coordinates with respect to ABC. For $X \in \{A, B, C\}$, the locus of the counterpart of P in the X-*pap* triangles for varying ϕ is a line through X. This can be seen from the fact that the quadrangles $AC_aA_2B_a$ in all Tucker hexagons are similar. Because the sums of coordinates of these points given in (9) are equal, we find that the A-counterpart of P, namely, P evaluated in $A_2C_aB_a$, say P_{A-pap}, has coordinates

$$\begin{aligned} P_{A-pap} \sim& u \cdot A_2 + v \cdot C_a + w \cdot B_a \\ \sim& u(-(S_A - S_\phi) : b^2 : c^2) + v(S_B + S_\phi : b^2 : 0) + w(S_C + S_\phi : 0 : c^2) \\ \sim& (-S_Au + S_Bv + S_Cw + (u+v+w)S_\phi : b^2(u+v) : c^2(u+w)). \end{aligned}$$

From this, it is clear that P_{A-pap} lies on the line $A\widetilde{P}$ where

$$\widetilde{P} = \left(\frac{a^2}{v+w} : \frac{b^2}{w+u} : \frac{c^2}{u+v}\right).$$

Likewise, we consider the counterparts of P in the B-*pap* and C-*pap* triangles $C_bB_2A_b$ and $B_cA_cC_2$. By symmetry, the loci of P_{B-pap} and P_{C-pap} are the B- and C-cevians of $\widetilde{P}$.

Proposition 2. *For every ϕ, the counterparts of P in the three pap-triangles of the Tucker hexagon $\mathcal{T}_\phi$ form a triangle perspective with ABC at the point $\widetilde{P}$.*

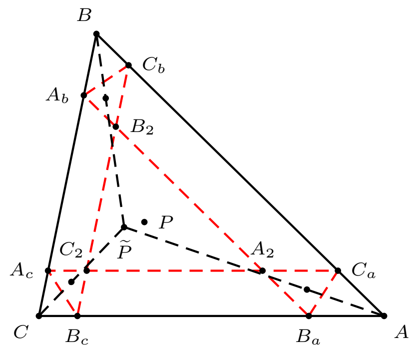

Figure 5

6. Circumcenters of *apa* triangles

As with the *pap*-triangles in the preceding section, we name the triangle $A_1B_cC_b$ formed by the **a**ntiparallel B_cC_b, the **p**arallel A_bC_b, and the **a**ntiparallel A_cB_c the A-*apa* triangle. The other two *apa*-triangles are $A_cB_1C_a$ and $A_bB_aC_1$. Unlike the *pap*-triangles, these are in general not similar to ABC. They are nevertheless isosceles triangles. We have the following interesting results on the circumcenters.

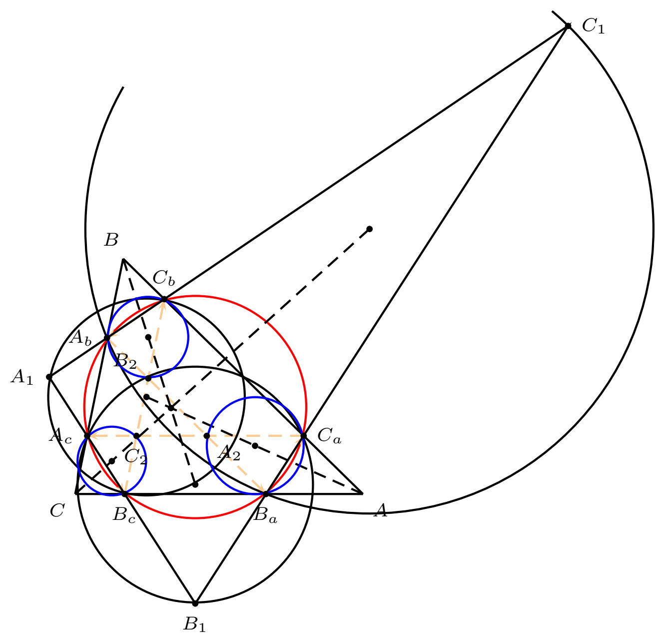

Figure 6

We note that the quadrangles $BA_cO_{B-apa}C_a$ for all possible ϕ are homothetic through B. Therefore, the locus of O_{B-apa} is a line through B. To identify this line, it is sufficient to find O_{B-apa} for one ϕ. Thus, for one special Tucker hexagon, we take the one with $C_a = A$ and $A_c = C$. Then the B-*apa* triangle is the isosceles triangle erected on side b and having a base angle of B, and its circumcenter

O_{B-apa} is the apex of the isosceles triangle erected on the same side with base angle $2B - \frac{\pi}{2}$. Using the identity [4]

$$S^2 = S_{AB} + S_{AC} + S_{BC},$$

we find that

$$\begin{aligned} O_{B-apa} =&(S_C + S_{2B-\frac{\pi}{2}} : -b^2 : S_A + S_{2B-\frac{\pi}{2}})\\ =&(a^2(a^2S_A + b^2S_B) : b^2(S_{BB} - SS) : c^2(b^2S_B + c^2S_C)), \end{aligned}$$

after some calculations. From this, we see that the O_{B-apa} lies on the line BN^*, where

$$N^* = \left(\frac{a^2}{b^2S_B + c^2S_C} : \frac{b^2}{a^2S_A + c^2S_C} : \frac{c^2}{c^2S_C + b^2S_B}\right)$$

is the isogonal conjugate of the nine point center N. Therefore, the locus of O_{B-apa} for all Tucker hexagons is the B-cevian of N^*. By symmetry, we see that the loci of O_{A-apa} and O_{C-apa} are the A- and C-cevians of N^* respectively. This, incidentally, is the same as the perspector of the circumcenters of the *pap*-triangles in the previous section.

Proposition 3. *For $X \in \{A, B, C\}$, the line joining the circumcenters of the X-*pap-*triangle and the X-*apa-*triangle passes through X. These three lines intersect at the isogonal conjugate of the nine point center of triangle ABC.*

7. More circumcenters of isosceles triangles

From the center $T = K^*(\overline{\phi})$ of the Tucker circle and the vertices of the Tucker hexagon $\mathcal{T}_\phi$, we obtain six isosceles triangles. Without giving details, we present some interesting results concerning the circumcenters of these isosceles triangles.

(1) The circumcenters of the isosceles triangles TB_aC_a, TC_bA_b and TA_cB_c form a triangle perspective with ABC at

$$K^*(\overline{2\phi}) = (a^2(S_A + S \cdot \tan 2\phi) : b^2(S_B + S \cdot \tan 2\phi) : c^2(S_C + S \cdot \tan 2\phi)).$$

See Figure 7, where the Tucker hexagon $\mathcal{T}_{2\phi}$ and Tucker circle $\mathcal{C}_{2\phi}$ are also indicated.

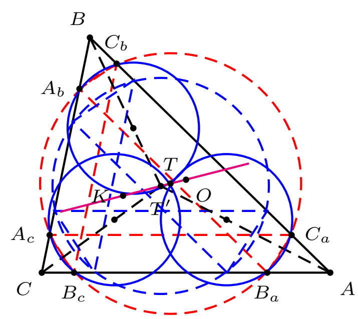

Figure 7

[4] Here, S_{XY} stands for the product S_XS_Y.

(2) The circumcenters of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b form a triangle perspective with ABC at

$$\left(\frac{a^2}{S^2(3S^2 - S_{BC}) + 2a^2S^2 \cdot S_\phi + (S^2 + S_{BC})S_{\phi\phi}} : \cdots : \cdots\right).$$

See Figure 8.

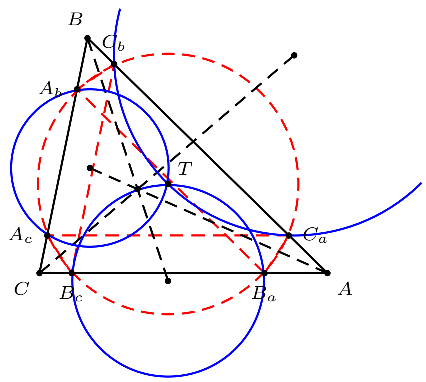

Figure 8

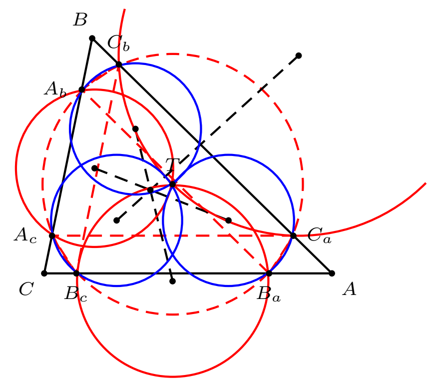

Figure 9

(3) The three lines joining the circumcenters of TB_aC_a, TA_bA_c; ... are concurrent at the point

$$(a^2(S^2(3S^2 - S_{\omega A}) + 2S^2(S_\omega + S_A)S_\phi + (2S^2 - S_{BC} + S_{AA})S_{\phi\phi}) : \cdots : \cdots).$$

See Figure 9.

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