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Collineations, Conjugacies, and Cubics

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Abstract. If F is an involution and φ a suitable collineation, then $\varphi \circ F \circ \varphi^{-1}$ is an involution; this form includes well-known conjugacies and new conjugacies, including *aleph, beth, complementary,* and *anticomplementary*. If $Z(U)$ is the self-isogonal cubic with pivot U, then φ carries $Z(U)$ to a pivotal cubic. Particular attention is given to the Darboux and Lucas cubics, D and L, and conjugacy-preserving mappings between D and L are formulated.

1. Introduction

The defining property of the kind of mapping called *collineation* is that it carries lines to lines. Matrix algebra lends itself nicely to collineations as in [1, Chapter XI] and [5]. In order to investigate collineation-induced conjugacies, especially with regard to triangle centers, suppose that an arbitrary point P in the plane of $\triangle ABC$ has homogeneous trilinear coordinates $p : q : r$ relative to $\triangle ABC$, and write

$$A = 1:0:0, \quad B = 0:1:0, \quad C = 0:0:1,$$

so that

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose now that suitably chosen points $P_i = p_i : q_i : r_i$ and $P'_i = p'_i : q'_i : r'_i$ for $i = 1, 2, 3, 4$ are given and that we wish to represent the unique collineation φ that maps each P_i to P'_i. (Precise criteria for "suitably chosen" will be determined soon.) Let

$$\mathbb{P} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}, \qquad \mathbb{P}' = \begin{pmatrix} p'_1 & q'_1 & r'_1 \\ p'_2 & q'_2 & r'_2 \\ p'_3 & q'_3 & r'_3 \end{pmatrix}.$$

We seek a matrix $\mathbb{M}$ such that $\varphi(X) = X\mathbb{M}$ for every point $X = x : y : z$, where X is represented as a 1×3 matrix:

$$X = \begin{pmatrix} x & y & z \end{pmatrix}$$

In particular, we wish to have

$$\mathbb{PM} = \mathbb{DP}' \quad \text{and} \quad P_4\mathbb{M} = \begin{pmatrix} gp'_4 & gq'_4 & gr'_4 \end{pmatrix},$$

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where

$$\mathbb{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$$

for some multipliers d, e, f, g. By homogeneity, we can, and do, put $g = 1$. Then substituting $\mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$ for $\mathbb{M}$ gives $P_4\mathbb{P}^{-1}\mathbb{D} = \mathbb{P}'_4(\mathbb{P}')^{-1}$. Writing out both sides leads to

$$\begin{aligned} d &= \frac{(q'_2r'_3 - q'_3r'_2)p'_4 + (r'_2p'_3 - r'_3p'_2)q'_4 + (p'_2q'_3 - p'_3q'_2)r'_4}{(q_2r_3 - q_3r_2)p_4 + (r_2p_3 - r_3p_2)q_4 + (p_2q_3 - p_3q_2)r_4}, \\ e &= \frac{(q'_3r'_1 - q'_1r'_3)p'_4 + (r'_3p'_1 - r'_1p'_3)q'_4 + (p'_3q'_1 - p'_1q'_3)r'_4}{(q_3r_1 - q_1r_3)p_4 + (r_3p_1 - r_1p_3)q_4 + (p_3q_1 - p_1q_3)r_4}, \\ f &= \frac{(q'_1r'_2 - q'_2r'_1)p'_4 + (r'_1p'_2 - r'_2p'_1)q'_4 + (p'_1q'_2 - p'_2q'_1)r'_4}{(q_1r_2 - q_2r_1)p_4 + (r_1p_2 - r_2p_1)q_4 + (p_1q_2 - p_2q_1)r_4}. \end{aligned}$$

The point $D := d : e : f$ is clearly expressible as quotients of determinants:

$$D = \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_2 & q'_2 & r'_2 \\ p'_3 & q'_3 & r'_3 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}} : \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_3 & q'_3 & r'_3 \\ p'_1 & q'_1 & r'_1 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_3 & q_3 & r_3 \\ p_1 & q_1 & r_1 \end{vmatrix}} : \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_1 & q'_1 & r'_1 \\ p'_2 & q'_2 & r'_2 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

With $\mathbb{D}$ determined[1], we write

$$\mathbb{M} = \mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$$

and are now in a position to state the conditions to be assumed about the eight initial points:

(i) $\mathbb{P}$ and $\mathbb{P}'$ are nonsingular;

(ii) the denominators in the expressions for d, e, f are nonzero;

(iii) $def \neq 0$.

Conditions (i) and (ii) imply that the collineation φ is given by $\varphi(X) = X\mathbb{M}$, and (iii) ensures that $\varphi^{-1}(X) = X\mathbb{M}^{-1}$. A collineation φ satisfying (i)-(iii) will be called *regular*. If φ is regular then clearly φ^{-1} is regular.

If the eight initial points are centers (*i.e.*, triangle centers) for which no three P_i are collinear and no three P'_i are collinear, then for every center X, the image $\varphi(X)$ is a center. If P_1, P_2, P_3 are respectively the A-, B-, C- vertices of a central triangle [3, pp. 53-57] and P_4 is a center, and if the same is true for P'_i for $i = 1$, 2, 3, 4, then in this case, too, φ carries centers to centers.

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[1]A geometric realization of D follows. Let $\hat{P}$ denote the circle

$$(p_1\alpha + p_2\beta + p_3\gamma)(a\alpha + b\beta + c\gamma) + p_4(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

and let $\hat{Q}$, $\hat{R}$, $\hat{P}'$, $\hat{Q}'$, $\hat{R}'$ be the circles likewise formed from the points P_i and P'_i. Following [3, p.225], let Λ and Λ' be the radical centers of circles $\hat{P}, \hat{Q}, \hat{R}$ and $\hat{P}', \hat{Q}', \hat{R}'$, respectively. Then D is the trilinear quotient Λ/Λ'.

The representation $\varphi(X) = X\mathbb{M}$ shows that for $X = x : y : z$, the image $\varphi(X)$ has the form

$$f_1x + g_1y + h_1z : f_2x + g_2y + h_2z : f_3x + g_3y + h_3z.$$

Consequently, if Λ is a curve homogeneous of degree $n \geq 1$ in α, β, γ, then $\varphi(\Lambda)$ is also a curve homogeneous of degree n in α, β, γ. In particular, φ carries a circumconic onto a conic that circumscribes the triangle having vertices $\varphi(A)$, $\varphi(B)$, $\varphi(C)$, and likewise for higher order curves. We shall, in §5, concentrate on cubic curves.

Example 1. Suppose

$$P = p : q : r, \qquad U = u : v : w, \qquad U' = u' : v' : w'$$

are points, none lying on a sideline of $\triangle ABC$, and U' is not on a sideline of the cevian triangle of P (whose vertices are the rows of matrix $\mathbb{P}'$ shown below). Then the collineation φ that carries ABC to $\mathbb{P}'$ and U to U' is regular. We have

$$\mathbb{P}' = \begin{pmatrix} 0 & q & r \\ p & 0 & r \\ p & q & 0 \end{pmatrix}, \quad \text{and} \quad (\mathbb{P}')^{-1} = \frac{1}{|\mathbb{P}'|}\begin{pmatrix} -p & q & r \\ p & -q & r \\ p & q & -r \end{pmatrix},$$

leading to

$$\varphi(X) = X\mathbb{M} = p(ey + fz) : q(fz + dx) : r(dx + ey), \tag{1}$$

where

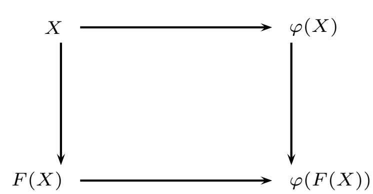
$$d : e : f = \frac{1}{u}\left(-\frac{u'}{p} + \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{v}\left(\frac{u'}{p} - \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{w}\left(\frac{u'}{p} + \frac{v'}{q} - \frac{w'}{r}\right). \tag{2}$$

Example 2. Continuing from Example 1, φ^{-1} is the collineation given by

$$\varphi^{-1}(X) = X\mathbb{M}^{-1} = \frac{1}{d}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{e}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{f}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right). \tag{3}$$

2. Conjugacies induced by collineations

Suppose F is a mapping on the plane of $\triangle ABC$ and φ is a regular collineation, and consider the following diagram:



On writing $\varphi(X)$ as P, we have $m(P) = \varphi(F(\varphi^{-1}(P)))$. If $F(F(X)) = X$, then $m(m(P)) = P$; in other words, if F is an involution, then m is an involution. We turn now to Examples 3-10, in which F is a well-known involution and φ is the collineation in Example 1 or a special case thereof. In Examples 11 and 12, φ is complementation and anticomplementation, respectively.

Example 3. For any point $X = x : y : z$ not on a sideline of $\triangle ABC$, the isogonal conjugate of X is given by

$$F(X) = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Suppose P, U, φ are as in Example 1. The involution m given by $m(X) = \varphi(F(\varphi^{-1}(X)))$ will be formulated: equation (3) implies

$$F(\varphi^{-1}(X)) = \frac{d}{-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}} : \frac{e}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} : \frac{f}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}},$$

and substituting these coordinates into (1) leads to

$$m(X) = m_1 : m_2 : m_3, \tag{4}$$

where

$$m_1 = m_1(p, q, r, x, y, z) = p\left(\frac{e^2}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} + \frac{f^2}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}}\right) \tag{5}$$

and m_2 and m_3 are determined cyclically from m_1; for example, $m_2(p, q, r, x, y, z) = m_1(q, r, p, y, z, x)$.

In particular, if $U = 1 : 1 : 1$ and $U' = p : q : r$, then from equation (2), we have $d : e : f = 1 : 1 : 1$, and (5) simplifies to

$$m(X) = x\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right) : y\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right) : z\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right).$$

This is the P-Ceva conjugate of X, constructed [3, p. 57] as the perspector of the cevian triangle of P and the anticevian triangle of X.

Example 4. Continuing with isogonal conjugacy for F and with φ as in Example 3 (with $U = 1 : 1 : 1$ and $U' = p : q : r$), here we use φ^{-1} in place of φ, so that $m(X) = \varphi^{-1}(F(\varphi(X)))$. The result is (4), with

$$m_1 = -q^2r^2x^2 + r^2p^2y^2 + p^2q^2z^2 + (-q^2r^2 + r^2p^2 + p^2q^2)(yz + zx + xy).$$

In this case, $m(X)$ is the P-*aleph conjugate of* X.

Let

$$n(X) = \frac{1}{y+z} : \frac{1}{z+x} : \frac{1}{x+y}.$$

Then $X = n(X)$-aleph conjugate of X. Another easily checked property is that a necessary and sufficient condition that

$$X = X\text{-aleph conjugate of the incenter}$$

is that $X =$ incenter or else X lies on the conic $\beta\gamma + \gamma\alpha + \alpha\beta = 0$.

In [4], various triples $(m(X), P, X)$ are listed. A selection of these permuted to $(X, P, m(X))$ appears in Table 1. The notation X_i refers to the indexing of triangle centers in [4]. For example,

$$X_{57} = \frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c} = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2},$$

abbreviated in Table 1 and later tables as "57, $\tan\frac{A}{2}$". In Table 1 and the sequel, the area σ of $\triangle ABC$ is given by

$$16\sigma^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Table 1. Selected aleph conjugates

center, X	P	P-aleph conj. of X
$57, \tan\frac{A}{2}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$63, \cot A$	$2, \frac{1}{a}$	$1, 1$
$57, \tan\frac{A}{2}$	$174, \sec\frac{A}{2}$	$1, 1$
$2, \frac{1}{a}$	$86, \frac{bc}{b+c}$	$2, \frac{1}{a}$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$3, \cos A$
$43, ab+ac-bc$	$1, 1$	$9, b+c-a$
$610, \sigma^2 - a^2\cot A$	$2, \frac{1}{a}$	$19, \tan A$
$165, \tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 5. Here, F is reflection about the circumcenter:

$$F(x:y:z) = 2R\cos A - hx : 2R\cos B - hy : 2R\cos C - hz,$$

where $R =$ circumradius, and h normalizes[2] X. Keeping φ as in Example 4, we find

$$m_1(x,y,z) = 2abc(\cos B+\cos C)\left(\frac{x(b+c-a)}{p}+\frac{y(c+a-b)}{q}+\frac{z(a+b-c)}{r}\right) - 16\sigma^2 x,$$

which, via (4), defines the *P-beth conjugate of* X.

Table 2. Selected beth conjugates

center, X	P	P-beth conj. of X
$110, \frac{a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$
$6, a$	$101, \frac{a}{b-c}$	$6, a$
$4, \sec A$	$8, \csc^2\frac{A}{2}$	$40, \cos B+\cos C-\cos A-1$
$190, \frac{bc}{b-c}$	$9, b+c-a$	$292, a/(a^2-bc)$
$11, 1-\cos(B-C)$	$11, 1-\cos(B-C)$	$244, (1-\cos(B-C))\sin^2\frac{A}{2}$
$1, 1$	$99, \frac{bc}{b^2-c^2}$	$85, \frac{b^2c^2}{b+c-a}$
$10, \frac{b+c}{a}$	$100, \frac{1}{b-c}$	$73, \cos A(\cos B+\cos C)$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$56, 1-\cos A$

Among readily verifiable properties of beth-conjugates are these:

(i) $\varphi(X_3)$ is collinear with every pair $\{X, m(X)\}$.

(ii) Since each line $\mathcal{L}$ through X_3 has two points fixed under reflection in X_3, the line $\varphi(\mathcal{L})$ has two points that are fixed by m, namely $\varphi(X_3)$ and $\varphi(\mathcal{L}\cap\mathcal{L}^\infty)$.

[2] If $X \notin \mathcal{L}^\infty$, then $h = 2\sigma/(ax+by+cz)$; if $X \in \mathcal{L}^\infty$ and $xyz \neq 0$, then $h = 1/x+1/y+1/z$; otherwise, $h = 1$. For $X \notin \mathcal{L}^\infty$, the nonhomogeneous representation for X as the ordered triple (hx, hy, hz) gives the actual directed distances hx, hy, hz from X to sidelines BC, CA, AB, respectively.

(iii) When $P = X_{21}$, φ carries the Euler line $L(3,4,20,30)$ to $L(1,3,56,36)$, on which the m-fixed points are X_1 and X_{36}, and φ carries the line $L(1,3,40,517)$ to $L(21,1,58,1078)$, on which the m-fixed points are X_1 and X_{1078}.

(iv) If X lies on the circumcircle, then the X_{21}-beth conjugate, X', of X lies on the circumcircle. Such pairs (X, X') include (X_i, X_j) for these (i,j): $(99,741)$, $(100,106)$, $(101,105)$, $(102,108)$, $(103,934)$, $(104,109)$, $(110,759)$.

(v) $P = P$-beth conjugate of X if and only if $X = P \cdot X_{56}$ (trilinear product).

Example 6. Continuing Example 5 with φ^{-1} in place of φ leads to the P-*gimel conjugate of* X, defined via (4) by

$$m_1(x,y,z) = 2abc\left(-\frac{\cos A}{p} + \frac{\cos B}{q} + \frac{\cos C}{r}\right)S - 8\sigma^2 x,$$

where $S = x(bq+cr) + y(cr+ap) + z(ap+bq)$.

It is easy to check that if $P \in \mathcal{L}^\infty$, then $m(X_1) = X_1$.

Table 3. Selected gimel conjugates

center, X	P	P-gimel conjugate of X
$1, 1$	$3, \cos A$	$1, 1$
$3, \cos A$	$283, \frac{\cos A}{\cos B+\cos C}$	$3, \cos A$
$30, \cos A - 2\cos B\cos C$	$8, \csc^2\frac{A}{2}$	$30, \cos A - 2\cos B\cos C$
$4, \sec A$	$21, \frac{1}{\cos B+\cos C}$	$4, \sec A$
$219, \cos A\cot\frac{A}{2}$	$63, \cot A$	$6, a$

Example 7. For distinct points $X' = x' : y' : z'$ and $X = x : y : z$, neither lying on a sideline of $\triangle ABC$, the X'-Hirst inverse of X is defined [4, Glossary] by

$$y'z'x^2 - x'^2yz : z'x'y^2 - y'^2zx : x'y'z^2 - z'^2xy.$$

We choose $X' = U = U' = 1:1:1$. Keeping φ as in Example 4, for $X \neq P$ we obtain m as in expression (4), with

$$m_1(x,y,z) = p\left(\frac{y}{q} - \frac{z}{r}\right)^2 + x\left(\frac{2x}{p} - \frac{y}{q} - \frac{z}{r}\right).$$

In this example, $m(X)$ defines the P-*daleth conjugate of* X. The symbol ω in Table 5 represents the Brocard angle of $\triangle ABC$.

Table 4. Selected daleth conjugates

center, X	P	P-daleth conjugate of X
$518, b^2+c^2-a(b+c)$	$1, 1$	$37, b+c$
$1, 1$	$1, 1$	$44, b+c-2a$
$511, \cos(A+\omega)$	$3, \cos A$	$216, \sin 2A\cos(B-C)$
$125, \cos A\sin^2(B-C)$	$4, \sec A$	$125, \cos A\sin^2(B-C)$
$511, \cos(A+\omega)$	$6, a$	$39, a(b^2+c^2)$
$672, a(b^2+c^2-a(b+c))$	$6, a$	$42, a(b+c)$
$396, \cos(B-C)+2\cos(A-\frac{\pi}{3})$	$13, \csc(A+\frac{\pi}{3})$	$30, \cos A - 2\cos B\cos C$
$395, \cos(B-C)+2\cos(A+\frac{\pi}{3})$	$14, \csc(A-\frac{\pi}{3})$	$30, \cos A - 2\cos B\cos C$

Among properties of daleth conjugacy that can be straightforwardly demonstrated is that for given P, a point X satisfies the equation

$$P = P\text{-daleth conjugate of } X$$

if and only if X lies on the trilinear polar of P.

Example 8. Continuing Example 7, we use φ^{-1} in place of φ and define the resulting image $m(X)$ as the *P-he conjugate of X.*[3] We have m as in (4) with

$$\begin{aligned} m_1(x,y,z) &= -p(y+z)^2 + q(z+x)^2 + r(x+y)^2 \\ &+ \frac{qr}{p}(x+y)(x+z) - \frac{rp}{q}(y+z)(y+x) - \frac{pq}{r}(z+x)(z+y). \end{aligned}$$

Table 5. Selected he conjugates

center, X	P	P-he conjugate of X
$239, bc(a^2 - bc)$	$2, \frac{1}{a}$	$9, b + c - a$
$36, 1 - 2\cos A$	$6, a$	$43, \csc B + \csc C - \csc A$
$514, \frac{b-c}{a}$	$7, \sec^2 \frac{A}{2}$	$57, \tan \frac{A}{2}$
$661, \cot B - \cot C$	$21, \frac{1}{\cos B + \cos C}$	$3, \cos A$
$101, \frac{a}{b-c}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 9. The X_1-Ceva conjugate of X not lying on a sideline of is $\triangle ABC$ is the point

$$-x(-x + y + z) : y(x - y + z) : z(x + y - z).$$

Taking this for F and keeping φ as in Example 4 leads to

$$m_1(x,y,z) = p(x^2q^2r^2 + 2p^2(ry - qz)^2 - pqr^2xy - pq^2rxz),$$

which via m as in (4) defines the *P-waw conjugate of X.*

Table 6. Selected waw conjugates

center, X	P	P-waw conjugate of X
$37, b + c$	$1, 1$	$354, (b - c)^2 - ab - ac$
$5, \cos(B - C)$	$2, \frac{1}{a}$	$141, bc(b^2 + c^2)$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$142, b + c - \frac{(b-c)^2}{a}$
$53, \tan A \cos(B - C)$	$4, \sec A$	$427, (b^2 + c^2) \sec A$
$51, a^2 \cos(B - C)$	$6, a$	$39, a(b^2 + c^2)$

Example 10. Continuing Example 9 with φ^{-1} in place of φ gives

$$m_1(x,y,z) = p(y+z)^2 - ry^2 - qz^2 + (p - r)xy + (p - q)xz,$$

which via m as in (4) defines the *P-zayin conjugate of X.* When $P =$ incenter, this conjugacy is isogonal conjugacy. Other cases are given in Table 7.

[3]The fifth letter of the Hebrew alphabet is *he*, homophonous with *hay*.

Table 7. Selected zayin conjugates

center, X	P	P-zayin conjugate of X
$9, b+c-a$	$2, \frac{1}{a}$	$9, b+c-a$
$101, \frac{a}{b-c}$	$2, \frac{1}{a}$	$661, \cot B - \cot C$
$108, \frac{\sin A}{\sec B - \sec C}$	$3, \cos A$	$656, \tan B - \tan C$
$109, \frac{\sin A}{\cos B - \cos C}$	$4, \sec A$	$656, \tan B - \tan C$
$43, ab+ac-bc$	$6, a$	$43, ab+ac-bc$
$57, \tan\frac{A}{2}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$40, \cos B + \cos C - \cos A - 1$	$8, \csc^2\frac{A}{2}$	$40, \cos B + \cos C - \cos A - 1$

Example 11. The complement of a point X not on $\mathcal{L}^\infty$ is the point X' satisfying the vector equation

$$\overrightarrow{X'X_2} = \frac{1}{2}\overrightarrow{X_2X}.$$

If $X = x : y : z$, then

$$X' = \frac{by+cz}{a} : \frac{cz+ax}{b} : \frac{ax+by}{c}. \tag{6}$$

If $X \in \mathcal{L}^\infty$, then (6) defines the complement of X. The mapping $\varphi(X) = X'$ is a collineation. Let $P = p : q : r$ be a point not on a sideline of $\triangle ABC$, and let

$$F(X) = \frac{1}{px} : \frac{1}{qy} : \frac{1}{rz},$$

the P-isoconjugate of X. Then m as in (4) is given by

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax-by+cz)} + \frac{c^2}{r(ax+by-cz)}\right)$$

and defines the *P-complementary conjugate of X*. The X_1-complementary conjugate of X_2, for example, is the symmedian point of the medial triangle, X_{141}, and X_{10} is its own X_1-complementary conjugate. Moreover, X_1-complementary conjugacy carries $\mathcal{L}^\infty$ onto the nine-point circle. Further examples follow:

Table 8. Selected complementary conjugates

center X	P	P-complementary conjugate of X
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$141, bc(b^2+c^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$3, \cos A$
$10, \frac{b+c}{a}$	$4, \sec A$	$5, \cos(B-C)$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$141, bc(b^2+c^2)$	$7, \sec^2\frac{A}{2}$	$142, b+c-\frac{(b-c)^2}{a}$
$9, b+c-a$	$9, b+c-a$	$141, bc(b^2+c^2)$
$2, \frac{1}{a}$	$19, \tan A$	$5, \cos(B-C)$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

Example 12. The anticomplement of a point X is the point X'' given by

$$X'' = \frac{-ax+by+cz}{a} : \frac{ax-by+cz}{b} : \frac{ax+by-cz}{c}.$$

Keeping F and φ as in Example 11, we have $\varphi^{-1}(X) = X''$ and define m by $m = \varphi^{-1} \circ F \circ \varphi$, Thus, $m(X)$ is determined as in (4) from

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax+cz)} + \frac{c^2}{r(ax+by)} - \frac{a^2}{p(by+cz)}\right).$$

Here, $m(X)$ defines the *P-anticomplementary conjugate of X*. For example, the centroid is the X_1-anticomplementary conjugate of X_{69} (the symmedian point of the anticomplementary triangle), and the Nagel point, X_8, is its own self X_1-anticomplementary conjugate. Moreover, X_1-anticomplementary conjugacy carries the nine-point circle onto $\mathcal{L}^\infty$. Further examples follow:

Table 9. Selected anticomplementary conjugates

center, X	P	P-anticomplementary conj. of X
$3, \cos A$	$1, 1$	$4, \sec A$
$5, \cos(B-C)$	$1, 1$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$69, bc(b^2+c^2-a^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$4, \sec A$	$4, \sec A$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$5, \cos(B-C)$	$19, \tan A$	$2, \frac{1}{a}$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

3. The Darboux cubic, D

This section formulates a mapping Ψ on the plane of $\triangle ABC$; this mapping preserves two pivotal properties of the Darboux cubic D. In Section 4, $\Psi(D)$ is recognized as the Lucas cubic. In Section 5, collineations will be applied to D, carrying it to cubics having two pivotal configurations with properties analogous to those of D.

The Darboux cubic is the locus of a point X such that the pedal triangle of X is a cevian triangle. The pedal triangle of X has for its A-vertex the point in which the line through X perpendicular to line BC meets line BC, and likewise for the B- and C- vertices. We denote these three vertices by X_A, X_B, X_C, respectively. To say that their triangle is a cevian triangle means that the lines AX_A, BX_B, CX_C concur. Let $\Psi(P)$ denote the point of concurrence. In order to obtain a formula for Ψ, we begin with the pedal triangle of P:

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 0 & \beta+\alpha c_1 & \gamma+\alpha b_1 \\ \alpha+\beta c_1 & 0 & \gamma+\beta a_1 \\ \alpha+\gamma b_1 & \beta+\gamma a_1 & 0 \end{pmatrix},$$

where $a_1 = \cos A$, $b_1 = \cos B$, $c_1 = \cos C$. Then

$$\begin{aligned} BX_B \cap CX_C =&(\alpha+\beta c_1)(\alpha+\gamma b_1) : (\beta+\gamma a_1)(\alpha+\beta c_1) : (\gamma+\beta a_1)(\alpha+\gamma b_1), \\ CX_C \cap AX_A =&(\alpha+\gamma b_1)(\beta+\alpha c_1) : (\beta+\gamma a_1)(\beta+\alpha c_1) : (\gamma+\alpha b_1)(\beta+\gamma a_1), \\ AX_A \cap BX_B =&(\alpha+\beta c_1)(\gamma+\alpha b_1) : (\beta+\alpha c_1)(\gamma+\beta a_1) : (\gamma+\alpha b_1)(\gamma+\beta a_1). \end{aligned}$$

Each of these three points is $\Psi(X)$. Multiplying and taking the cube root gives the following result:

$$\Psi(X) = \psi(\alpha,\beta,\gamma,a,b,c) : \psi(\beta,\gamma,\alpha,b,c,a) : \psi(\gamma,\alpha,\beta,c,a,b),$$

where

$$\psi(\alpha,\beta,\gamma,a,b,c) = [(\alpha+\beta c_1)^2(\alpha+\gamma b_1)^2(\beta+\alpha c_1)(\gamma+\alpha b_1)]^{1/3}.$$

The Darboux cubic is one of a family of cubics $Z(U)$ given by the form (e.g., [3, p.240])

$$u\alpha(\beta^2-\gamma^2) + v\beta(\gamma^2-\alpha^2) + w\gamma(\alpha^2-\beta^2) = 0, \tag{7}$$

where the point $U = u : v : w$ is called the pivot of $Z(U)$, in accord with the collinearity of U, X, and the isogonal conjugate, X^{-1}, of X, for every point $X = \alpha : \beta : \gamma$ on $Z(U)$. The Darboux cubic is $Z(X_{20})$; that is,

$$(a_1 - b_1c_1)\alpha(\beta^2-\gamma^2) + (b_1 - c_1a_1)\beta(\gamma^2-\alpha^2) + (c_1 - a_1b_1)\gamma(\alpha^2-\beta^2) = 0.$$

This curve has a secondary pivot, the circumcenter, X_3, in the sense that if X lies on D, then so does the reflection of X in X_3. Since X_3 itself lies on D, we have here a second system of collinear triples on D.

The two types of pivoting lead to chains of centers on D:

$$X_1 \xrightarrow{\text{refl}} X_{40} \xrightarrow{\text{isog}} X_{84} \xrightarrow{\text{refl}} \cdots \tag{8}$$

$$X_3 \xrightarrow{\text{isog}} X_4 \xrightarrow{\text{refl}} X_{20} \xrightarrow{\text{isog}} X_{64} \xrightarrow{\text{refl}} \cdots. \tag{9}$$

Each of the centers in (8) and (9) has a trilinear representation in polynomials with all coefficients integers. One wonders if all such centers on D can be generated by a finite collection of chains using reflection and isogonal conjugation as in (8) and (9).

4. The Lucas cubic, L

Transposing the roles of pedal and cevian triangles in the description of D leads to the Lucas cubic, L, i.e., the locus of a point $X = \alpha : \beta : \gamma$ whose cevian triangle is a pedal triangle. Mimicking the steps in Section 3 leads to

$$\Psi^{-1}(X) = \lambda(\alpha,\beta,\gamma,a,b,c) : \lambda(\beta,\gamma,\alpha,b,c,a) : \lambda(\gamma,\alpha,\beta,c,a,b),$$

where $\lambda(\alpha,\beta,\gamma,a,b,c) =$

$$\{[\alpha^2 - (\alpha a_1 - \gamma c_1)(\alpha a_1 - \beta b_1)]([(\alpha\beta + \gamma(\alpha a_1 - \beta b_1)][(\alpha\gamma + \beta(\alpha a_1 - \gamma c_1)]\}^{1/3}.$$

It is well known [1, p.155] that “the feet of the perpendiculars from two isogonally conjugate points lie on a circle; that is, isogonal conjugates have a common pedal circle ... ” Consequently, L is self-cyclocevian conjugate [3, p. 226]. Since L is also self-isotomic conjugate, certain centers on L are generated in chains:

$$X_7 \xrightarrow{\text{isot}} X_8 \xrightarrow{\text{cycl}} X_{189} \xrightarrow{\text{isot}} X_{329} \xrightarrow{\text{cycl}} \cdots \tag{10}$$

$$X_2 \xrightarrow{\text{cycl}} X_4 \xrightarrow{\text{isot}} X_{69} \xrightarrow{\text{cycl}} X_{253} \xrightarrow{\text{isot}} X_{20} \xrightarrow{\text{cycl}} \cdots . \tag{11}$$

The mapping Ψ, of course, carries D to L, isogonal conjugate pairs on D to cyclocevian conjugate pairs on L, reflection-in-circumcenter pairs on D to isotomic conjugate pairs on L, and chains (8) and (9) to chains (10) and (11).

5. Cubics of the form $\varphi(Z(U))$

Every line passing through the pivot of the Darboux cubic D meets D in a pair of isogonal conjugates, and every line through the secondary pivot X_3 of D meets D in a reflection-pair. We wish to obtain generalizations of these pivotal properties by applying collineations to D. As a heuristic venture, we apply to D trilinear division by a point $P = p : q : r$ for which $pqr \neq 0$: the set D/P of points X/P as X traverses D is easily seen to be the cubic

$$\begin{aligned} &(a_1 - b_1c_1)px(q^2y^2 - r^2z^2) + (b_1 - c_1a_1)qy(r^2z^2 - p^2x^2) \\ &+(c_1 - a_1b_1)rz(p^2x^2 - q^2y^2) = 0. \end{aligned}$$

This is the self-P-isoconjugate cubic with pivot X_{20}/P and secondary pivot X_3/P. The cubic D/P, for some choices of P, passes through many “known points,” of course, and this is true if for D we substitute any cubic that passes through many “known points”.

The above preliminary venture suggests applying a variety of collineations to various cubics $Z(U)$. To this end, we shall call a regular collineation φ a ***tricentral collineation*** if there exists a mapping m_1 such that

$$\varphi(\alpha : \beta : \gamma) = m_1(\alpha : \beta : \gamma) : m_1(\beta : \gamma : \alpha) : m_1(\gamma : \alpha : \beta) \tag{12}$$

for all $\alpha : \beta : \gamma$. In this case, φ^{-1} has the form given by

$$n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta),$$

hence is tricentral.

The tricentral collineation (12) carries $Z(U)$ in (7) to the cubic $\varphi(Z(U))$ having equation

$$u\hat{\alpha}(\hat{\beta}^2 - \hat{\gamma}^2) + v\hat{\beta}(\hat{\gamma}^2 - \hat{\alpha}^2) + w\hat{\gamma}(\hat{\alpha}^2 - \hat{\beta}^2) = 0, \tag{13}$$

where

$$\hat{\alpha} : \hat{\beta} : \hat{\gamma} = n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta).$$

Example 13. Let

$$\varphi(\alpha : \beta : \gamma) = p(\beta + \gamma) : q(\gamma + \alpha) : r(\alpha + \beta),$$

so that

$$\varphi^{-1}(\alpha : \beta : \gamma) = -\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} - \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} + \frac{\beta}{q} - \frac{\gamma}{r}.$$

In accord with (13), the cubic $\varphi(Z(U))$ has equation

$$\frac{u\alpha}{p}\left(-\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\beta}{q}-\frac{\gamma}{r}\right)+\frac{v\beta}{q}\left(\frac{\alpha}{p}-\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\gamma}{r}-\frac{\alpha}{p}\right)$$
$$+\quad\frac{w\gamma}{r}\left(\frac{\alpha}{p}+\frac{\beta}{q}-\frac{\gamma}{r}\right)\left(\frac{\alpha}{p}-\frac{\beta}{q}\right)=0.$$

Isogonic conjugate pairs on $Z(U)$ are carried as in Example 3 to P-Ceva conjugate pairs on $\varphi(Z(U))$. Indeed, each collinear triple X, U, X^{-1} is carried to a collinear triple, so that $\varphi(U)$ is a pivot for $\varphi(Z(U))$.

If $U = X_{20}$, so that $Z(U)$ is the Darboux cubic, then collinear triples X, X_3, $\tilde{X}$, where $\tilde{X}$ denotes the reflection of X in X_3, are carried to collinear triples $\varphi(X)$, $\varphi(X_3)$, $\varphi(\tilde{X})$, where $\varphi(\tilde{X})$ is the P-beth conjugate of X, as in Example 5.

Example 14. Continuing Example 13 with φ^{-1} in place of φ, the cubic $\varphi^{-1}(Z(U))$ is given by

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)+s(v,w,u,q,r,p,\beta,\gamma,\alpha)+s(w,u,v,r,p,q,\gamma,\alpha,\beta)=0,$$

where

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)=up(\beta+\gamma)(q^2(\gamma+\alpha)^2-r^2(\alpha+\beta)^2).$$

Collinear triples X, U, X^{-1} on $Z(U)$ yield collinear triples on $\varphi^{-1}(Z(U))$, so that $\varphi^{-1}(U)$ is a pivot for $\varphi^{-1}(Z(U))$. The point $\varphi^{-1}(X^{-1})$ is the P-aleph conjugate of X, as in Example 4.

On the Darboux cubic, collinear triples $X, X_3, \tilde{X}$, yield collinear triples $\varphi^{-1}(X)$, $\varphi^{-1}(X_3)$, $\varphi^{-1}(\tilde{X})$, this last point being the P-gimel conjugate of X, as in Example 6.

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