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Equilateral Chordal Triangles

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Abstract. When a circle intersects each of the sidelines of a triangle in two points, we can pair the intersections in such a way that three chords not along the sidelines bound a triangle, which we call a *chordal triangle*. In this paper we show that equilateral chordal triangles are homothetic to Morley's triangle, and identify all cases.

1. Chordal triangles

Let $T = ABC$ be the triangle of reference, and let a circle γ intersect side a in points B_a and C_a, side b in A_b and C_b and side c in A_c and B_c. The chords $a' = C_bB_c$, $b' = A_cC_a$ and $c' = A_bB_a$ enclose a triangle T', which we call a *chordal triangle*. See Figure 1. We begin with some preliminary results. In writing these the expression (ℓ_1, ℓ_2) denotes the directed angle from ℓ_1 to ℓ_2.

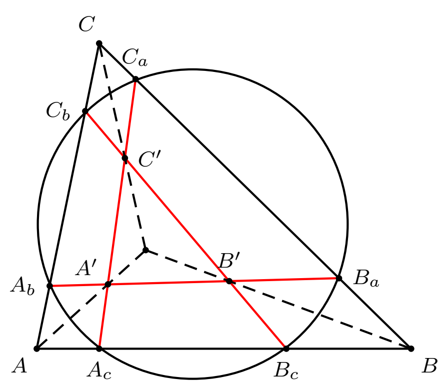

Figure 1

Proposition 1. *The sides of the chordal triangle T' satisfy*

$$(a', a) + (b', b) + (c', c) = 0 \pmod{\pi}.$$

Proof. Note that $(a', c) = (B_cC_b, B_cA)$ and

$$(c', b) = -(A_bA, A_bB_a) = (B_cB_a, B_cC_b) \pmod{\pi}$$

while also

$$(b', a) = -(C_aC, C_aA_c) = (B_cA_c, B_cB_a) \pmod{\pi}.$$

We conclude that $(a', c) + (c', b) + (b', a) = 0 \pmod{\pi}$, and the proposition follows from the fact that the internal directed angles of a triangle have sum π. □

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Proposition 2. *The triangle T' is perspective to ABC.*

Proof. From Pascal's hexagon theorem applied to $C_aB_aA_bC_bB_cA_c$ we see that the points of intersection $C_aB_a \cap C_bB_c$, $B_cA_c \cap B_aA_b$ and $A_bC_b \cap A_cC_a$ are collinear. Therefore, triangles ABC and $A'B'C'$ are line perspective, and by Desargues' two-triangle theorem, they are point perspective as well. □

The triangle T'' enclosed by the lines $a'' = (a \cap b') \cup (a' \cap b)$ and similarly defined b'' and c'' is also a chordal triangle, which we will call the *alternative chordal triangle* of T'. [1]

Proposition 3. *The corresponding sides of T' and T'' are antiparallel with respect to triangle T.*

Proof. From the fact that $B_cA_cA_bC_b$ is a cyclic quadrilateral, immediately we see $\angle AB_cC_b = \angle AA_bA_c$, so that a' and a'' are antiparallel. By symmetry this proves the proposition. □

We now see that there is a family of chordal triangles homothetic to T'. From a starting point on one of the sides of ABC we can construct segments to the next sides alternately parallel to corresponding sides of T' and T''.[2] Extending the segments parallel to T' we get a chordal triangle homothetic to T'. [3]

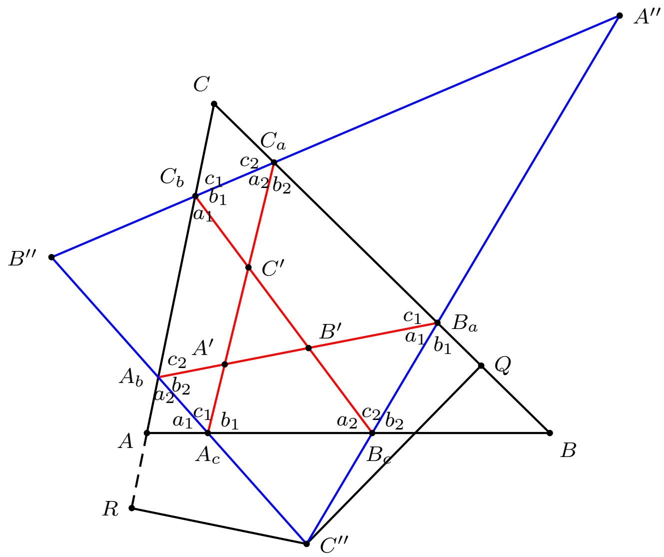

Figure 2

[1]This is the triangle enclosed by the lines A_bA_c, B_cB_a and C_aC_b in Figure 2. The definition of T'' from T and T' is exactly dual to the definition of 'desmic mate' (see [1, §4]). This yields also that T, T' and T'' are perspective through one perspector, which will be shown differently later this section, in order to keep this paper self contained.

[2]This is very similar to the well known construction of the Tucker hexagon.

[3]In fact it is easy to see that starting with any pair of triangles T' and T'' satisfying Propositions 1 and 3, we get a family of chordal triangles with this construction.

With the knowledge of Propositions 1-3 we can indicate angles as in Figure 2. In this figure we have also drawn altitudes $C''Q$ and $C''R$ to BC and AC respectively.

Note that

$$\begin{aligned} C''Q &= \sin(b_1)\sin(b_2)\csc(C'')\cdot A_bB_a, \\ C''R &= \sin(a_1)\sin(a_2)\csc(C'')\cdot A_bB_a. \end{aligned}$$

This shows that the (homogeneous) normal coordinates[4] for C'' are of the form

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \cdots).$$

From this we see that T and T'' have perspector

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \csc(c_1)\csc(c_2)).$$

Clearly this perspector is independent from choice of T' or T'', and depends only on the angles a_1, a_2, b_1, b_2, c_1 and c_2.

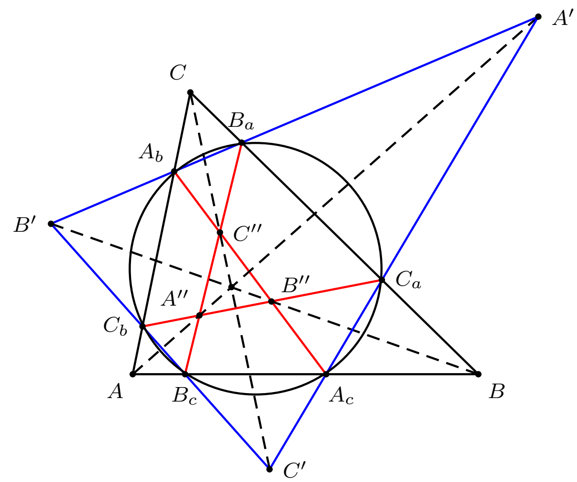

Figure 3

Proposition 4. *All chordal triangles homothetic to a chordal triangle T', as well as all chordal triangles homothetic to the alternative chordal triangle of T', are perspective to T through one perspector.*

2. Equilateral chordal triangles

Jean-Pierre Ehrmann and Bernard Gibert have given a magnificently elegant characterization of lines parallel to sides of Morley's trisector triangle.

Proposition 5. [2, Proposition 5] *A line ℓ is parallel to a side of Morley's trisector triangle if and only if*

$$(\ell, a) + (\ell, b) + (\ell, c) = 0 \pmod{\pi}.$$

[4]These are traditionally called (homogeneous) trilinear coordinates.

An interesting consequence of Proposition 5 in combination with Proposition 1 is that Morley triangles of chordal triangles are homothetic to the Morley triangle of ABC. Furthermore, equilateral chordal triangles themselves are homothetic to Morley's triangle. This means that they are not in general constructible by ruler and compass.

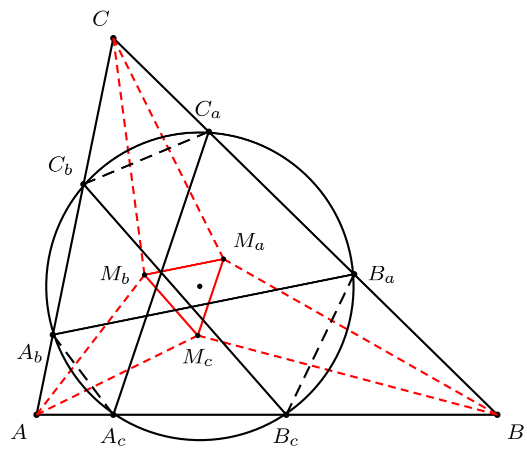

Figure 4

With this knowledge we can identify all equilateral chordal triangles. First we can specify the angles (a', c), (c', b), and (b', a). There are six possibilities. Now we can fix one point, say B_a, and use the specified angles and Proposition 3 to find the other vertices of hexagon $C_aB_aA_bC_bB_cA_c$. The rest is easy.

We can now identify all equilateral chordal triangles by (homogeneous) normal coordinates. As an example we will study the family

$$(a', c) = \frac{2}{3}B + \frac{1}{3}C, \quad (c', b) = \frac{2}{3}A + \frac{1}{3}B, \quad (b', a) = \frac{2}{3}C + \frac{1}{3}A.$$

From the derivation of Proposition 4 we see that the perspector of this family has normal coordinates

$$\left(\csc\frac{2B+C}{3}\csc\frac{B+2C}{3} : \csc\frac{2A+C}{3}\csc\frac{A+2C}{3} : \csc\frac{2A+B}{3}\csc\frac{A+2B}{3}\right).$$

Writing (TU) for the directed arc from T to U, and defining

$$\begin{array}{lll} t_a = (C_bB_c), & t_b = (A_cC_a), & t_c = (B_aA_b), \\ u_a = (C_aB_a), & u_b = (A_bC_b), & u_c = (B_cA_c), \end{array}$$

we find the following system of equations

$$\begin{array}{ll} (C_bA_c) = t_a + u_c = \frac{4}{3}B + \frac{2}{3}C, & (A_bB_c) = u_b + t_a = \frac{4}{3}C + \frac{2}{3}B, \\ (B_aC_b) = t_c + u_b = \frac{4}{3}A + \frac{2}{3}B, & (C_aA_b) = u_a + t_c = \frac{4}{3}B + \frac{2}{3}A, \\ (A_cB_a) = t_b + u_a = \frac{4}{3}C + \frac{2}{3}A, & (B_cC_a) = u_c + t_b = \frac{4}{3}A + \frac{2}{3}C. \end{array}$$

This system can be solved with one parameter τ to be

$$t_a = \frac{4(B+C)}{3} - 2\tau \quad t_b = \frac{4(C+A)}{3} - 2\tau \quad t_c = \frac{4(A+B)}{3} - 2\tau$$

$$u_a = -\frac{2A}{3} + 2\tau \qquad u_b = -\frac{2B}{3} + 2\tau \qquad u_c = -\frac{2C}{3} + 2\tau$$

The coordinates of the centers of these circles are now given by[5]

$$\left(\pm\cos\left(\frac{A}{3}+\tau\right) : \pm\cos\left(\frac{B}{3}+\tau\right) : \pm\cos\left(\frac{C}{3}+\tau\right)\right).$$

Assuming all cosines positive, these centers describe a line, which passes (take $\tau = 0$) through the perspector of the adjoint Morley triangle and ABC, in [3,4] numbered as X_{358}. By taking $\tau = \frac{\pi}{2}$ we see the line also passes through the point

$$\left(\sin\frac{A}{3} : \sin\frac{B}{3} : \sin\frac{C}{3}\right).$$

Hence, the equation of this line through the centers of the circles is

$$\sum_{\text{cyclic}}\left(\sin\frac{B}{3}\cos\frac{C}{3} - \cos\frac{B}{3}\sin\frac{C}{3}\right)x = 0,$$

or simply

$$\sum_{\text{cyclic}} \sin\frac{B-C}{3}x = 0.$$

One can find the other families of equilateral chordal triangles by adding and/or subtracting appropriate multiples of $\frac{\pi}{3}$ to the inclinations of the sides of T' with respect to T. The details are left to the reader.

References

[1] K. R. Dean and F. M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115–120.

[2] J.-P. Ehrmann and B. Gibert, A Morley configuration, *Forum Geom.*, 1 (2001) 51–58.

[3] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/˜ck6/encyclopedia/.

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[5] We have to be careful with this type of conclusion. We cannot blindly give signs to the coordinates. In particular, we cannot blindly follow the signs of the cosines below - if we would add 360 degrees to u_a, this would yield a change of sign for the first coordinate for the same figure. To establish signs, one can shuffle the hexagon $C_aB_aA_bC_bB_cA_c$ in such a way that the central angles on the segments on the sides are all positive and the sum of central angles is exactly 2π. From this we can draw conclusions on the location of the center with respect to the sides.