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The Napoleon Configuration

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Abstract. It is an elementary fact in triangle geometry that the two Napoleon triangles are equilateral and have the same centroid as the reference triangle. We recall some basic properties of the Fermat and Napoleon configurations, and use them to study equilateral triangles bounded by cevians. There are two families of such triangles, the triangles in the two families being oppositely oriented. The locus of the circumcenters of the triangles in each family is one of the two Napoleon circles, and the circumcircles of each family envelope a conchoid of a circle.

1. The Fermat-Napoleon configuration

Consider a reference triangle ABC, with side lengths a, b, c. Let F_a^+ be the point such that CBF_a^+ is equilateral with the same orientation as ABC; similarly for F_b^+ and F_c^+. See Figure 1.

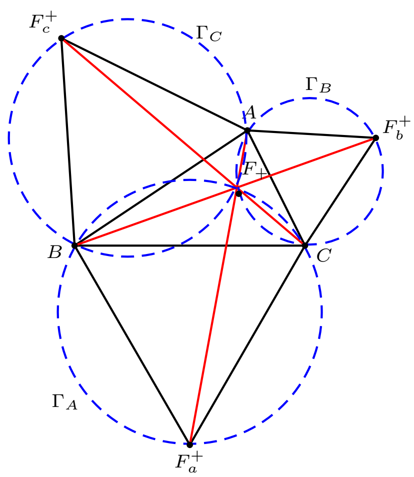

Figure 1. The Fermat configuration

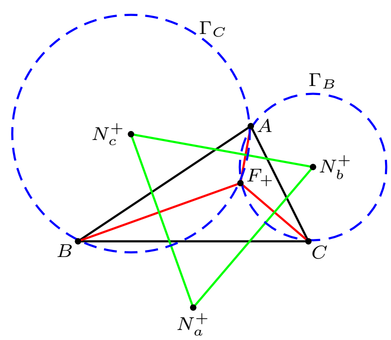

Figure 2. The Napoleon configuration

The triangle $F_a^+F_b^+F_c^+$ is called the *first Fermat triangle*. It is an elementary fact that triangles $F_a^+F_b^+F_c^+$ and ABC are perspective at the *first Fermat point* F_+. We define similarly the *second Fermat triangle* $F_a^-F_b^-F_c^-$ in which CBF_a^-, ACF_b^- and BAF_c^- are equilateral triangles with opposite orientation of ABC. This is perspective with ABC at the *second Fermat point* F_-. [1] Denote by Γ_A the circumcircle of CBF_a^+, and N_a^+ its center; similarly for Γ_B, Γ_C, N_b^+, N_c^+. The

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[1] In [1], these are called the isogonic centers, and are referenced as X_{13} and X_{14}.

triangle $N_a^+N_b^+N_c^+$ is called the *first Napoleon triangle*, and is perspective with ABC at the *first Napoleon point* N_+. Similarly, we define the *second Napoleon triangle* $N_a^-N_b^-N_c^-$ perspective with ABC at the *second Napoleon point* N_-. [2] See Figure 2. Note that N_a^- is the antipode of F_a^+ on Γ_A.

We summarize some of the important properties of the Fermat and Napoleon points.

Theorem 1. *Let ABC be a triangle with side lengths a, b, c and area $\triangle$.*

(1) *The first Fermat point F_+ is the common point to Γ_A, Γ_B and Γ_C.*
(2) *The segments AF_a^+, BF_b^+, CF_c^+ have the same length ℓ given by*

$$\ell^2 = \frac{1}{2}(a^2 + b^2 + c^2 + 4\sqrt{3}\triangle).$$

(3) *The first Napoleon triangle $N_a^+N_b^+N_c^+$ is equilateral with the same orientation as ABC. Its circumradius is $\frac{\ell}{3}$.*
(4) *The Fermat and Napoleon triangles have the same centroid G as ABC.*
(5) *The first Fermat point lies on the circumcircle of the second Napoleon triangle. We shall call this circle the* second Napoleon circle.
(6) *The lines $N_b^+N_c^+$ and AF_+ are respectively the line of centers and the common chord of Γ_B and Γ_C. They are perpendicular.*

Remarks. (i) Similar statements hold for the second Fermat and Napoleon points F_- and N_-, with appropriate changes of signs.

(ii) (4) is an easy corollary of the following important result: Given a triangle $A'B'C'$ with ABC', BCA', CAB' positively similar. Thus ABC and $A'B'C'$ have the same centroid. See, for example, [3, p.462].

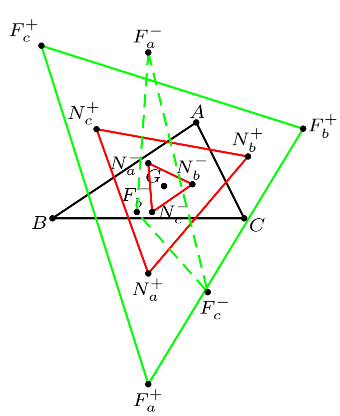

Figure 3. The Fermat and Napoleon triangles

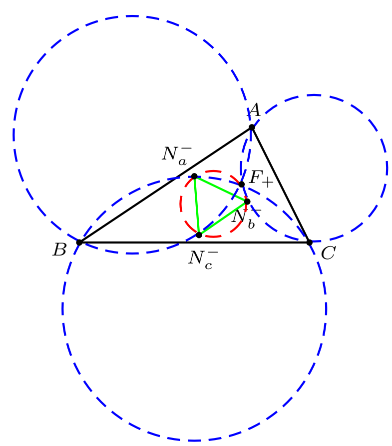

Figure 4. The Fermat point on the second Napoleon circle

[2] In [1], the Napoleon points as X_{17} and X_{18}.

2. Equilateral triangles bounded by cevians

Let $A_1B_1C_1$ be an equilateral triangle, with the same orientation as ABC and whose sides are cevian lines in ABC, i.e. A lies on B_1C_1, B lies on C_1A_1, C lies on A_1B_1. See Figure 5. Thus, CB is seen from A_1 at an angle $\frac{\pi}{3}$, *i.e.*, $\angle CA_1B = \frac{\pi}{3}$, and A_1 lies on Γ_A. Similarly B_1 lies on Γ_B and C_1 lies on Γ_C. Conversely, let A_1 be any point on Γ_A. The line A_1B intersects Γ_C at B and C_1, the line A_1C intersects Γ_B at C and B_1.

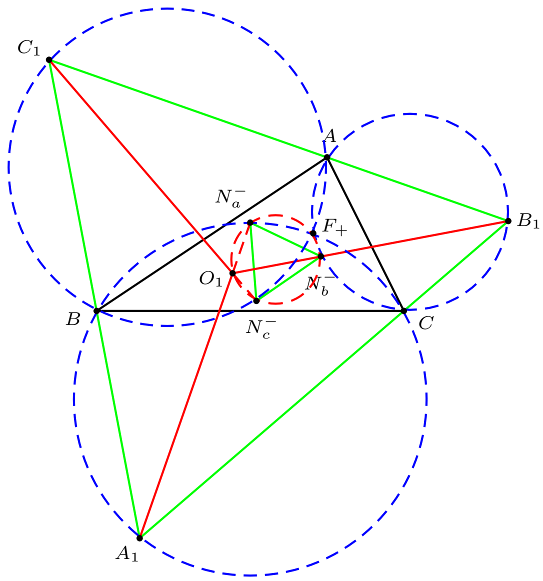

Figure 5. Equilateral triangle bounded by cevians

Three of the angles of the quadrilateral $A_1B_1AC_1$ are $\frac{\pi}{3}$; so A lies on B_1C_1 and $A_1B_1C_1$ is equilateral with the same orientation as ABC. We obtain an equilateral triangle bounded by cevians. There is an infinity of such triangles.

Let O_1 be the center of $A_1B_1C_1$. BO_1 is seen from A_1 at an angle $\frac{\pi}{6} \pmod{\pi}$; similarly for BN_a^-. The line A_1O_1 passes through N_a^-. Similarly the lines B_1O_1 and C_1O_1 pass through N_b^- and N_c^- respectively. It follows that $N_b^-N_c^-$ and B_1C_1 are seen from O_1 at the same angle $\frac{2\pi}{3} = -\frac{\pi}{3} \pmod{\pi}$, and the point O_1 lies on the circumcircle of $N_a^-N_b^-N_c^-$. Thus we have:

Theorem 2. *The locus of the center of equilateral triangles bounded by cevians, and with the same orientation as ABC, is the second Napoleon circle.*

Similarly, the locus of the center of equilateral triangles bounded by cevians, and with the opposite orientation of ABC, is the first Napoleon circle.

3. Pedal curves and conchoids

We recall the definitions of pedal curves and conchoids from [2].

Definitions. Given a curve $\mathcal{C}$ and a point O,

(1) the *pedal curve* of $\mathcal{C}$ with respect to O is the locus of the orthogonal projections of O on the tangent lines of $\mathcal{C}$;
(2) for a positive number k, the *conchoid* of $\mathcal{C}$ with respect to O and with *offset* k is the locus of the points P for which there exists M on $\mathcal{C}$ with O, M, P collinear and $MP = k$.

For the constructions of normal lines, we have

Theorem 3. *Let $\mathcal{P}_O$ be the pedal curve of $\mathcal{C}$ with respect to O. For any point M on $\mathcal{C}$, if P is the projection of O on the tangent to $\mathcal{C}$ at M, and Q is such that $OPMQ$ is a rectangle, then the line PQ is normal to $\mathcal{P}_O$ at P.*

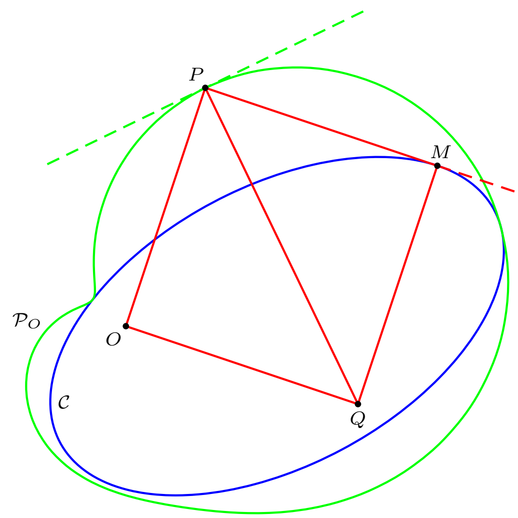

Figure 6. The normal to a pedal curve

Theorem 4. *Let $\mathcal{C}_{O,k}$ be the conchoid of $\mathcal{C}$ with respect to O and offset k. For any point P on $\mathcal{C}_{O,k}$, if M is the intersection of the line OP with $\mathcal{C}$, then the mormal lines to $\mathcal{C}_{O,k}$ at P and to $\mathcal{C}$ at M intersect on the perpendicular to OP at O.*

4. Envelope of the circumcircles

Consider one of the equilateral triangles with the same orientation of ABC. Let $\mathcal{C}_1$ be the circumcircle of $A_1B_1C_1$, R_1 its radius. Its center O_1 lies on the Napoleon circle and the vertex A_1 lies on the circle Γ_A. The latter two circles pass through F_+ and N_a^-. The angles $\angle N_a^- A_1 F_+$ and $\angle N_a^- O_1 F_+$ have constant magnitudes. The shape of triangle $A_1O_1F_+$ remains unchanged when O_1 traverses the second Napoleon circle $\mathcal{N}$. The ratio $\dfrac{O_1A_1}{O_1F_+} = \dfrac{R_1}{O_1F_+}$ remains constant, say, λ.

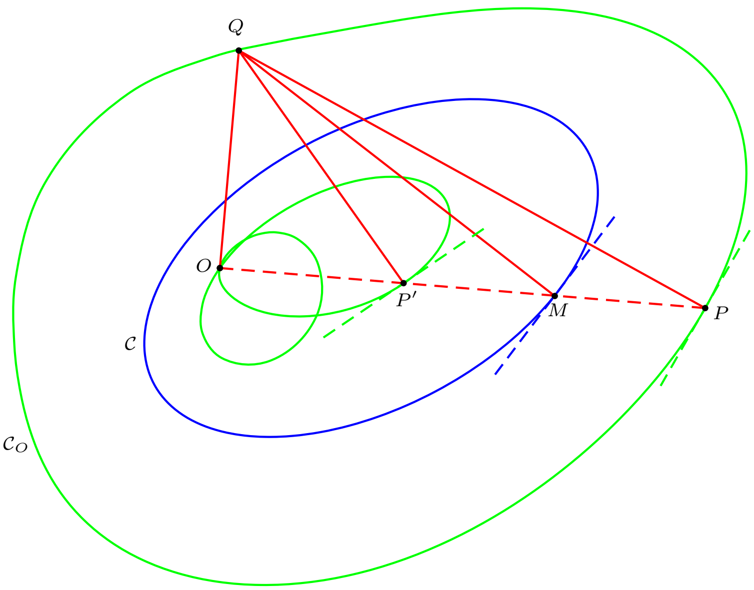

Figure 7. The normal to a conchoid

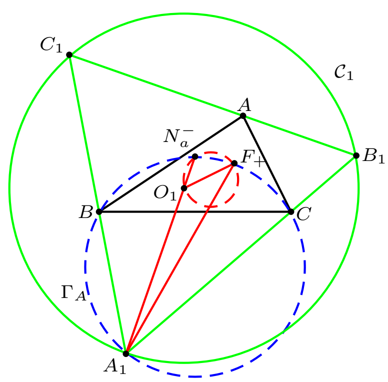

Figure 8. Equilateral triangle bounded by cevians and its circumcircle

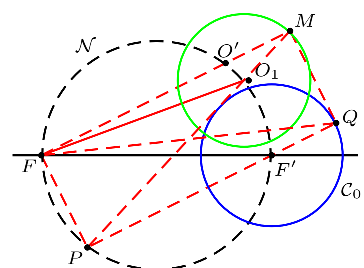

Figure 9. The pedal of $\mathcal{C}_0$ with respect to F

For convenience we denote by $\mathcal{N}$ the Napoleon circle which is the locus of O_1, F the Fermat point lying on this circle, F' the antipode of F on $\mathcal{N}$, and $\mathcal{C}_0$ the particular position of $\mathcal{C}_1$ when O_1 and F' coincide. See Figure 7. Let P be any point on $\mathcal{N}$, the line PF' intersects $\mathcal{C}_0$ at Q and Q' (F' between P and Q), we construct the point M such that $FPQM$ is a rectangle. The locus $\mathcal{P}_F$ of M is the pedal curve of $\mathcal{C}_0$ with respect to F and, by Theorem 3, the line MP is the normal to $\mathcal{P}_F$ at M. The line MP intersects $\mathcal{N}$ at P and O_1 and the circle through M with center O_1 is tangent to $\mathcal{P}_F$ at M.

The triangles FMO_1 and FQF' are similar since $\angle FMO_1 = \angle FQF'$ and $\angle FO_1M = \angle FF'Q$.[3] It follows that $\dfrac{O_1M}{O_1F} = \dfrac{F'Q}{F'F} = \lambda$, and $O_1M = R_1$. The circle through M with center O_1 is one in the family of circle for which we search the envelope.

Furthermore, the line FM intersects $\mathcal{N}$ at F and O', and $O'MQF'$ is a rectangle. Thus, $O'M = F'Q$, the radius of $\mathcal{C}_0$. It follows that M lies on the external branch of the conchoid of $\mathcal{N}$ with respect to F and the length $R =$ radius of $\mathcal{C}_0$.

By the same reasoning for the point Q', we obtain M' on $\mathcal{P}_F$, but on the internal branch of the conchoid. Each circle $\mathcal{C}_1$ touches both branches of the conchoid.

Theorem 5. *Let $\mathcal{F}$ be the family of circumcircles of equilateral triangles bounded by cevians whose locus of centers is the Napoleon circle $\mathcal{N}$ passing through the Fermat point F. The envelope of this family $\mathcal{F}$ is the pedal with respect to F of the circle $\mathcal{C}_0$ of $\mathcal{F}$ whose center is the antipode of F on $\mathcal{N}$, i.e. the conchoid of $\mathcal{N}$ with respect to F and offset the radius of $\mathcal{C}_0$. Each circle of $\mathcal{F}$ is bitangent to the envelope.*

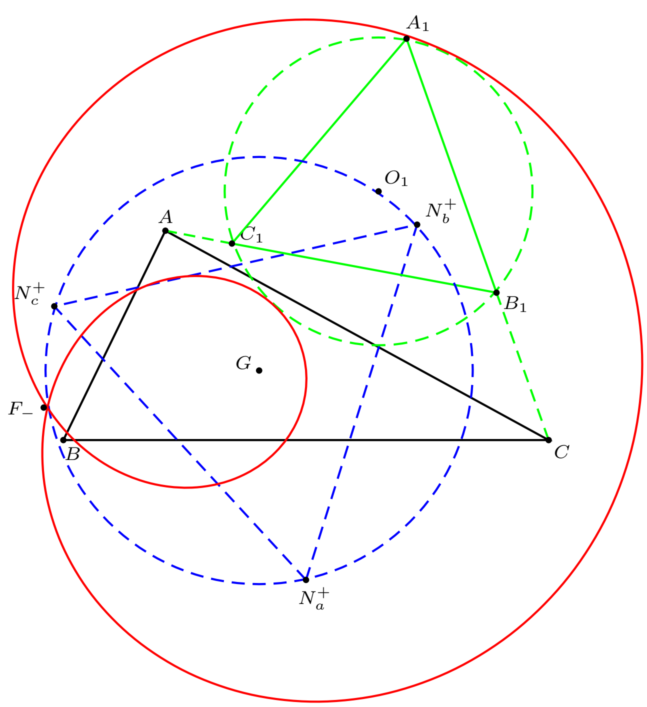

Figure 10. The envelope of the circumcirles ($\lambda < 1$)

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[3] FP is seen at the same angle from O_1 and from F'.

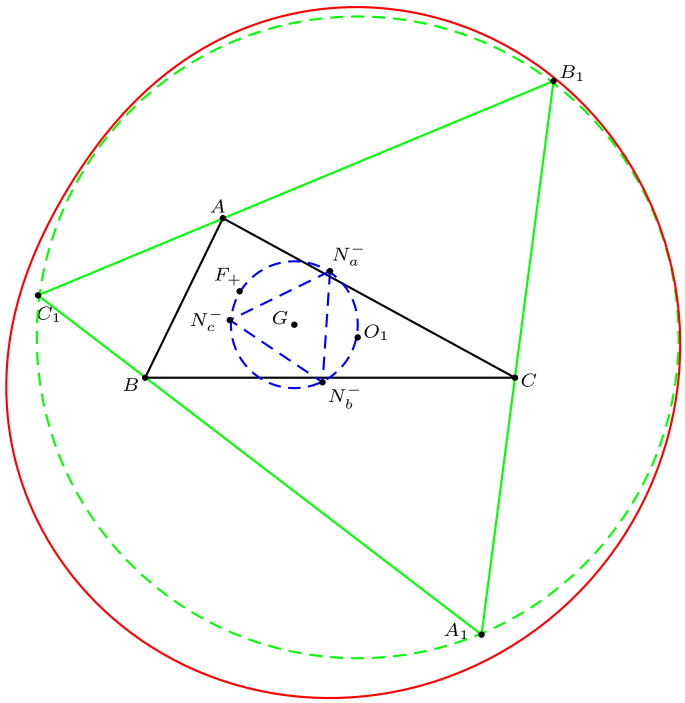

Figure 11. The envelope of the circumcirles ($\lambda > 1$)

Let i be the inversion with respect to a circle $\mathcal{C}$ whose center is F and such that $\mathcal{C}_0$ is invariant under it. The curve $i(\mathcal{P}_F)$ is the image of $\mathcal{C}_0$ by the reciprocal polar transformation with respect to $\mathcal{C}$, i.e., a conic with one focus at F. This conic is :

(1) a hyperbola for $\lambda < 1$ (F is exterior at $\mathcal{C}_0$) ;
(2) a parabola for $\lambda = 1$ (F lies on $\mathcal{C}_0$) ;
(3) an ellipse for $\lambda > 1$ (F is interior at $\mathcal{C}_0$).

So the envelope $\mathcal{P}$ of the circumcircles $\mathcal{C}_1$ is the inverse of this conic with respect to one of its foci, i.e., a conchoid of circle which is :

(1) a limaçon of Pascal for $\lambda < 1$: the hyperbola $i(\mathcal{P})$ as two asymptotes, so F is a node on $\mathcal{P}$;
(2) a cardioid for $\lambda = 1$: the parabola $i(\mathcal{P})$ is tangent to the line at infinity, so F is a cusp on $\mathcal{P}$;
(3) a curve without singularity for $\lambda > 1$: all points of the ellipse $i(\mathcal{P})$ are at finite distance.

We illustrate (1) and (3) in Figures 12 and 13. [4] It should be of great interest to see if always $\lambda > 1$ for F_+ (and < 1 for F_-). We think that the answer is affirmative, and that $\lambda = 1$ is possible if and only if A, B, C are collinear.

[4]Images of inversion of the limaçon of Pascal and the cardioid can also be found in the websites http://www-history.mcs.st-andrews.ac.uk/history/Curves and http://xahlee.org/SpecialPlaneCurves.

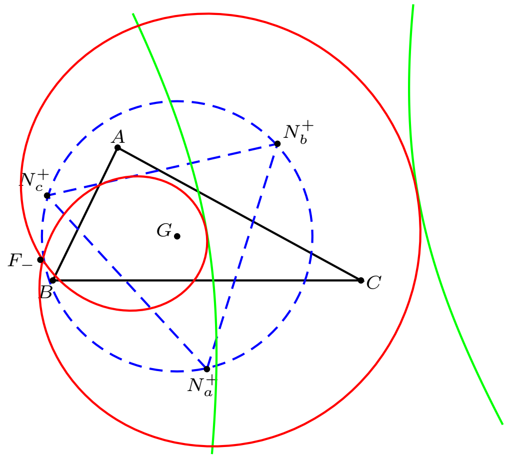

Figure 12. The inverse of the envelope ($\lambda < 1$)

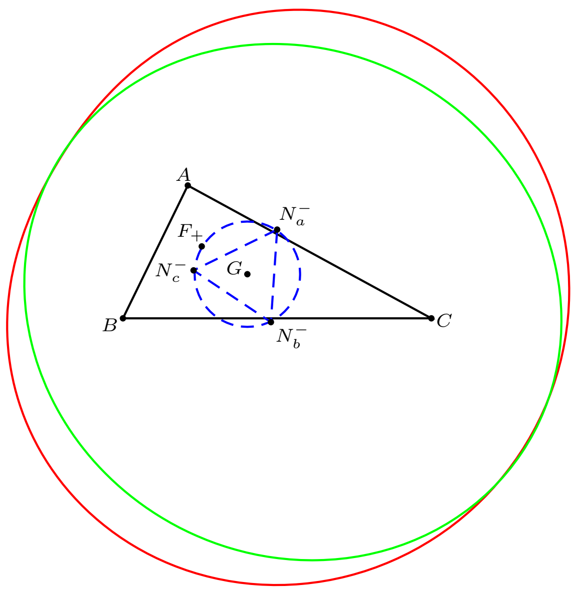

Figure 13. The inverse of the envelope ($\lambda > 1$)

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, http://www2.evansville.edu/ck6/encyclopedia/ (2000).

[2] J. D. Lawrence, *A Catalog of Special Plane Curves*, Dover, 1972.

[3] E. Rouché et Ch. de Comberousse, *Traité de Géométrie*, Gauthier-Villars, Paris, 6e edition, 1891.

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