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The Lemoine Cubic and Its Generalizations

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Abstract. For a given triangle, the Lemoine cubic is the locus of points whose cevian lines intersect the perpendicular bisectors of the corresponding sides of the triangle in three collinear points. We give some interesting geometric properties of the Lemoine cubic, and study a number of cubics related to it.

1. The Lemoine cubic and its constructions

In 1891, Lemoine published a note [5] in which he very briefly studied a cubic curve defined as follows. Let M be a point in the plane of triangle ABC. Denote by M_a the intersection of the line MA with the perpendicular bisector of BC and define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is the cubic in question. We shall denote this cubic by $\mathcal{K}(O)$, and follow Neuberg [8] in referring to it as the Lemoine cubic. Lemoine claimed that the circumcenter O of the reference triangle was a triple point of $\mathcal{K}(O)$. As pointed out in [7], this statement is false. The present paper considerably develops and generalizes Lemoine's note.

We use homogeneous barycentric coordinates, and adopt the notations of [4] for triangle centers. Since the second and third coordinates can be obtained from the first by cyclic permutations of a, b, c, we shall simply give the first coordinates. For convenience, we shall also write

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Thus, for example, the circumcenter is $X_3 = [a^2 S_A]$.

Figure 1 shows the Lemoine cubic $\mathcal{K}(O)$ passing through A, B, C, the orthocenter H, the midpoints A', B', C' of the sides of triangle ABC, the circumcenter O, and several other triangle centers such as $X_{32} = [a^4]$, $X_{56} = \left[\frac{a^2}{b+c-a}\right]$ and its extraversions. [1] Contrary to Lemoine's claim, the circumcenter is a node. When M traverses the cubic, the line $\mathcal{L}_M$ envelopes the Kiepert parabola with focus

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[1]The three extraversions of a point are each formed by changing in its homogeneous barycentric coordinates the signs of one of a, b, c. Thus, $X_{56a} = \left(\frac{a^2}{b+c+a} : \frac{b^2}{c-a-b} : \frac{c^2}{-a+b-c}\right)$, and similarly for X_{56b} and X_{56c}.

$F = X_{110} = \left[\frac{a^2}{b^2-c^2}\right]$ and directrix the Euler line. The equation of the Lemoine cubic is

$$\sum_{\text{cyclic}} a^4 S_A yz(y-z) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

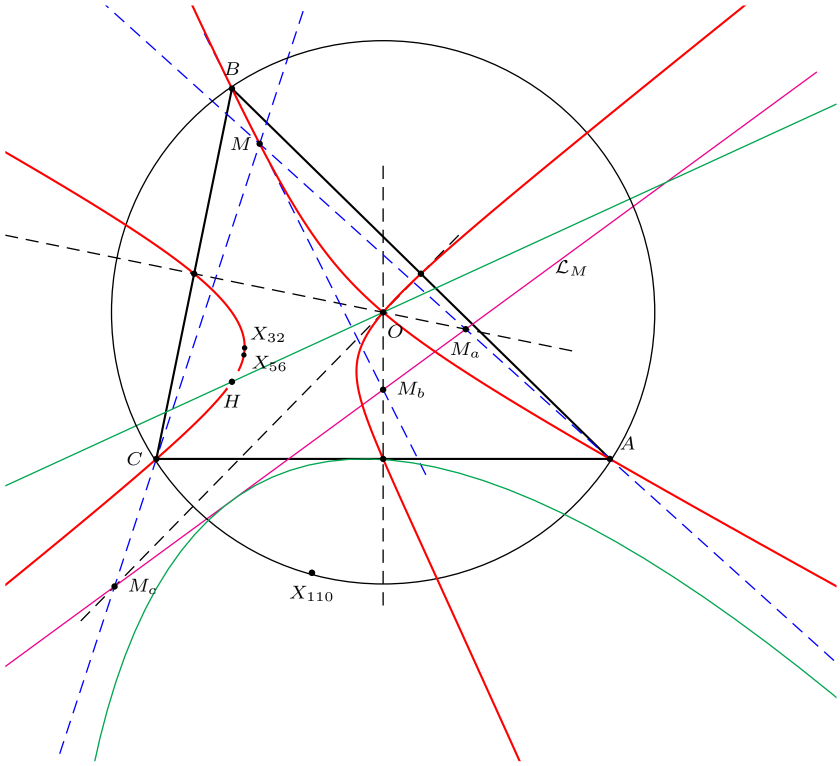

Figure 1. The Lemoine cubic with the Kiepert parabola

We give two equivalent constructions of the Lemoine cubic.

Construction 1. For any point Q on the line GK, the trilinear polar q of Q meets the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. [2] The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

For $Q = (a^2+t : b^2+t : c^2+t)$, this point of concurrency is

$$M = \left(\frac{a^2+t}{b^2c^2+(b^2+c^2-a^2)t} : \frac{b^2+t}{c^2a^2+(c^2+a^2-b^2)t} : \frac{c^2+t}{a^2b^2+(a^2+b^2-c^2)t}\right).$$

[2]The tripolar q envelopes the Kiepert parabola.

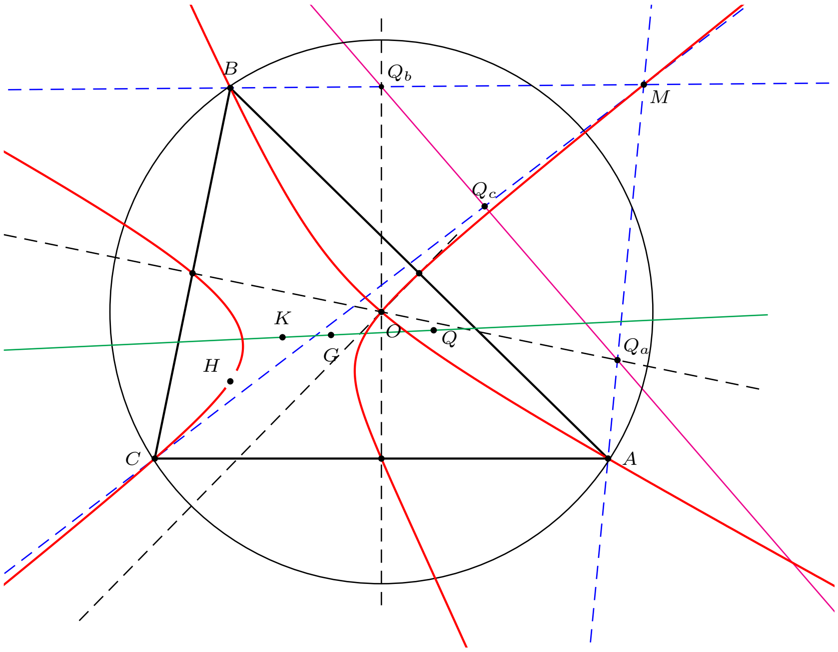

Figure 2. The Lemoine cubic as a locus of perspectors (Construction 1)

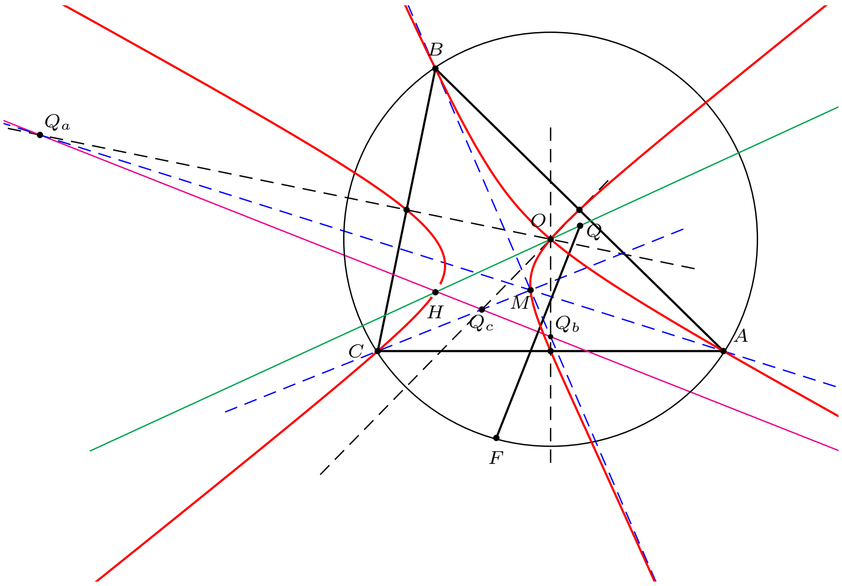

Figure 3. The Lemoine cubic as a locus of perspectors (Construction 2)

This gives a parametrization of the Lemoine cubic. This construction also yields the following points on $\mathcal{K}(O)$, all with very simple coordinates, and are not in [4].

i	$Q = X_i$	$M = M_i$
69	S_A	$\frac{S_A}{b^4+c^4-a^4}$
86	$\frac{1}{b+c}$	$\frac{1}{a(b+c)-(b^2+bc+c^2)}$
141	b^2+c^2	$\frac{b^2+c^2}{b^4+b^2c^2+c^4-a^4}$
193	$b^2+c^2-3a^2$	$S_A(b^2+c^2-3a^2)$

Construction 2. For any point Q on the Euler line, the perpendicular bisector of FQ intersects the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

See Figure 3 and Remark following Construction 4 on the construction of tangents to $\mathcal{K}(O)$.

2. Geometric properties of the Lemoine cubic

Proposition 1. *The Lemoine cubic has the following geometric properties.*

(1) *The two tangents at O are parallel to the asymptotes of the Jerabek hyperbola.*
(2) *The tangent at H passes through the center $X_{125} = [(b^2 - c^2)^2 S_A]$ of the Jerabek hyperbola.* [3]
(3) *The tangents at A, B, C concur at $X_{184} = [a^4 S_A]$, the inverse of X_{125} in the Brocard circle.*
(4) *The asymptotes are parallel to those of the orthocubic,* i.e., *the pivotal isogonal cubic with pivot H.*
(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(O)$ and the altitudes lie on the circle with diameter OH.* [4] *The triangles $A'B'C'$ and $H_AH_BH_C$ are perspective at a point*

$$Z_1 = \left[a^4 S_A(a^4 + b^4 + c^4 - 2a^2(b^2 + c^2))\right]$$

on the cubic. [5]
(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(O)$ and the sidelines of the medial triangle form a triangle perspective with $H_AH_BH_C$ at a point*

$$Z_2 = \left[\frac{a^4 S_A^2}{3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2}\right]$$

on the cubic. [6]
(7) *$\mathcal{K}(O)$ intersects the circumcircle of ABC at the vertices of the circumnormal triangle of ABC.* [7]

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[3] This is also tangent to the Jerabek hyperbola at H.

[4] In other words, these are the projections of O on the altitudes. The coordinates of H_A are

$$\left(\frac{2a^4 S_A}{a^2(b^2 + c^2) - (b^2 - c^2)^2} : S_C : S_B\right).$$

[5] Z_1 is the isogonal conjugate of X_{847}. It lies on a large number of lines, 13 using only triangle centers from [4], for example, X_2X_{54}, X_3X_{49}, X_4X_{110}, X_5X_{578}, $X_{24}X_{52}$ and others.

[6] This point Z_2 is not in the current edition of [4]. It lies on the lines X_3X_{64}, X_4X_{122} and $X_{95}X_{253}$.

[7] These are the points U, V, W on the circumcircle for which the lines UU^*, VV^*, WW^* (joining each point to its own isogonal conjugate) all pass through O. As such, they are, together with the vertices, the intersections of the circumcircle and the McCay cubic, the isogonal cubic with pivot the circumcenter O. See [3, p.166, §6.29].

We illustrate (1), (2), (3) in Figure 4, (4) in Figure 5, (5), (6) in Figure 6, and (7) in Figure 7 below.

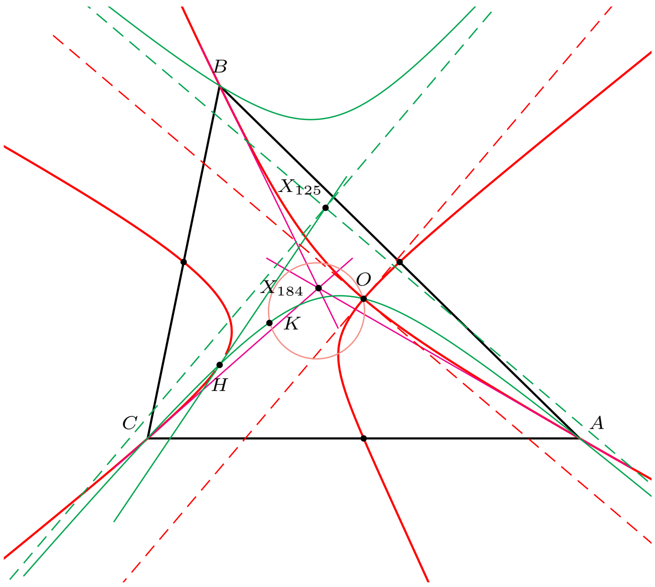

Figure 4. The tangents to the Lemoine cubic at O and the Jerabek hyperbola

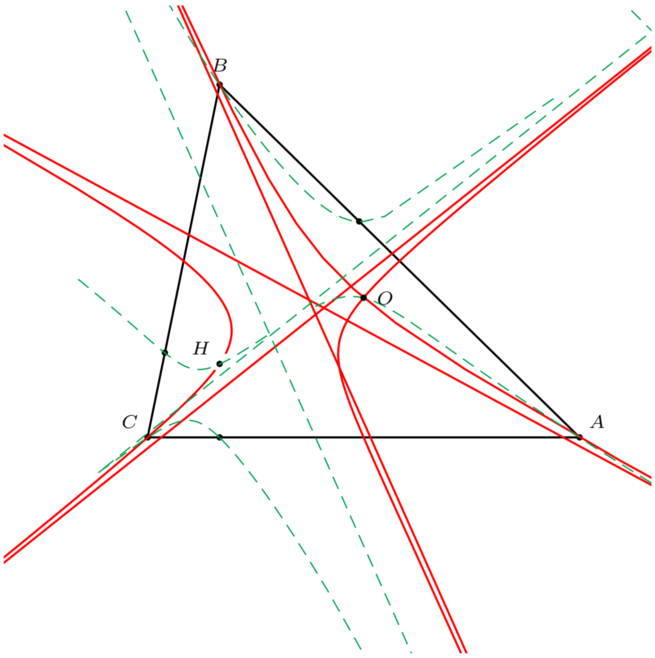

Figure 5. The Lemoine cubic and the orthocubic have parallel asymptotes

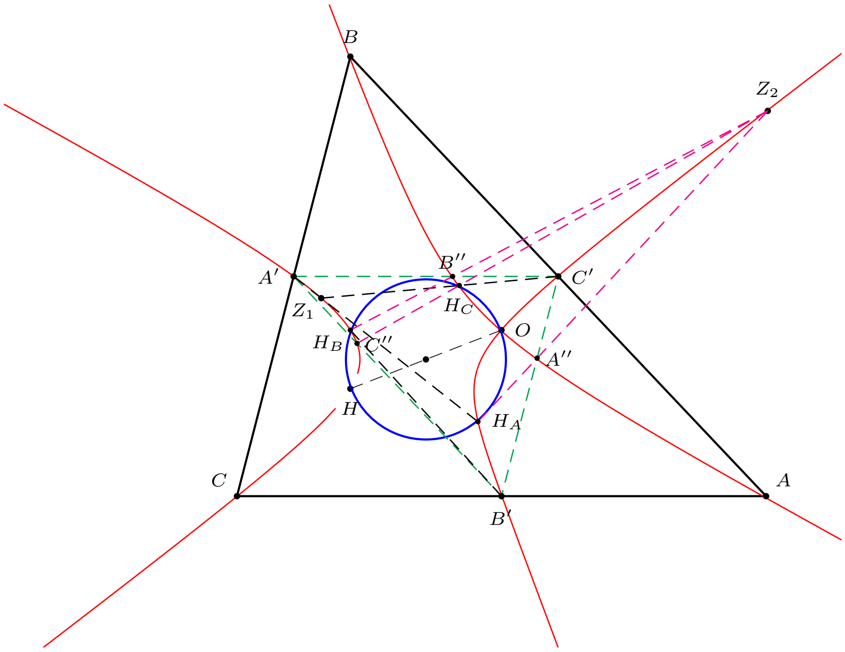

Figure 6. The perspectors Z_1 and Z_2

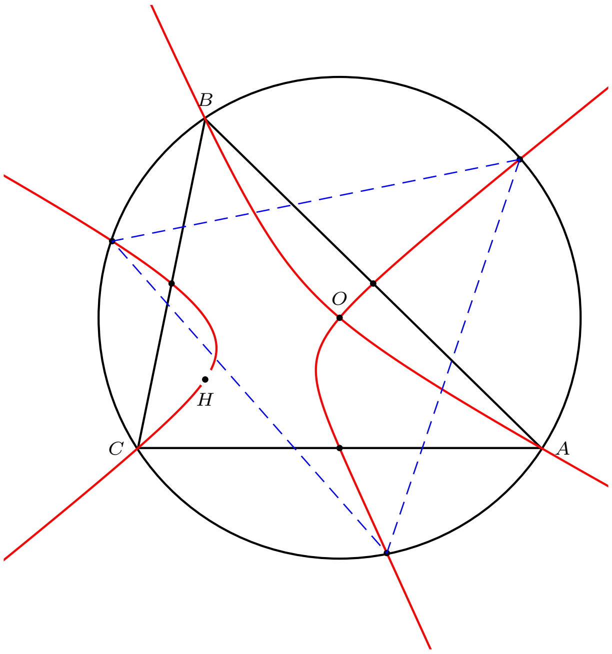

Figure 7. The Lemoine cubic with the circumnormal triangle

3. The generalized Lemoine cubic

Let P be a point distinct from H, not lying on any of the sidelines of triangle ABC. Consider its pedal triangle $P_aP_bP_c$. For every point M in the plane, let $M_a = PP_a \cap AM$. Define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is a cubic $\mathcal{K}(P)$ called the generalized Lemoine cubic associated with P. This cubic passes through A, B, C, H, P_a, P_b, P_c, and P which is a node. Moreover, the line $\mathcal{L}_M$ envelopes the inscribed parabola with directrix the line HP and focus F the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of the line HP. [8] The perspector S is the second intersection of the Steiner circum-ellipse with the line through F and the Steiner point $X_{99} = \left[\frac{1}{b^2-c^2}\right]$.

With $P = (p : q : r)$, the equation of $\mathcal{K}(P)$ is

$$\sum_{\text{cyclic}} x\left(r(c^2p + S_Br)y^2 - q(b^2p + S_Cq)z^2\right) + \left(\sum_{\text{cyclic}} a^2p(q-r)\right) xyz = 0.$$

The two constructions in §1 can easily be adapted to this more general situation.

Construction 3. For any point Q on the trilinear polar of S, the trilinear polar q of Q meets the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Construction 4. For any point Q on the line HP , the perpendicular bisector of FQ intersects the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Remark. The tangent at M to $\mathcal{K}(P)$ can be constructed as follows: the perpendicular at Q to the line HP intersects the perpendicular bisector of FQ at N, which is the point of tangency of the line through Q_a, Q_b, Q_c with the parabola. The tangent at M to $\mathcal{K}(P)$ is the tangent at M to the circum-conic through M and N. Given a point M on the cubic, first construct $M_a = AM \cap PP_a$ and $M_b = BM \cap PP_b$, then Q the reflection of F in the line M_aM_b, and finally apply the construction above.

Jean-Pierre Ehrmann has noticed that $\mathcal{K}(P)$ can be seen as the locus of point M such that the circum-conic passing through M and the infinite point of the line PM is a rectangular hyperbola. This property gives another very simple construction of $\mathcal{K}(P)$ or the construction of the "second" intersection of $\mathcal{K}(P)$ and any line through P.

Construction 5. A line ℓ_P through P intersects BC at P_1. The parallel to ℓ_P at A intersects HC at P_2. AB and P_1P_2 intersect at P_3. Finally, HP_3 intersects ℓ_P at M on the cubic $\mathcal{K}(P)$.

Most of the properties of the Lemoine cubic $\mathcal{K}(O)$ also hold for $\mathcal{K}(P)$ in general.

[8] Construction of F : draw the perpendicular at A to the line HP and reflect it about a bisector passing through A. This line meets the circumcircle at A and F.

Proposition 2. *Let $\mathcal{K}(P)$ be the generalized Lemoine cubic.*

(1) *The two tangents at P are parallel to the asymptotes of the rectangular circum-hyperbola passing through P.*
(2) *The tangent at H to $\mathcal{K}(P)$ is the tangent at H to the rectangular circum-hyperbola which is the isogonal image of the line OF. The asymptotes of this hyperbola are perpendicular and parallel to the line HP.*
(3) *The tangents at A, B, C concur if and only if P lies on the Darboux cubic.* [9]
(4) *The asymptotes are parallel to those of the pivotal isogonal cubic with pivot the anticomplement of P.*
(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(P)$ with the altitudes are on the circle with diameter HP. The triangles $P_aP_bP_c$ and $H_AH_BH_C$ are perspective at a point on $\mathcal{K}(P)$.* [10]
(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(P)$ and the sidelines of $P_aP_bP_c$ form a triangle perspective with $H_AH_BH_C$ at a point on the cubic.*

Remarks. (1) The tangent of $\mathcal{K}(P)$ at H passes through the center of the rectangular hyperbola through P if and only if P lies on the isogonal non-pivotal cubic $\mathcal{K}_H$

$$\sum_{\text{cyclic}} x\,(c^2y^2 + b^2z^2) - \Phi\, x\,y\,z = 0$$

where

$$\Phi = \frac{\sum_{\text{cyclic}} \left(2b^2c^2(a^4 + b^2c^2) - a^6(2b^2 + 2c^2 - a^2)\right)}{4S_AS_BS_C}.$$

We shall study this cubic in §6.3 below.

(2) The polar conic of P can be seen as a degenerate rectangular hyperbola. If $P \neq X_5$, the polar conic of a point is a rectangular hyperbola if and only if it lies on the line PX_5. From this, there is only one point (apart from P) on the curve whose polar conic is a rectangular hyperbola. Very obviously, the polar conic of H is a rectangular hyperbola if and only if P lies on the Euler line. If $P = X_5$, all the points in the plane have a polar conic which is a rectangular hyperbola. This very special situation is detailed in §4.2.

4. Special Lemoine cubics

4.1. *$\mathcal{K}(P)$ with concuring asymptotes.* The three asymptotes of $\mathcal{K}(P)$ are concurrent if and only if P lies on the cubic $\mathcal{K}_{conc}$

$$\sum_{\text{cyclic}} \left(S_B\left(c^2(a^2+b^2) - (a^2-b^2)^2\right)y - S_C\left(b^2(a^2+c^2) - (a^2-c^2)^2\right)z\right)x^2$$
$$- 2(a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

[9]The Darboux cubic is the isogonal cubic with pivot the de Longchamps point X_{20}.

[10]The coordinates of this point are $(p^2(-S_Ap + S_Bq + S_Cr) + a^2pqr : \cdots : \cdots)$.

The three asymptotes of $\mathcal{K}(P)$ are all real if and only if P lies inside the Steiner deltoid $\mathcal{H}_3$. [11] For example, the point $X_{76} = \left[\frac{1}{a^2}\right]$ lies on the cubic $\mathcal{K}_{conc}$ and inside the Steiner deltoid. The cubic $\mathcal{K}(X_{76})$ has three real asymptotes concurring at a point on X_5X_{76}. See Figure 8. On the other hand, the de Longchamps point X_{20} also lies on $\mathcal{K}_{conc}$, but it is not always inside $\mathcal{H}_3$. See Figure 10. The three asymptotes of $\mathcal{K}(X_{20})$, however, intersect at the real point X_{376}, the reflection of G in O.

We shall study the cubic $\mathcal{K}_{conc}$ in more detail in §6.1 below.

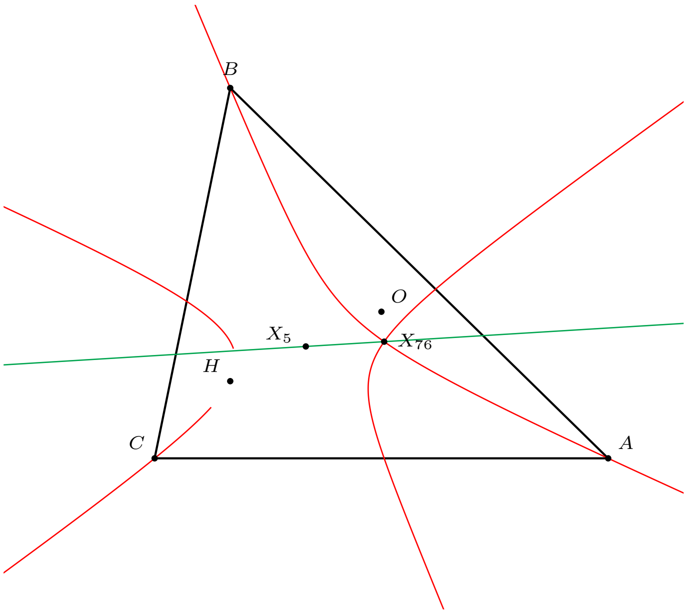

Figure 8. $\mathcal{K}(X_{76})$ with three concurring asymptotes

4.2. $\mathcal{K}(P)$ *with asymptotes making* $60°$ *angles with one another.* $\mathcal{K}(P)$ has three real asymptotes making $60°$ angles with one another if and only if P is the nine-point center X_5. See Figure 9. The asymptotes of $\mathcal{K}(X_5)$ are parallel again to those of the McCay cubic and their point of concurrence is [12]

$$Z_3 = \left[a^2((b^2-c^2)^2 - a^2(b^2+c^2))(a^4 - 2a^2(b^2+c^2) + b^4 - 5b^2c^2 + c^4)\right].$$

[11]Cf. Cundy and Parry [1] have shown that for a pivotal isogonal cubic with pivot P, the three asymptotes are all real if and only if P lies inside a certain "critical deltoid" which is the anticomplement of $\mathcal{H}_3$, or equivalently, the envelope of axes of inscribed parabolas.

[12]Z_3 is not in the current edition of [4]. It is the common point of several lines, e.g. X_5X_{51}, $X_{373}X_{549}$ and $X_{511}X_{547}$.

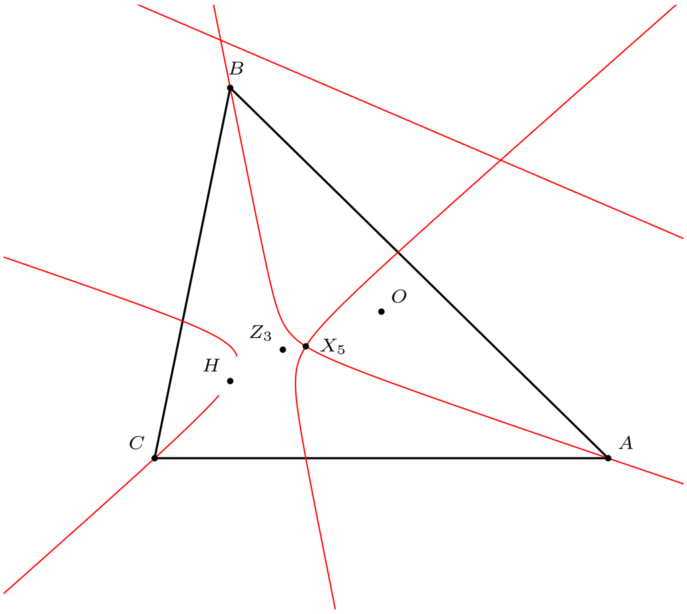

Figure 9. $\mathcal{K}(X_5)$ with three concurring asymptotes making 60° angles

4.3. *Generalized Lemoine isocubics.* $\mathcal{K}(P)$ is an isocubic if and only if the points P_a, P_b, P_c are collinear. It follows that P must lie on the circumcircle. The line through P_a, P_b, P_c is the Simson line of P and its trilinear pole R is the root of the cubic. When P traverses the circumcircle, R traverses the Simson cubic. See [2]. The cubic $\mathcal{K}(P)$ is a conico-pivotal isocubic: for any point M on the curve, its isoconjugate M^* (under the isoconjugation with fixed point P) lies on the curve and the line MM^* envelopes a conic. The points M and M^* are obtained from two points Q and Q' (see Construction 4) on the line HP which are inverse with respect to the circle centered at P going through F, focus of the parabola in §2. (see remark in §5 for more details)

5. The construction of nodal cubics

In §3, we have seen how to construct $\mathcal{K}(P)$ which is a special case of nodal cubic. More generally, we give a very simple construction valid for any nodal circum-cubic with a node at P, intersecting the sidelines again at any three points P_a, P_b, P_c. Let R_a be the trilinear pole of the line passing through the points $AB \cap PP_b$ and $AC \cap PP_c$. Similarly define R_b and R_c. These three points are collinear on a line $\mathcal{L}$ which is the trilinear polar of a point S. For any point Q on the line $\mathcal{L}$, the trilinear polar q of Q meets PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the sought cubic and, as usual, q envelopes the inscribed conic γ with perspector S.

Remarks. (1) The tangents at P to the cubic are those drawn from P to γ. These tangents are

(i) real and distinct when P is outside γ and is a "proper" node,
(ii) imaginary when P is inside γ and is an isolated point, or
(iii) identical when P lies on γ and is a cusp, the cuspidal tangent being the tangent at P to γ.

It can be seen that this situation occurs if and only if P lies on the cubic tangent at P_a, P_b, P_c to the sidelines of ABC and passing through the points $BC \cap P_bP_c$, $CA \cap P_cP_a$, $AB \cap P_aP_b$. In other words and generally speaking, there is no cuspidal circum-cubic with a cusp at P passing through P_a, P_b, P_c.

(2) When P_a, P_b, P_c are collinear on a line ℓ, the cubic becomes a conico-pivotal isocubic invariant under isoconjugation with fixed point P: for any point M on the curve, its isoconjugate M^* lies on the curve and the line MM^* envelopes the conic Γ inscribed in the anticevian triangle of P and in the triangle formed by the lines AP_a, BP_b, CP_c. The tangents at P to the cubic are tangent to both conics γ and Γ.

6. Some cubics related to $\mathcal{K}(P)$

6.1. *The cubic $\mathcal{K}_{conc}$* . The circumcubic $\mathcal{K}_{conc}$ considered in §4.1 above contains a large number of interesting points: the orthocenter H, the nine-point center X_5, the de Longchamps point X_{20}, X_{76}, the point

$$Z_4 = [a^2 S_A^2 (a^2(b^2 + c^2) - (b^2 - c^2)^2)]$$

which is the anticomplement of X_{389}, the center of the Taylor circle. [13] The cubic $\mathcal{K}_{conc}$ also contains the traces of X_{69} on the sidelines of ABC, the three cusps of the Steiner deltoid, and its contacts with the altitudes of triangle ABC. [14] Z is also the common point of the three lines each joining the trace of X_{69} on a sideline of ABC and the contact of the Steiner deltoid with the corresponding altitude. See Figure 10.

Proposition 3. *The cubic $\mathcal{K}_{conc}$ has the following properties.*

(1) *The tangents at A, B, C concur at X_{53}, the Lemoine point of the orthic triangle.*
(2) *The tangent at H is the line HK.*
(3) *The tangent at X_5 is the Euler line of the orthic triangle, the tangential being the point Z_4.* [15]
(4) *The asymptotes of $\mathcal{K}_{conc}$ are parallel to those of the McCay cubic and concur at a point* [16]

$$Z_5 = \left[a^2(a^2(b^2 + c^2) - (b^2 - c^2)^2)(2S_A^2 + b^2c^2)\right].$$

[13] The point Z_4 is therefore the center of the Taylor circle of the antimedial triangle. It lies on the line X_4X_{69}.

[14] The contact with the altitude AH is the reflection of its trace on BC about the midpoint of AH.

[15] This line also contains X_{51}, X_{52} and other points.

[16] Z_5 is not in the current edition of [4]. It is the common point of quite a number of lines, e.g. X_3X_{64}, X_5X_{51}, $X_{113}X_{127}$, $X_{128}X_{130}$, and $X_{140}X_{185}$. The three asymptotes of the McCay cubic are concurrent at the centroid G.

(5) *$\mathcal{K}_{conc}$ intersects the circumcircle at A, B, C and three other points which are the antipodes of the points whose Simson lines pass through X_{389}.*

We illustrate (1), (2), (3) in Figure 11, (4) in Figure 12, and (5) in Figure 13.

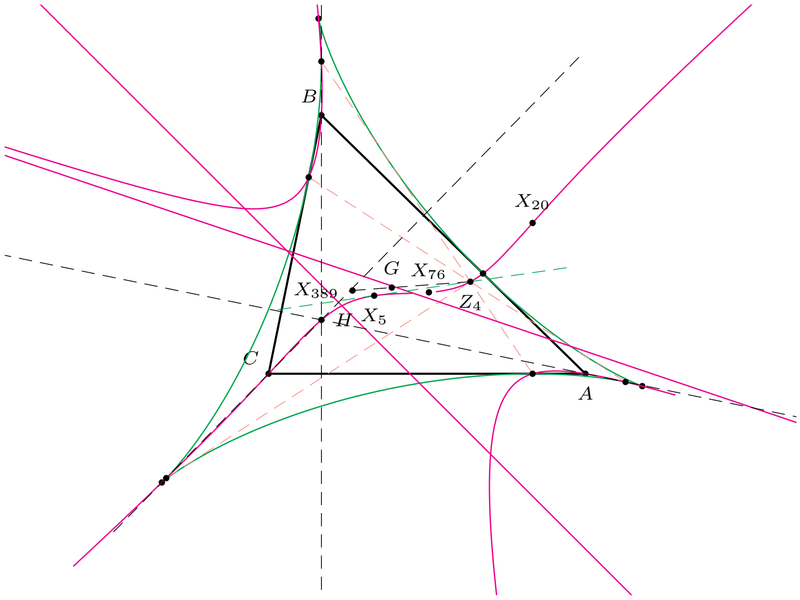

Figure 10. $\mathcal{K}_{conc}$ with the Steiner deltoid

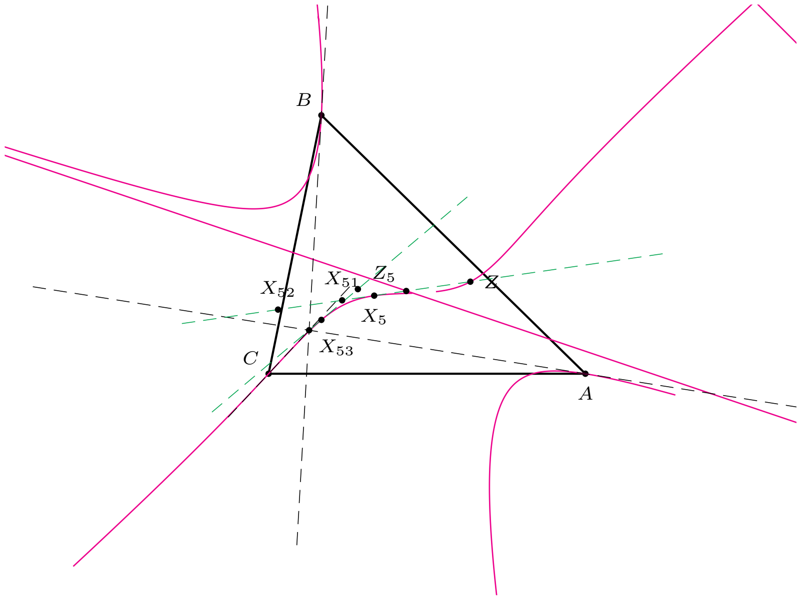

Figure 11. Tangents of $\mathcal{K}_{conc}$

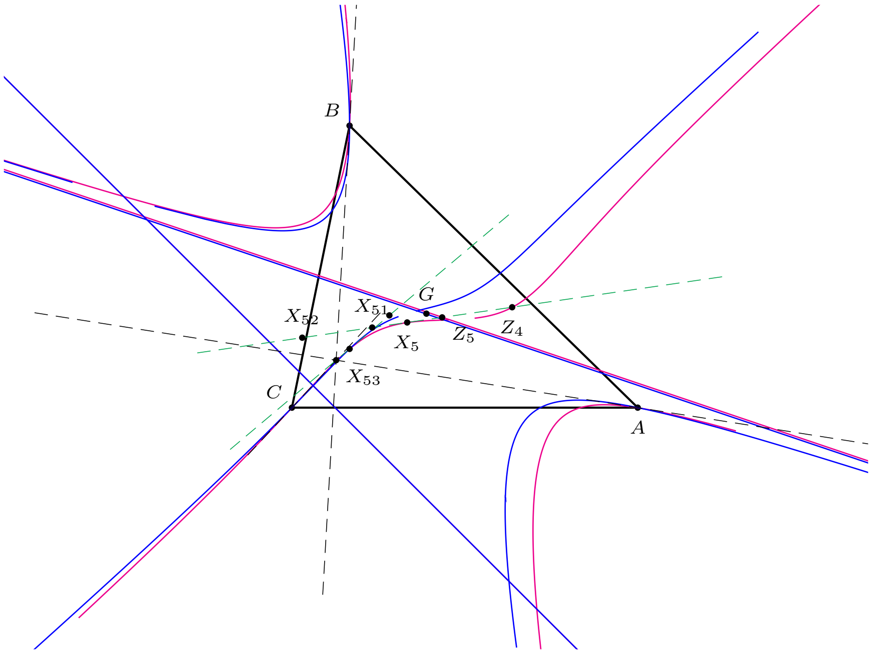

Figure 12. $\mathcal{K}_{conc}$ with the McCay cubic

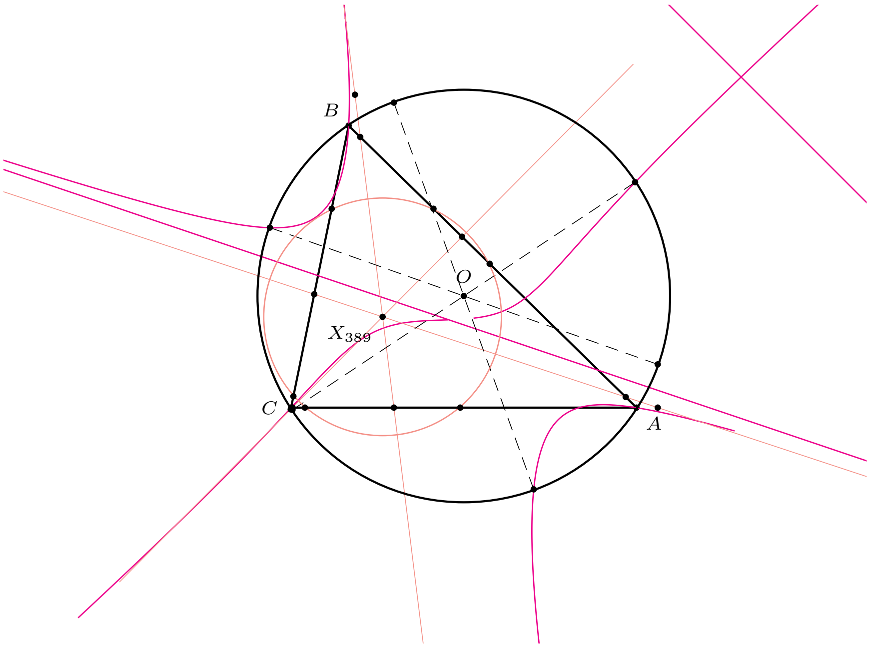

Figure 13. $\mathcal{K}_{conc}$ with the circumcircle and the Taylor circle

6.2. *The isogonal image of* $\mathcal{K}(O)$. Under isogonal conjugation, $\mathcal{K}(O)$ transforms into another nodal circum-cubic

$$\sum_{\text{cyclic}} b^2c^2\,x\,(S_By^2 - S_Cz^2) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

The node is the orthocenter H. The cubic also passes through O, X_8 (Nagel point) and its extraversions, X_{76}, $X_{847} = Z_1^*$, and the traces of $X_{264} = \left[\frac{1}{a^2 S_A}\right]$. The tangents at H are parallel to the asymptotes of the Stammler rectangular hyperbola[17]. The three asymptotes are concurrent at the midpoint of GH, [18] and are parallel to those of the McCay cubic.

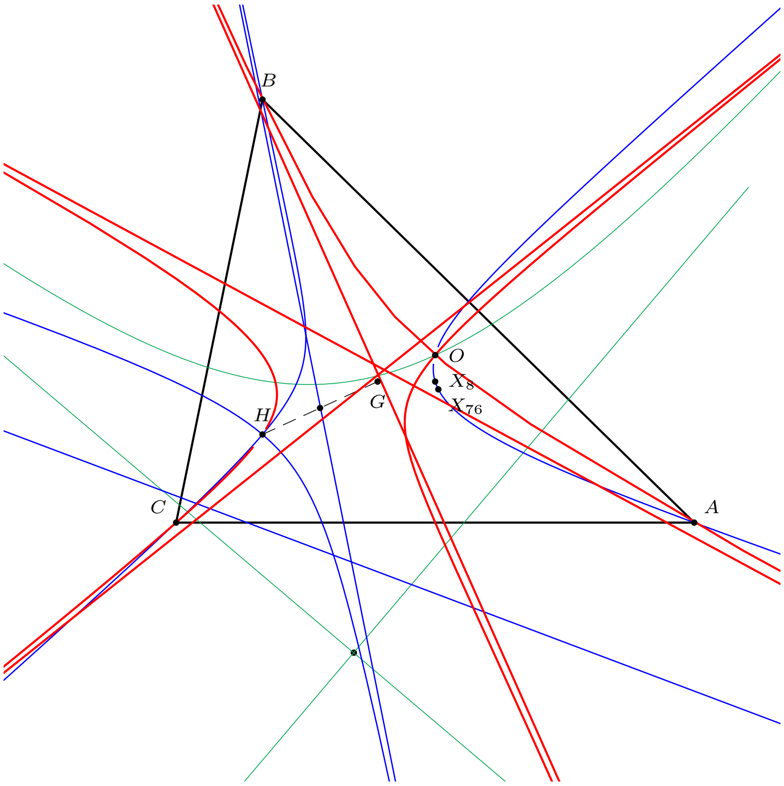

Figure 14. The Lemoine cubic and its isogonal

This cubic was already known by J. R. Musselman [6] although its description is totally different. We find it again in [9] in a different context. Let P be a point on the plane of triangle ABC, and P_1, P_2, P_3 the orthogonal projections of P on the perpendicular bisectors of BC, CA, AB respectively. The locus of P such that the triangle $P_1P_2P_3$ is in perspective with ABC is the Stammler hyperbola and the locus of the perspector is the cubic which is the isogonal transform of $\mathcal{K}(O)$. See Figure 15.

[17] The Stammler hyperbola is the rectangular hyperbola through the circumcenter, incenter, and the three excenters. Its asymptotes are parallel to the lines through X_{110} and the two intersections of the Euler line and the circumcircle

[18] This is $X_{381} = [a^2(a^2+b^2+c^2) - 2(b^2-c^2)^2]$.

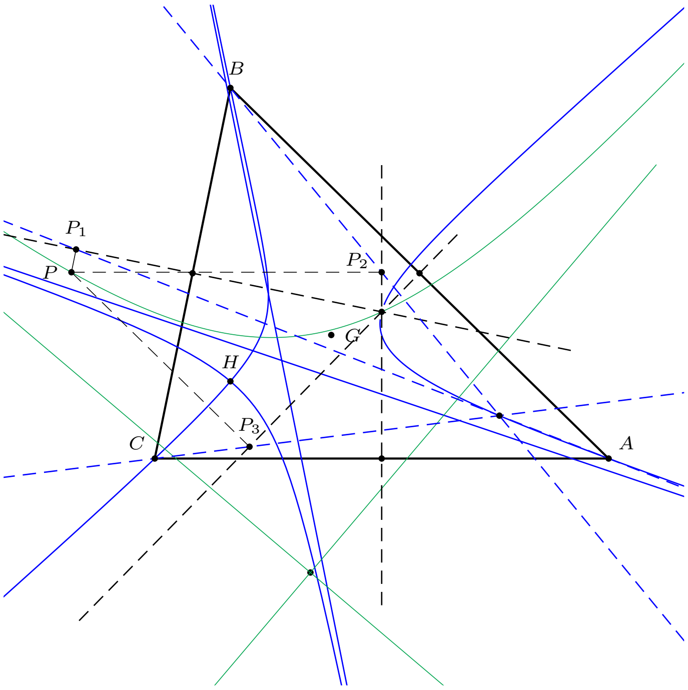

Figure 15. The isogonal of $\mathcal{K}(O)$ with the Stammler hyperbola

6.3. *The cubic $\mathcal{K}_H$.* Recall from Remark (1) following Proposition 2 that the tangent at H to $\mathcal{K}(P)$ passes through the center of the rectangular circum-hyperbola passing through P if and only if P lies on the cubic $\mathcal{K}_H$. This is a non-pivotal isogonal circum-cubic with root at G. See Figure 14.

Proposition 4. *The cubic $\mathcal{K}_H$ has the following geometric properties.*

(1) *$\mathcal{K}_H$ passes through A, B, C, O, H, the three points H_A, H_B, H_C and their isogonal conjugates H_A^*, H_B^*, H_C^*.* [19]
(2) *The three real asymptotes are parallel to the sidelines of ABC.*
(3) *The tangents of $\mathcal{K}_H$ at A, B, C are the sidelines of the tangential triangle. Hence, $\mathcal{K}_H$ is tritangent to the circumcircle at the vertices A, B, C.*
(4) *The tangent at A (respectively B, C) and the asymptote parallel to BC (respectively CA, AB) intersect at a point $\widetilde{A}$ (respectively $\widetilde{B}$, $\widetilde{C}$) on $\mathcal{K}_H$.*
(5) *The three points $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ are collinear on the perpendicular $\mathcal{L}$ to the line OK at the inverse of X_{389} in the circumcircle.*[20]

[19] The points H_A, H_B, H_C are on the circle, diameter OH. See Proposition 1(5). Their isogonal conjugates are on the lines OA. OB, OC respectively.

[20] In other words, the line $\mathcal{L}$ is the inversive image of the circle with diameter OX_{389}. Hence, $\widetilde{A}$ is the common point of $\mathcal{L}$ and the tangent at A to the circumcircle, and the parallel through $\widetilde{A}$ to BC is an asymptote of $\mathcal{K}_H$.

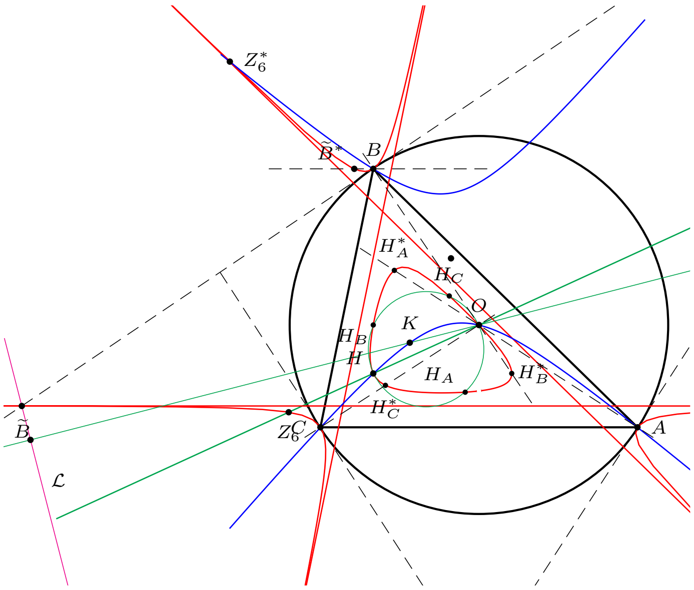

Figure 16. The cubic $\mathcal{K}_H$ with the Jerabek hyperbola

(6) *The isogonal conjugate of $\widetilde{A}$ is the "third" intersection of $\mathcal{K}_H$ with the parallel to BC through A; similarly for the isogonal conjugates of $\widetilde{B}$ and $\widetilde{C}$.*

(7) *The third intersection with the Euler line, apart from O and H, is the point* [21]

$$Z_6 = \left[\frac{(b^2-c^2)^2 + a^2(b^2+c^2-2a^2)}{(b^2-c^2)\,S_A}\right].$$

(8) *The isogonal conjugate of Z_6 is the sixth intersection of $\mathcal{K}_H$ with the Jerabek hyperbola.*

We conclude with another interesting property of the cubic $\mathcal{K}_H$. Recall that the polar circle of triangle ABC is the unique circle with respect to which triangle ABC is self-polar. This is in the coaxal system generated by the circumcircle and the nine-point circle. It has center H, radius ρ given by

$$\rho^2 = 4R^2 - \frac{1}{2}(a^2+b^2+c^2),$$

and is real only when triangle ABC is obtuse angled. Let $\mathcal{C}$ be the concentric circle with radius $\frac{\rho}{\sqrt{2}}$.

Proposition 5. *$\mathcal{K}_H$ is the locus of point M whose pedal circle is orthogonal to circle $\mathcal{C}$.*

[21]This is not in [4]. It is the homothetic of X_{402} (Gossard perspector) in the homothety with center G, ratio 4 or, equivalently, the anticomplement of the anticomplement of X_{402}.

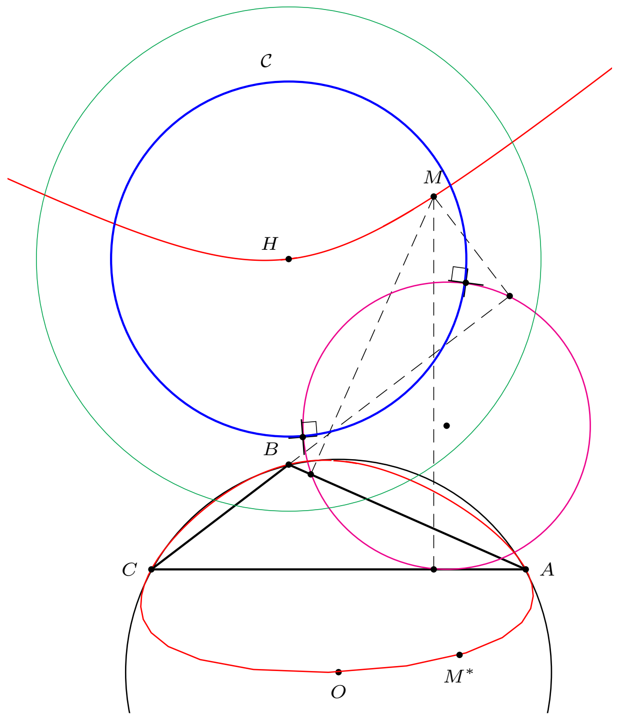

Figure 17. The cubic $\mathcal{K}_H$ for an obtuse angled triangle

In fact, more generally, every non-pivotal isogonal cubic can be seen, in a unique way, as the locus of point M such that the pedal circle of M is orthogonal to a fixed circle, real or imaginary, proper or degenerate.

References

[1] H. M. Cundy and C. F. Parry, Some cubic curves associated with a triangle, *Journal of geometry*, 53 (1995) 41–66.
[2] J.P. Ehrmann and B. Gibert, The Simson cubic, *Forum Geom.*, 1 (2001) 107 – 114.
[3] C. Kimberling, Triangle Centers and Central Triangles,*Congressus Numerantium*, 129 (1998) 1–295.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.
[5] E. Lemoine, *A. F.* (Association Française pour l'Avancement des Sciences) (1891) 149, and (1892) 124.
[6] J. R. Musselman, Some loci connected with a triangle, *Amer. Math. Monthly*, 47 (1940) pp. 354–361.
[7] J. Neuberg et A. Mineur, Sur la cubique de Lemoine, *Mathesis* 39 (1925) 64–65 .
[8] J. Neuberg, Sur les cubiques de Darboux, de Lemoine et de Thomson, *Annales Société Sc. Bruxelles* 44 (1925) 1–10.
[9] P. Yiu, Hyacinthos, message 1299, August 28, 2000.

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