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A Conic Through Six Triangle Centers

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Abstract. We show that there is a conic through the two Fermat points, the two Napoleon points, and the two isodynamic points of a triangle.

1. Introduction

It is always interesting when several significant triangle points lie on some sort of familiar curve. One recently found example is June Lester's circle, which passes through the circumcenter, nine-point center, and inner and outer Fermat (isogonic) points. See [8], also [6]. The purpose of this note is to demonstrate that there is a conic, apparently not previously known, which passes through six classical triangle centers.

Clark Kimberling's book [6] lists 400 centers and innumerable collineations among them as well as many conic sections and cubic curves passing through them. The list of centers has been vastly expanded and is now accessible on the internet [7]. Kimberling's definition of triangle center involves trilinear coordinates, and a full explanation would take us far afield. It is discussed both in his book and journal publications, which are readily available [4, 5, 6, 7]. Definitions of the Fermat (isogonic) points, isodynamic points, and Napoleon points, while generally known, are also found in the same references. For an easy construction of centers used in this note, we refer the reader to Evans [3]. Here we shall only require knowledge of certain collinearities involving these points. When points $X, Y, Z, \dots$ are collinear we write $\mathcal{L}(X, Y, Z, \dots)$ to indicate this and to denote their common line.

2. A conic through six centers

Theorem 1. *The inner and outer Fermat, isodynamic, and Napoleon points lie on a conic section.*

Proof. Let O denote the circumcenter of a triangle, H its orthocenter, and G its centroid. Denote the inner Fermat point by F_+, the inner isodynamic point by J_+, and the inner Napoleon point by N_+. Similarly denote the outer Fermat, isodynamic, and Napoleon points by F_-, J_-, and N_-.

Consider the hexagon whose vertices are F_+, N_+, J_+, F_-, N_-, and J_-. Kimberling lists many collineations of triangle centers which are readily verified when

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the centers are given in homogeneous trilinear coordinates. Within the list are these collinearities involving the sides of the hexagon and classical centers on the Euler line: $\mathcal{L}(H, N_+, J_+)$, $\mathcal{L}(H, N_-, J_-)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(G, J_+, F_-)$, and $\mathcal{L}(G, J_-, F_+)$. These six lines pass through opposite sides of the hexagon and concur in pairs at H, O, and G. But we know that H, O, and G are collinear, lying on the Euler line. So, by the converse of Pascal's theorem there is a conic section through the six vertices of the hexagon. □

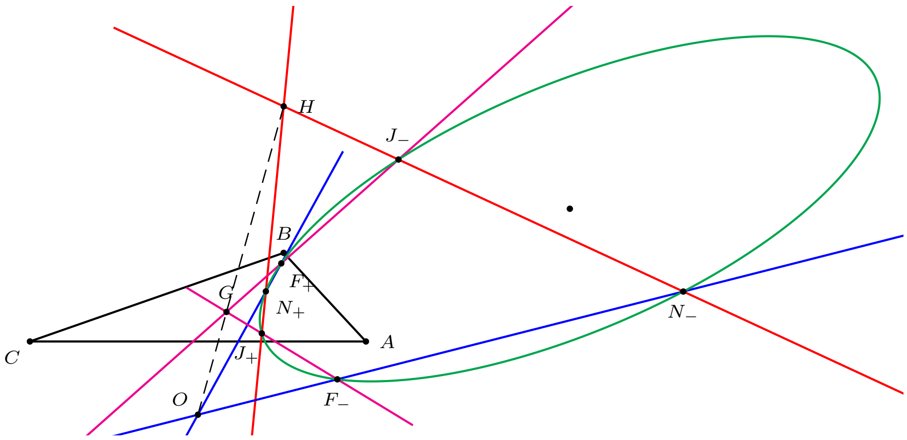

Figure 1. The conic through $F_\pm$, $N_\pm$ and $J_\pm$

Remark. In modern texts one sometimes sees Pascal's theorem stated as an "if and only if" theorem, omitting proper attribution for its converse, first proved independently by Braikenridge and by MacLaurin (See [2]). In the proof above, the Euler line plays the role of the Pascal line for the hexagon.

In Figure 1 the conic is shown as an ellipse, but it can also take the shape of a parabola or hyperbola. Since its announcement, several geometers have contributed knowledge about it. Peter Yff has calculated the equation of this conic [9], Paul Yiu has found criteria for it to be an ellipse, parabola, or a hyperbola [10], [1] and John H. Conway has generalized the conic [1].

3. Another conic

From Kimberling's list of collinearities, there is at least one more set of six points to which similar reasoning applies. We assume the reader is familiar with the concept of isogonal conjugate, fully explained in [6, 7].

Theorem 2. *The inner and outer Fermat (isogonic) and Napoleon points along with the isogonal conjugates of the Napoleon points all lie on a conic consisting of two lines intersecting at the center of the nine-point circle.*

[1]This conic is an ellipse, a parabola, or a hyperbola according as the Brocard angle is less than, equal to, or greater than $\arctan \frac{1}{3}$.

Proof. Denote the isogonal conjugates of the inner and outer Napoleon points by N_+^* and N_-^* respectively. Consider the hexagon with vertices F_+, F_-, N_+, N_-, N_+^*, and N_-^*. Kimberling lists these collinearities: $\mathcal{L}(G, N_+, N_-^*)$, $\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(H, F_+, N_+^*)$, $\mathcal{L}(H, F_-, N_-^*)$, so the converse of Pascal's theorem applies with the role of the Pascal line played by the Euler line, $\mathcal{L}(O, G, H)$. The conic is degenerate, consisting of two lines $L(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$, meeting at the nine-point center N_p . □

Second proof. The two collinearities $\mathcal{L}(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$ are in Kimberling's list, which *a fortiori* says that the six points in question lie on the degenerate conic consisting of the two lines. See Figure 2.

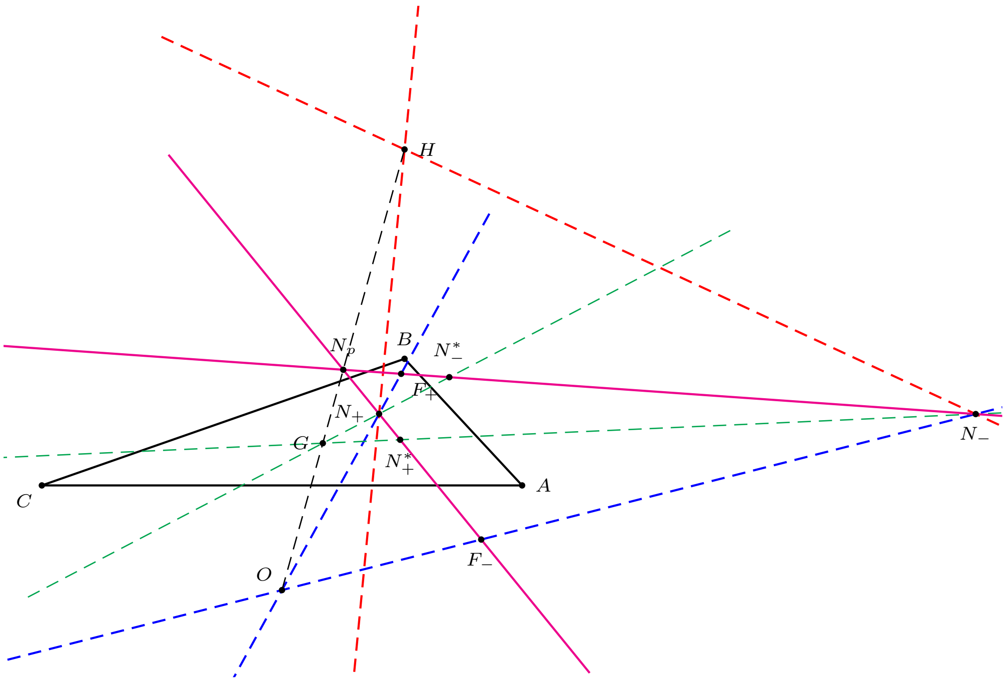

Figure 2. The degenerate conic through $F_\pm$, $N_\pm$ and $N_\pm^*$

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