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Paper-folding and Euler's Theorem Revisited

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Abstract. Given three points O, G, I, we give a simple construction by paper-folding for a triangle having these points as circumcenter, centroid, and incenter. If two further points H and N are defined by $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$, we prove that this procedure is successful if and only if I lies inside the circle on GH as diameter and differs from N. This locus for I is also independently derived from a famous paper of Euler, by complementing his calculations and properly discussing the reality of the roots of an algebraic equation of degree 3.

1. Introduction

The so-called *Modern Geometry of the Triangle* can be said to have been founded by Leonhard Euler in 1765, when his article [2] entitled *Easy Solution to some Very Difficult Geometrical Problems* was published in St. Petersburg. In this famous paper the distances between the main notable points of the triangle (centroid G, circumcenter O, orthocenter H, incenter I) are calculated in terms of the side lengths, so that several relationships regarding their mutual positions can be established. Among Euler's results, two have become very popular and officially bear his name: the vector equation $\mathbf{OH} = 3\mathbf{OG}$, implying the collinearity of G, O, H on the Euler line, and the scalar equation $OI^2 = R(R - 2r)$ involving the radii of the circumcircle and the incircle. Less attention has been given to the last part of the paper, though it deals with the problem Euler seems most proud to have solved in a very convenient [1] way, namely, the "determination of the triangle" from its points O, G, H, I. If one wants to avoid the "tedious calculations" which had previously prevented many geometricians from success, says Euler in his introduction, "everything comes down to choosing proper quantities". This understatement hides Euler's masterly use of symmetric polynomials, for which he adopts a cleverly chosen basis and performs complicated algebraic manipulations.

A modern reader, while admiring Euler's far-sightedness and skills, may dare add a few critical comments:

(1) Euler's §31 is inspired by the correct intuition that, given O, G, H, the location of I cannot be free. In fact he establishes the proper algebraic conditions but does not tell what they geometrically imply, namely that I must always lie inside the circle on GH as diameter. Also, a trivial mistake

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[1]Latin: commodissime.

leads Euler to a false conclusion; his late editor's formal correction[2] does not lead any further.

(2) As for the determination of the triangle, Euler reduces the problem of finding the side lengths to solving an algebraic equation of degree 3. However, no attention is given to the crucial requirements that the three roots - in order to be side lengths - be real positive and the triangle inequalities hold. On the other hand, Euler's equation clearly suggests to a modern reader that the problem cannot be solved by ruler and compass.

(3) In Euler's words (§20) the main problem is described as follows: *Given the positions of the four points ..., to construct the triangle*. But finding the side lengths does not imply determining the location of the triangle, given that of its notable points. The word *construct* also seems improperly used, as this term's traditional meaning does not include solving an algebraic equation. It should rather refer, if not to ruler and compass, to some alternative geometrical techniques.

The problem of the locus of the incenter (and the excenters) has been independently settled by Andrew P. Guinand in 1982, who proved in his nice paper [5] that I must lie inside the critical circle on GH as diameter [3] (Theorem 1) and, conversely, any point inside this circle - with a single exception - is eligible for I (Theorem 4). In his introduction, Guinand does mention Euler's paper, but he must have overlooked its final section, as he claims that in all previous researches "the triangle was regarded as given and the properties of the centers were investigated" while in his approach "the process is reversed".

In this paper we give an alternative treatment of Euler's problem, which is independent both of Euler's and Guinand's arguments. Euler's crucial equation, as we said, involves the side lengths, while Guinand discusses the cosines of the angles. We deal, instead, with the coefficients for equations of the sides. But an independent interest in our approach may be found in the role played by the Euler point of the triangle, a less familiar notable point. [4] Its properties are particularly suitable for reflections and suggest a most natural paper-folding reconstruction procedure. Thus, while the first part (locus) of the following theorem is well-known, the construction mentioned in the last statement is new:

Theorem 1. *Let O, G, I be three distinct points. Define two more points H, N on the line OG by letting $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$. Then there exists a nondegenerate, nonequilateral triangle $\mathcal{T}$ with centroid G, circumcenter O, orthocenter H, and incenter I, if and only if I lies inside the circle on GH as diameter and differs from N. In this case the triangle $\mathcal{T}$ is unique and can be reconstructed by paper-folding, starting with the points O, G, I.*

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[2] A. Speiser in [2, p.155, footnote].

[3] This is also known as the orthocentroidal circle. See [7]. This term is also used by Varilly in [9]. The author thanks the referee for pointing out this paper also treats this subject.

[4] This point is the focus of the Kiepert parabola, indexed as X_{110} in [7], where the notable points of a triangle are called triangle centers.

We shall find the sides of the triangle sides as proper creases, *i.e.*, reflecting lines, which simultaneously superimpose two given points onto two given lines. This can be seen as constructing the common tangents to two parabolas, whose foci and directrices are given. Indeed, the extra power of paper-folding, with respect to ruler-and-compass, consists in the feasibility of constructing such lines. See [4, 8].

The reconstruction of a triangle from three of its points (e.g. one vertex, the foot of an altitude and the centroid G) is the subject of an article of William Wernick [10], who in 1982 listed 139 triplets, among which 41 corresponded to problems still unsolved. Our procedure solves items 73, 80, and 121 of the list, which are obviously equivalent. [5] It would not be difficult to make slight changes in our arguments in order to deal with one of the excenters in the role of the incenter I.

As far as we know, paper-folding, which has been successfully applied to trisecting an angle and constructing regular polygons, has never yet produced any significant contribution to the geometry of the triangle.

This paper is structured as follows: in §2 we reformulate the well-known properties of the Simson line of a triangle in terms of side reflections and apply them to paper-folding. In §3 we introduce the Euler point E and study its properties. The relative positions of E, O, G are described by analytic geometry. This enables us to establish the locus of E and a necessary and sufficient condition for the existence of the triangle. [6] An immediate paper-folding construction of the triangle from E, O, G is then illustrated. In §4 we use complex variables to relate points E and I. In §5 a detailed ruler-and-compass construction of E from I, O, G is described.[7] The expected incenter locus is proved in §6 by reducing the problem to the former results on E, so that the proof of Theorem 1 is complete. In §7 we take up Euler's standpoint and interpret his formulas to find once more the critical circle locus as a necessary condition. Finally, we discuss the discriminant of Euler's equation and complete his arguments by supplying the missing algebraic calculations which imply sufficiency. Thus a third, independent, proof of the first part of Theorem 1 is achieved.

2. Simson lines and reflections

In this section we shall reformulate well-known results on the Simson line in terms of reflections, so that applications to paper-folding constructions will be natural. The following formulation was suggested by a paper of Longuet-Higgins [6].

Theorem 2. *Let H be the orthocenter, $\mathcal{C}$ the circumcircle of a triangle $\mathcal{T} = A_1A_2A_3$.*

(i) *For any point P, let P_i denote the reflection of P across the side A_jA_h of $\mathcal{T}$. (Here, i, j, h is a permutation of 1, 2, 3). Then the points P_i are collinear on a line $r = r(P)$ if and only if P lies on $\mathcal{C}$. In this case H lies on r.*

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[5] Given I and two of O, G, H.

[6] Here too, as in the other approaches, the discussion amounts to evaluating the sign of a discriminant.

[7] A ruler and compass construction always entails a paper-folding construction. See [4, 8].

(ii) *For any line r, let r_i denote the reflection of r across the side A_jA_h. Then the lines r_i are concurrent at a point $P = P(r)$ if and only if H lies on r. In this case P lies on $\mathcal{C}$. When P describes an arc of angle α on $\mathcal{C}$, $r(P)$ rotates in the opposite direction around H by an angle $-\frac{\alpha}{2}$.*

All these statements are easy consequences of well-known properties of the Simson line, which is obviously parallel to $r(P)$. See, for example, [1, Theorems 2.5.1, 2.7.1,2]. This theorem defines a bijective mapping $P \mapsto r(P)$. Thus, given any line e through H, there exists a unique point E on $\mathcal{C}$ such that $r(E) = e$.

We now recall the basic assumption of paper-folding constructions, namely the possibility of determining a line, *i.e.*, folding a crease, which simultaneously reflects two given points A, B onto points which lie on two given lines a, b. It is proved in [4, 8] that this problem has either one or three solutions. We shall discuss later how these two cases can be distinguished, depending on the relative positions of the given points and lines. For the time being, we are interested in the case that three such lines (creases) are found. The following result is a direct consequence of Theorem 2.

Corollary 3. *Given two points A, B and two (nonparallel) lines a, b, assume that there exist three different lines r such that A (respectively B) is reflected across r onto a point A' (respectively B') lying on a (respectively b). These lines are the sides of a triangle $\mathcal{T}$ such that*

(i) *a and b intersect at the orthocenter H of $\mathcal{T}$;*
(ii) *A and B lie on the circumcircle of $\mathcal{T}$;*
(iii) *the directed angle $\angle AOB$ is twice the directed angle from b to a. Here, O denotes the circumcenter of $\mathcal{T}$.*

3. The Euler point

We shall now consider a notable point whose behaviour under reflections makes it especially suitable for paper-folding applications. The Euler point E is the unique point which is reflected across the three sides of the triangle onto the Euler line OG. Equivalently, the three reflections of the Euler line across the sides are concurrent at E. [8]

We first prove that for any nonequilateral, nondegenerate triangle with prescribed O and G (hence also H), the Euler point E lies outside a region whose boundary is a cardioid, a closed algebraic curve of degree 4, which is symmetric with respect to the Euler line and has the centroid G as a double-point (a cusp; see Figure 3). If we choose cartesian coordinates such that such that $G = (0, 0)$ and $O = (-1, 0)$ (so that $H = (2, 0)$), then this curve is represented by

$$(x^2 + y^2 + 2x)^2 - 4(x^2 + y^2) = 0 \quad \text{or} \quad \rho = 2(1 - \cos\theta). \tag{1}$$

Since this cardioid is uniquely determined by the choice of the two (different) points G, O, we shall call it the GO-cardioid. As said above, we want to prove that the locus of Euler point E for a triangle is the exterior of the GO-cardioid.

[8] This point can also be described as the Feuerbach point of the tangential triangle.

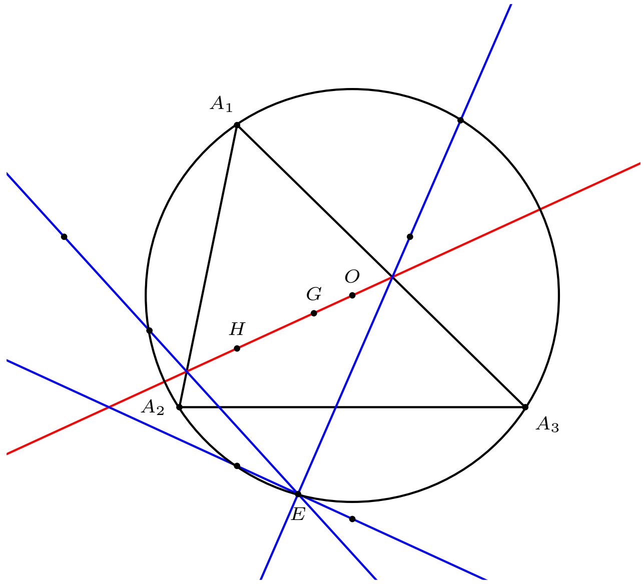

Figure 1. The Euler point of a triangle

Theorem 4. *Let G, O, E be three distinct points. Then there exists a triangle $\mathcal{T}$ whose centroid, circumcenter and Euler point are G, O, E, respectively, if and only if E lies outside the GO-cardioid. In this case the triangle $\mathcal{T}$ is unique and can be constructed by paper-folding, from the points G, O, E .*

Proof. Let us first look at isosceles (nonequilateral) triangles, which can be treated within ruler-and-compass geometry. [9] Here, by symmetry, the Euler point E lies on the Euler line; indeed, by definition, it must be one of the vertices, say $A_3 = E = (e, 0)$. Then being external to the GO-cardioid is equivalent to lying outside the segment GH_O, where $H_O = (-4, 0)$ is the symmetric of H with respect to O. Now the side A_1A_2 must reflect the orthocenter H into the point $E_O = (-2-e, 0)$, symmetric of E with respect to O, and therefore its equation is $x = -\frac{e}{2}$. This line has two intersections with the circumcircle $(x+1)^2 + y^2 = (e+1)^2$ if and only if $e(e+4) > 0$, which is precisely the condition for E to be outside GH_O. Conversely, given any two distinct points O, G, define H and H_O by $\mathbf{GH_O} = -2\mathbf{GH} = 4\mathbf{GO}$. Then for any choice of E on line OG, outside the segment GH_O, we can construct an isosceles triangle having E, O, G, H as its notable points as follows: first construct the (circum)-circle centered at O, through E, and let E_O be diametrically opposite to E. Then, under our assumptions on E, the perpendicular bisector of HE_O intersects the latter circle at two points, say A_1, A_2, and the isosceles triangle $\mathcal{T} = A_1A_2E$ fulfills our requirements.

We now deal with the nonisosceles case. Let $E = (u, v)$, $v \neq 0$ be the Euler point of a triangle $\mathcal{T}$. By definition, E is reflected across the three sides of the triangle into points E' which lie on the line $y = 0$. Now the line which reflects

[9]The case of the isosceles triangle is also studied separately by Euler in [2, §§25–29].

$E(u,v)$ onto a point $E'(t,0)$ has equation $2(u-t)x+2vy-(u^2+v^2-t^2)=0$. If the same line must also reflect (according to Theorem 2) point $E_O=(-u-2,-v)$ onto the line $x=2$ which is orthogonal to the Euler line through $H(2,0)$, then a direct calculation yields the following condition:

$$t^3-3(u^2+v^2)t+2u(u^2+v^2)-4v^2=0. \tag{2}$$

Hence we find three different reflecting lines if and only if this polynomial in t has three different real roots. The discriminant is

$$\Delta(u,v)=108v^2((u^2+v^2+2u)^2-4(u^2+v^2)).$$

Since $v\neq 0$, the inequality $\Delta(u,v)>0$ holds only if and only if E lies outside the cardioid, as we wanted.

The preceding argument can be also used for sufficiency: the assumed locus of E guarantees that (2) has three real roots. Therefore, three different lines exist which simultaneously reflect E onto line $a=OH$ and E_O onto the line b through H, perpendicular to OH. According to Corollary 3, these three lines are the sides of a triangle $\mathcal{T}$ which fulfills our requirements. In fact, H is the intersection of lines a and b and therefore H is the orthocenter of $\mathcal{T}$; a and b are perpendicular, hence E and E_O must be diametrically opposite points on the circumcircle of $\mathcal{T}$, so that their midpoint O is the circumcenter of $\mathcal{T}$. The three sides reflect E onto the x-axis, that is the Euler line of $\mathcal{T}$. Hence, by definition, E is the Euler point of $\mathcal{T}$. Since a polynomial of degree 3 cannot have more than 3 roots, the triangle is uniquely determined. □

Let us summarize the procedure for the reconstruction of the sides from the points O, G, E:

(1) Construct points H and E_O such that $\mathbf{GH}=-2\mathbf{GO}$ and $\mathbf{OE_O}=-\mathbf{OE}$.
(2) Construct line a through O, H and line b through H, perpendicular to a.
(3) Construct three lines that simultaneously reflect E on to a and E_O on to b.

4. Coordinates

The preceding results regarding the Euler point E are essential in dealing with the incenter I. In fact we shall construct E from G, O and I, so that Theorem 1 will be reduced to Theorem 3. To this end, we introduce the Gauss plane and produce complex variable equations relating I and E.[10] The cartesian coordinates will be different from the one we used in §3, but this seems unavoidable if we want to simplify calculations. A point $Z=(x,y)$ will be represented by the complex number $z=x+iy$. We write $Z=z$ and sometimes indicate operations as if they were acting directly on points rather than on their coordinates. We also write $z^*=x-iy$ and $|z|^2=x^2+y^2$.

Let $A_i=a_i$ be the vertices of a nondegenerate, nonequilateral triangle $\mathcal{T}$. Without loss of generality, we can assume for the circumcenter that $O=0$ and $|a_i|=1$, so that $a_i^{-1}=a_i^*$. Now the orthocenter H and the Euler point E have the following simple expressions in terms of elementary symmetric polynomials σ_1, σ_2, σ_3.

[10] A good reference for the use of complex variables in Euclidean geometry is [3].

$$H = a_1 + a_2 + a_3 = \sigma_1,$$
$$E = \frac{a_1a_2 + a_2a_3 + a_3a_1}{a_1 + a_2 + a_3} = \frac{\sigma_2}{\sigma_1}.$$

The first formula is trivial, as $G = \frac{1}{3}\sigma$ and $H = 3G$. As for E, the equation for a side, say A_1A_2, is $z + a_1a_2z^* = a_1 + a_2$, and the reflection across this line takes a point $T = t$ on to $T' = a_1 + a_2 - a_1a_2t^*$. An easy calculation shows that E' lies on the Euler line $z\sigma_1^* - z^*\sigma_1 = 0$. This holds for all sides, and this property characterizes E by Theorem 2. Notice that $\sigma_1 \neq 0$, as we have assumed $G \neq O$.[11]

We now introduce $\sigma_3 = a_1a_2a_3$, $k = |OH|$ and calculate

$$\sigma_1^* = \sigma_2\sigma_3^{-1}, \qquad |\sigma_3|^2 = 1, \quad |\sigma_1|^2 = |\sigma_2|^2 = \sigma_1\sigma_2\sigma_3^{-1} = k^2.$$

Hence,

$$\sigma_3 = \frac{\sigma_1\sigma_2}{k^2} = \left(\frac{\sigma_1}{k}\right)^2 \cdot \frac{\sigma_2}{\sigma_1} = \frac{H}{|H|} \cdot E.$$

In order to deal with the incenter I, let $B_i = b_i$ denote the (second) intersection of the circumcircle with the internal angle bisector of A_i. Then $b_i^{-1} = b_i^*$ and $b_i^2 = a_ja_k$, and $b_1b_2b_3 = -a_1a_2a_3$. Since I is the orthocenter of triangle $B_1B_2B_3$, we have, as above, $I = b_1 + b_2 + b_3$. Likewise, we define

$$\tau_1 = b_1 + b_2 + b_3, \quad \tau_2 = b_1b_2 + b_2b_3 + b_3b_1, \quad \tau_3 = b_1b_2b_3, \quad f = |OI|,$$

and calculate

$$\tau_1^* = \tau_2\tau_3^{-1}, \qquad |\tau_3|^2 = 1, \quad |\tau_1|^2 = |\tau_2|^2 = \tau_1\tau_2\tau_3^{-1} = f^2.$$

From the definition of b_i, we derive

$$\tau_2^2 = \sigma_3(\sigma_1 - 2\tau_1),$$
$$\tau_3 = -\sigma_3 = \left(\frac{\tau_3}{\tau_2}\right)^2 \cdot \frac{\tau_2^2}{\tau_3} = -\left(\frac{\tau_1}{f^2}\right)^2 (\sigma_1 - 2\tau_1).$$

Equivalently,

$$\sigma_3 = -\tau_3 = \left(\frac{\tau_1}{f}\right)^2 \cdot \frac{\sigma_1 - 2\tau_1}{f^2},$$
$$\left(\frac{H}{|H|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{H - 2I}{|I|^2},$$
$$\left(\frac{G}{|G|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{3G - 2I}{|I|^2}, \tag{3}$$

where the Euler equation $H = 3G$ has been used.

[11]The triangle is equilateral if $G = O$.

5. Construction of the Euler point

The last formulas suggest easy constructions of E from O, G (or H) and I. Since $H - 2I = 3G - 2I = G - 2(I - G)$, our attention moves from $\mathcal{T}$ to its antimedial triangle $\mathcal{T}^*$ (the midpoints of whose sides are the vertices of $\mathcal{T}$) and the homothetic mapping: $Z \mapsto G - 2(Z - G)$. Thus $I^* = 3G - 2I$ is the incenter of $\mathcal{T}^*$. Note that multiplying by a *unit* complex number $\cos\theta + i\sin\theta$ is equivalent to rotating around O by an angle θ. Since $G/|G|$ and $I/|I|$ are unit complex numbers, multiplication by $\dfrac{(I/|I|)^2}{(G/|G|)^2}$ represents a rotation which is the product of two reflections, first across the line OG, then across OI. Since $|I|^2 = f^2$, dividing f^2 by I^* is equivalent to inverting I^* in the circle with center O and radius OI. Altogether, we conclude that E can be constructed from O, G and I by the following procedure. See Figure 2.

(1) Construct lines OG and OI; construct I^* by the equation $\mathbf{GI}^* = -2\mathbf{GI}$.
(2) Construct the circle Ω centered at O through I. By inverting I^* with respect to this circle, construct F^*. Note that this inversion is possible if and only if $I^* \neq O$, or, equivalently, $I \neq N$. [12]
(3) Construct E: first reflect F^* in line OG, and then its image in line OI.

Note that all these steps can be performed by ruler and compass.

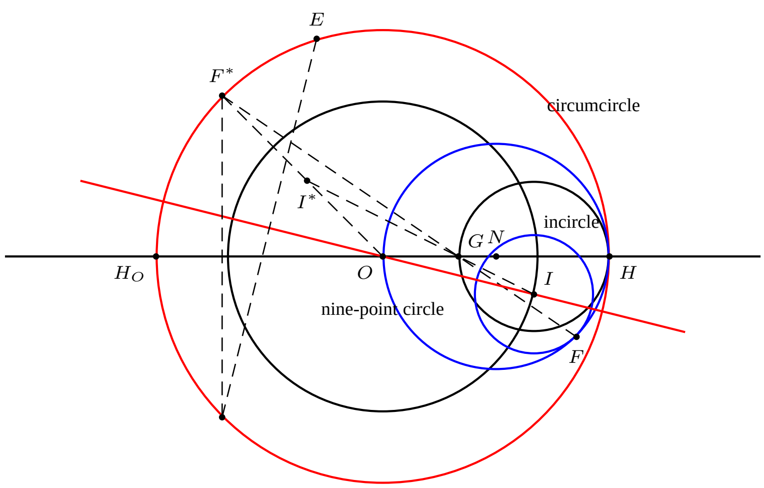

Figure 2. Construction of the Euler point from O, G, I

[12] It will appear that F^* is the Feuerbach point of $\mathcal{T}^*$. Thus, at this stage we have both the circumcircle (center O, through F^*) and the incircle of $\mathcal{T}$ (center I, through F, as defined by $\mathbf{GF}^* = -2\mathbf{GF}$).

6. The locus of incenter

As we know from §3, one can now apply paper-folding to O, G, E and produce the sides of $\mathcal{T}$. But in order to prove Theorem 1 we must show that the critical circle locus for I is equivalent to the existence of three different good creases. To this end we check that I lies inside the orthocentroidal GH-circle if and only if E lies outside the GO-cardioid. If we show that the two borders correspond under the transformation $I \mapsto E$ described by (3) for given O, G, H, then, by continuity, the two ranges, the interior of the circle and the exterior of the cardioid, will also correspond.

We first notice that the right side of (3) can be simplified when I lies on the GH-circle, as $|IO| = 2|IN| = |I^*O|$ implies that the inversion (step 2) does not affect I^*. In order to compare the transformation $I \mapsto E$ with our previous results, we must change scale and return to the cartesian coordinates used in §2, where $G = (0,0)$, $H = (2,0)$. If we set $I = (r,s)$, then $I^* = (-2r,-2s)$. The first reflection (across the Euler line) maps I^* on to $(-2r,2s)$; the second reflection takes place across line OI: $s(x+1)-(r+1)y=0$ and yields $E(u,v)$, where

$$(u,v) = \left(\frac{-2(r^3-3rs^2+r+2r^2-s^2)}{(r+1)^2+s^2}, \frac{-2s(3r^2-s^2+3r)}{(r+1)^2+s^2}\right).$$

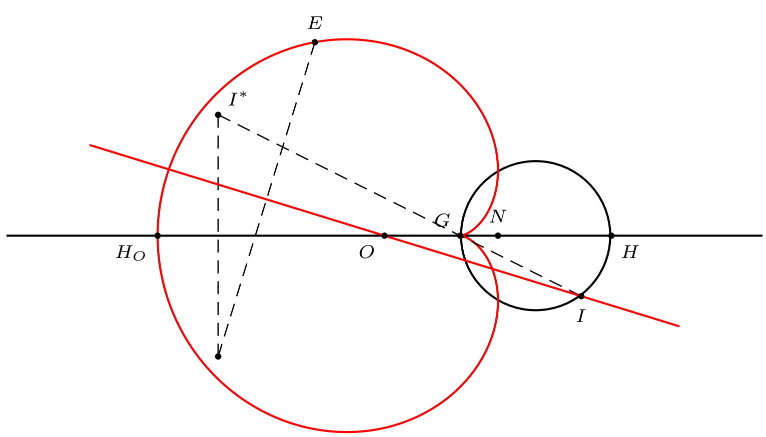

Figure 3. Construction of the GO-cardioid from the GH-circle

Notice that $I \neq O$ implies $(r+1)^2+s^2 \neq 0$. Then, by direct calculations, we have

$$((u^2+v^2+2u)^2-4(u^2+v^2))((r+1)^2+s^2) = 16(r^2+s^2-2r)(r^2+s^2+r)^2$$

and conclude that $I(r,s)$ lies on the GH-circle $x^2+y^2-2x=0$ whenever E lies on the GO-cardioid (1), as we wanted. Thus the proof of Theorem 1 is complete.

7. Euler's theorem revisited

We shall now give a different proof of the first part of Theorem 1 by exploiting Euler's original ideas and complementing his calculations.

Necessity. In [2] Euler begins (§§1-20) with a nonequilateral, nondegenerate triangle and calculates the "notable" lengths

$$|HI| = e, \quad |OI| = f, \quad |OG| = g, \quad |GI| = h, \quad |HO| = k$$

as functions of the side lengths a_1, a_2, a_3. From those expressions he derives a number of algebraic equalities and inequalities, whose geometrical interpretations he only partially studies. [13] In particular, in §31, by observing that some of his quantities can only assume positive values, Euler explicitly states that the two inequalities

$$k^2 < 2e^2 + 2f^2, \tag{4}$$

$$k^2 > 2e^2 + f^2 \tag{5}$$

must hold. However, rather than studying their individual geometrical meaning, he tries to combine them and wrongly concludes, owing to a trivial mistake, that the inequalities $19f^2 > 8e^2$ and $13f^2 < 19e^2$ are also necessary conditions. Speiser's correction of Euler's mistake [2, p.155, footnote] does not produce any interesting result. On the other hand, if one uses the main result $\mathbf{OH} = 3\mathbf{OG}$, defines the nine-point center N (by letting $\mathbf{OH} = 2\mathbf{ON}$) and applies elementary geometry (Carnot's and Apollonius's theorems), it is very easy to check that the two original inequalities (4) and (5) are respectively equivalent to

(4′) I is different from N, and
(5′) I lies inside the GH-circle.

These are precisely the conditions of Theorem 1. It is noteworthy that Euler, unlike Guinand, could not use Feuerbach's theorem.

Sufficiency. In §21 Euler begins with three positive numbers f, g, h and derives a real polynomial of degree 3, whose roots a_1, a_2, a_3 - in case they are sides of a triangle - produce indeed f, g, h for the notable distances. It remains to prove that, under the assumptions of Theorem 1, these roots are real positive and satisfy the triangle inequalities. In order to complete Euler's work, we need a couple of lemmas involving symmetric polynomials.

Lemma 5. *(a) Three real numbers a_1, a_2, a_3 are positive if and only if $\sigma_1 = a_1 + a_2 + a_3$, $\sigma_2 = a_1a_2 + a_2a_3 + a_3a_1$ and $\sigma_3 = a_1a_2a_3$ are all positive.*

[13] The famous result on the collinearity of O, G, H and the equation $\mathbf{OH} = 3\mathbf{OG}$ are explicitly described in [2]. The other famous formula $OI^2 = R(R - 2r)$ is not explicitly given, but can be immediately derived, by applying the well known formulas for the triangle area $\frac{1}{2}r(a_1 + a_2 + a_3) = \frac{a_1a_2a_3}{4R}$.

(b) Three positive real numbers a_1, a_2, a_3 satisfy the triangle inequalities $a_1+a_2 \geq a_3$, $a_2+a_3 \geq a_1$ and $a_3+a_1 \geq a_2$ if and only if [14]

$$\tau(a_1, a_2, a_3) = (a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3) \geq 0.$$

Now suppose we are given three different points I, O, N and define two more points G, H by $3\mathbf{OG} = 2\mathbf{ON} = \mathbf{OH}$. Assume that I is inside the GH-circle. If we let

$$m = |ON|, \quad n = |IN|, \quad f = |OI|,$$

then we have $m > 0$, $n > 0$, $f > 0$, $n+f-m \geq 0$, and also, according to Lemma 5(b), $\tau(f, m, n) \geq 0$. Moreover, the assumed locus of I within the critical circle implies, by Apollonius, $f - 2n > 0$ so that $f^2 - 4n^2 = b^2$ for some real $b > 0$. We now introduce the same quantities p, q, r of Euler, [15] but rewrite their defining relations in terms of the new variables m, n, f as follows:

$$\begin{aligned} n^2 r =& f^4, \\ 4n^2 q =& b^2 f^2, \\ 9h^2 =& (f-2n)^2 + 2((n+f)^2 - m^2), \\ 4n^2 p =& 27b^4 + 128n^2 b^2 + 144h^2 n^2. \end{aligned}$$

Notice that, under our assumptions, all these functions assume positive values, so that we can define three more positive quantities [16]

$$\sigma_1 = \sqrt{p}, \quad \sigma_2 = \frac{p}{4} + 2q + \frac{q^2}{r}, \quad \sigma_3 = q\sqrt{p}.$$

Now let a_1, a_2, a_3 be the (complex) roots of the polynomial $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$. The crucial point regards the discriminant

$$\begin{aligned} \Delta(a_1, a_2, a_3) =& (a_1 - a_2)^2 (a_2 - a_3)^2 (a_3 - a_1)^2 \\ =& \sigma_1^2 \sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2. \end{aligned}$$

By a tedious but straightforward calculation, involving a polynomial of degree 8 in m, n, f, one finds

$$n^2 \Delta(a_1, a_2, a_3) = b^4 \tau(f, m, n).$$

Since, by assumption, $n \neq 0$, this implies $\Delta(a_1, a_2, a_3) \geq 0$, so that a_1, a_2, a_3 are real. By Lemma 5(a), since σ_1, σ_2, $\sigma_3 > 0$, we also have a_1, a_2, $a_3 > 0$. A final calculation yields $\tau(a_1, a_2, a_3) = \dfrac{4pq^2}{r} > 0$, ensuring, by Lemma 5(b) again, that the triangle inequalities hold. Therefore, under our assumptions, there exists a triangle with a_1, a_2, a_3 as sides lengths, which is clearly nondegenerate and nonequilateral, and whose notable distances are f, m, n. Thus the alternative proof the first part of Theorem 1 is complete. Of course, the last statement on

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[14] The expression $\tau(a_1, a_2, a_3)$ appears under the square root in Heron's formula for the area of a triangle.

[15] These quantities read P, Q, R in [2, p.149].

[16] These quantities read p, q, r in [2, p.144].

construction is missing: the actual location of the triangle, in terms of the location of its notable points, cannot be studied by this approach.

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