Forum Geometricorum
Volume 2 (2002) 115–128.

FORUM GEOM
ISSN 1534-1178

Napoleon-like Configurations and Sequences of Triangles

Barukh Ziv

Abstract. We consider the sequences of triangles where each triangle is formed out of the apices of three similar triangles built on the sides of its predecessor. We show under what conditions such sequences converge in shape, or are periodic.

1. Introduction

The well-known geometrical configuration consisting of a given triangle and three equilateral triangles built on its sides, all outwardly or inwardly, has many interesting properties. The most famous is the theorem attributed to Napoleon that the centers of the three equilateral triangles built on the sides are vertices of another equilateral triangle [3, pp. 60–65]. Numerous works have been devoted to this configuration, including various generalizations [6, 7, 8] and converse problems [10].

Some authors [5, 9, 1] considered the iterated configurations where construction of various geometrical objects (e.g. midpoints) on the sides of polygons is repeated an arbitrary number of times. Douglass [5] called such constructions *linear polygon transformations* and showed their relation with circulant matrices. In this paper, we study the sequence of triangles obtained by a modification of such a configuration. Each triangle in the sequence is called a *base* triangle, and is obtained from its predecessor by two successive transformations: (1) the classical construction on the sides of the base triangle triangles similar to a given (*transformation*) triangle and properly oriented, (2) a normalization which is a direct similarity transformation on the apices of these new triangles so that one of the images lies on a fixed circle. The three points thus obtained become the vertices of the new base triangle. The normalization step is the feature that distinguishes the present paper from earlier works, and it gives rise to interesting results. The main result of this study is that under some general conditions the sequence of base triangles converges to an equilateral triangle (in a sense defined at the end of §2). When the limit does not exist, we study the conditions for periodicity. We study two types of sequences of triangles: in the first, the orientation of the transformation triangle is given a priori; in the second, it depends on the orientation of the base triangle.

Publication Date: October 4, 2002. Communicating Editor: Floor van Lamoen.
The author is grateful to Floor van Lamoen and the referee for their valuable suggestions in improving the completeness and clarity of the paper.

The rest of the paper is organized as follows. In §2, we explain the notations and definitions used in the paper. In §3, we give a formal definition of the transformation that generates the sequence. In §4, we study the first type of sequences mentioned above. In §5, we consider the exceptional case when the transformation triangle degenerates into three collinear points. In §6, we consider the second type of sequences mentioned above. In §7, we study a generalization for arbitrary polygons.

2. Terminology and definitions

We adopt the common notations of complex arithmetic. For a complex number z, Re(z) denotes its real part, Im(z) its imaginary part, $|z|$ its modulus, arg(z) its argument (chosen in the interval $(-\pi, \pi]$), and $\overline{z}$ its conjugate. The primitive complex n-th root of unity $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, is denoted by ζ_n. Specifically, we write $\omega = \zeta_3$ and $\eta = \zeta_6$. The important relation between the two is $\omega^2 + \eta = 0$.

A triangle is oriented if an ordering of its vertices is specified. It is positively (negatively) oriented if the vertices are ordered counterclockwise (clockwise). Two oriented triangles are directly (oppositely) similar if they have the same (opposite) orientation and pairs of corresponding vertices may be brought into coincidence by similarity transformations.

Throughout the paper, we coordinatized points in a plane by complex numbers, using the same letter for a point and its complex number coordinate. An oriented triangle is represented by an ordered triple of complex numbers. To obtain the orientation and similarity conditions, we define the following function $z : \mathbb{C}^3 \to \mathbb{C}$ on the set of all vectors $V = (A, B, C)$ by

$$z[V] = z(A, B, C) = \frac{C - A}{B - A}. \tag{1}$$

Triangle ABC is positively or negatively oriented according as arg($z(A, B, C)$) is positive or negative. Furthermore, every complex number z defines a class of directly similar oriented triangles ABC such that $z(A, B, C) = z$. In particular, if ABC is a positively (respectively negatively) oriented equilateral triangle, then $z(A, B, C) = \eta$ (respectively $\overline{\eta}$).

Finally, we define the convergence of triangles. An infinite sequence of triangles $(A_nB_nC_n)$ converges to a triangle ABC if the sequence of complex numbers $z(A_n, B_n, C_n)$ converges to $z(A, B, C)$.

3. The transformation f

We describe the transformations that generate the sequence of triangles we study in the paper. We start with a base triangle $A_0B_0C_0$ and a transformation triangle XYZ. Let G be the centroid of $A_0B_0C_0$, and Γ the circle centered at G and passing through the vertex farthest from G. (Figure 1a). For every $n > 0$, triangle $A_nB_nC_n$ is obtained from its predecessor $A_{n-1}B_{n-1}C_{n-1}$ by $f = f_2 \circ f_1$, where

(i) f_1 maps $A_{n-1}B_{n-1}C_{n-1}$ to $A'_nB'_nC'_n$, by building on the sides of triangle $A_{n-1}B_{n-1}C_{n-1}$, three triangles $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ similar to XYZ and all with the same orientation, [1] (Figure 1b);

(ii) f_2 transforms by similarity with center G the three points A'_n, B'_n, C'_n so that the image of the farthest point lies on the circle Γ, (Figure 1c).

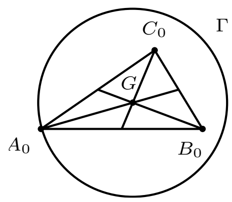

Figure 1a

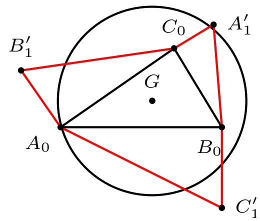

Figure 1b

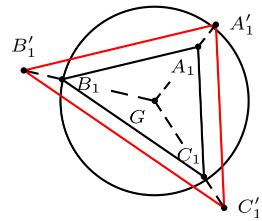

Figure 1c

The three points so obtained are the vertices of the next base triangle $A_nB_nC_n$. We call this the *normalization* of triangle $A'_nB'_nC'_n$. In what follows, it is convenient to coordinatize the vertices of triangle $A_0B_0C_0$ so that its centroid G is at the origin, and Γ is the unit circle. In this setting, normalization is simply division by

$$r_n = \max(|A'_n|, |B'_n|, |C'_n|).$$

It is easy to see that f may lead to a degenerate triangle. Figure 2 depicts an example of a triple of collinear points generated by f_1. Nevertheless, f is well defined, except only when $A_{n-1}B_{n-1}C_{n-1}$ degenerates into the point G. But it is readily verified that this happens only if triangle $A_{n-1}B_{n-1}C_{n-1}$ is equilateral, in which case we stipulate that $A_nB_nC_n$ coincides with $A_{n-1}B_{n-1}C_{n-1}$.

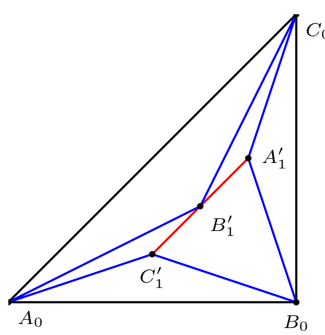

Figure 2

The normalization is a crucial part of this transformation. While preserving direct similarity of the triangles $A'_nB'_nC'_n$ and $A_nB_nC_n$, it prevents the latter from

[1]We deliberately do not specify the orientation of those triangles with respect to the transformation triangle, since they are specific for the different types of sequences we discuss later in this paper.

converging to a single point or diverging to infinity (since every triangle after normalization lies inside a fixed circle, and at least one of its vertices lies on the circle), and the convergence of triangles receives a definite geometrical meaning. Also, since f_1 and f_2 leave G fixed, we have a rather expected important property that G is a fixed point of the transformation.

4. The first sequence

We first keep the orientation of the transformation triangle fixed and independent from the base triangle.

Theorem 1. *Let $A_0B_0C_0$ be an arbitrary base triangle, and XYZ a non-degenerate transformation triangle. The sequence $(A_nB_nC_n)$ of base triangles generated by the transformation f in §3 (with $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ directly similar to XYZ) converges to the equilateral triangle with orientation opposite to XYZ, except when $A_0B_0C_0$, and the whole sequence, is equilateral with the same orientation as XYZ.*

Proof. Without loss of generality let XYZ be positively oriented. We treat the special cases first. The exceptional case stated in the theorem is verified straightforwardly; also it is obvious that we may exclude the cases where $A_nB_nC_n$ is positively oriented equilateral. Hence in what follows it is assumed that $z(A_0, B_0, C_0) \neq \eta$, and $r_n \neq 0$ for every n.

Let $z(X, Y, Z) = t$. Since for every n, triangle $B_{n-1}C_{n-1}A'_n$ is directly similar to XYZ, by (1)

$$A'_n = (1-t)B_{n-1} + tC_{n-1},$$

and similarly for B'_n and C'_n. After normalization,

$$V_n = \frac{1}{r_n} T V_{n-1}, \tag{2}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, T is the circulant matrix $\begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$, and $r_n = \max(|A'_n|, |B'_n|, |C'_n|)$. By induction,

$$V_n = \frac{1}{r_1 \cdots r_n} T^n V_0.$$

We use the standard diagonalization procedure to compute the powers of T. Since T is circulant, its eigenvectors are the columns of the Fourier matrix ([4, pp.72–73])

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{pmatrix},$$

and the corresponding eigenvalues are λ_0, λ_1, λ_2 are [2]

$$\lambda_j = (1-t)\omega^j + t\omega^{2j}, \tag{3}$$

[2] Interestingly enough, ordered triples $(\omega, \omega^2, \lambda_1)$ and $(\omega^2, \omega, \lambda_2)$ form triangles directly similar to XYZ.

for $j = 0, 1, 2$. With these, we have

$$T = F_3 U F_3^{-1},$$

where U is the diagonal matrix $\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

Let $S = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$ be a vector of points in the complex plane that is transformed into V_0 by the Fourier matrix, *i.e.*,

$$V_0 = F_3 S. \tag{4}$$

Since $A_0 + B_0 + C_0 = 3G = 0$, we get $s_0 = 0$, and

$$V_0 = s_1 F_{3,1} + s_2 F_{3,2}, \tag{5}$$

where $F_{3,j}$ is the j-th column of F_3. After n iterations,

$$V_n \sim T^n V_0 = F_3 U^n F_3^{-1}(s_1 F_{3,1} + s_2 F_{3,2}) = s_1 \lambda_1^n F_{3,1} + s_2 \lambda_2^n F_{3,2}. \tag{6}$$

According to (3) and the assumption that XYZ is negatively oriented,

$$|\lambda_2|^2 - |\lambda_1|^2 = \lambda_2 \overline{\lambda_2} - \lambda_1 \overline{\lambda_1} = 2\sqrt{3}\mathrm{Im}(t) < 0,$$

so that $\dfrac{|\lambda_2|}{|\lambda_1|} < 1$, and $\dfrac{|\lambda_2^n|}{|\lambda_1^n|} \to 0$ when $n \to \infty$. Also, expressing $z(A_0, B_0, C_0)$ in terms of s_1, s_2, we get

$$z(A_0, B_0, C_0) = \frac{s_1 \eta + s_2}{s_1 + s_2 \eta}, \tag{7}$$

so that $z(A_0, B_0, C_0) \neq \overline{\eta}$ implies $s_1 \neq 0$. Therefore,

$$\lim_{n\to\infty} z(A_n, B_n, C_n) = \lim_{n\to\infty} z[V_n] = z[F_{3,1}] = \eta.$$

□

Are there cases when the sequence converges after a finite number of iterations? Because the columns of the Fourier matrix F_3 are linearly independent, this may happen if and only if the second term in (6) equals 0. There are two cases:

(i) $s_2 = 0$: this, according to (7), corresponds to a base triangle $A_0B_0C_0$ which is equilateral and positively oriented;

(ii) $\lambda_2 = 0$: this, according to (3), corresponds to a transformation triangle XYZ which is isosceles with base angle $\frac{\pi}{6}$. In this case, one easily recognizes the triangle of the Napoleon theorem.

We give a geometric interpretation of the values s_1, s_2. Changing for a while the coordinates of the complex plane so that A_0 is at the origin, we get from (4):

$$|s_1| = |B_0 - C_0\eta|, \qquad |s_2| = |B_0 - C_0\overline{\eta}|,$$

and we have the following construction: On the side A_0C_0 of the triangle $A_0B_0C_0$ build two oppositely oriented equilateral triangles (Figure 3), then $|s_1| = B_0B'$,

$|s_2| = B_0B''$. After some computations, we obtain the following symmetric formula for the ratio $\frac{s_1}{s_2}$ in terms of the angles α, β, γ of triangle $A_0B_0C_0$:

$$\left|\frac{s_1}{s_2}\right|^2 = \frac{\sin\alpha\sin(\alpha+\frac{\pi}{3})+\sin\beta\sin(\beta+\frac{\pi}{3})+\sin\gamma\sin(\gamma+\frac{\pi}{3})}{\sin\alpha\sin(\alpha-\frac{\pi}{3})+\sin\beta\sin(\beta-\frac{\pi}{3})+\sin\gamma\sin(\gamma-\frac{\pi}{3})}.$$

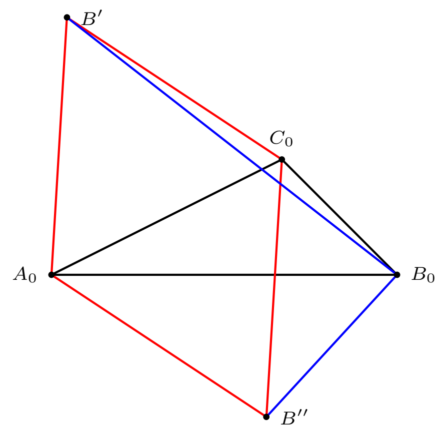

Figure 3

5. An exceptional case

In this section we consider the case t is a real number. Geometrically, it means that the transformation triangle XYZ degenerates into a triple of collinear points, so that A'_n, B'_n, C'_n divide the corresponding sides of triangle $A_{n-1}B_{n-1}C_{n-1}$ in the ratio $1-t:t$. (Figure 4 depicts an example for $t=\frac{1}{3}$). Can the sequence of triangles still converge in this case? To settle this question, notice that when t is real, λ_1 and λ_2 are complex conjugates, and rewrite (6) as follows:

$$V_n \sim \lambda_1^n \left(s_1F_{3,1} + \frac{\lambda_2^n}{\lambda_1^n}s_2F_{3,2}\right), \tag{8}$$

and because $\dfrac{\lambda_2}{\lambda_1}$ defines a *rotation*, it is clear that it does not have a limit unless $\dfrac{\lambda_2}{\lambda_1}=1$, in which case the sequence consists of directly similar triangles. Now, $\lambda_1=\lambda_2$ implies $t=\frac{1}{2}$, so we have the well-known result that the triangle is similar to its medial triangle [3, p. 19].

Next, we find the conditions under which the sequence has a finite period m. Geometrically, it means that m is the least number such that triangles $A_nB_nC_n$ and $A_{n+m}B_{n+m}C_{n+m}$ are directly similar for every $n\ge 0$. The formula (8) shows that it happens when $\dfrac{\lambda_2}{\lambda_1}=\zeta_m^k$, and k, m are co-prime. Plugging this into

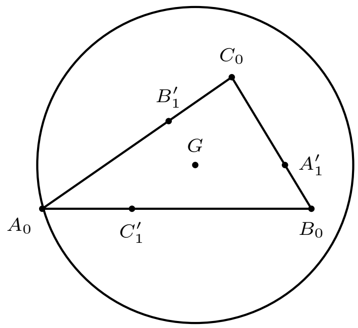

Figure 4a

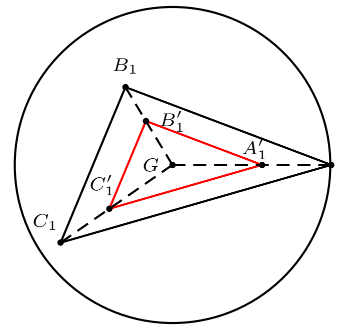

Figure 4b

(3) and solving for t, we conclude that the sequence of triangles with period m exists for t of the form

$$t(m) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan \frac{k\pi}{m}. \tag{9}$$

Several observations may be made from this formula. First, the periodic sequence with finite t exists for every $m \neq 2$. (The case $m = 2$ corresponds to transformation triangle with two coinciding vertices X, Y). The number of different sequences of a given period m is $\phi(m)$, Euler's totient function [2, pp.154–156]. Finally, the case $m = 1$ yields $t = \frac{1}{2}$, which is the case of medial triangles.

Also, several conclusions may be drawn about the position of corresponding triangles in a periodic sequence. Comparing (8) with (5), we see that triangle $A_mB_mC_m$ is obtained from triangle $A_0B_0C_0$ by a rotation about their common centroid G through angle $m \cdot \arg(\lambda_1)$. Because 2arg(λ_1) = 0 $\mathrm{mod} 2\pi$, it follows that $A_mB_mC_m$ coincides with $A_0B_0C_0$, or is a half-turn. In both cases, the triangle $A_{2m}B_{2m}C_{2m}$ will always coincide with $A_0B_0C_0$. We summarize these results in the following theorem.

Theorem 2. *Let a triangle $A_0B_0C_0$ and a real number t be given. The sequence $(A_nB_nC_n)$ of triangles constructed by first dividing the sides of each triangle in the ratio $1 - t : t$ and then normalizing consists of similar triangles with period m if and only if t satisfies (9) for some k relatively prime to m. In this case, triangles $A_nB_nC_n$ and $A_{n+2m}B_{n+2m}C_{n+2m}$ coincide for every $n \geq 0$. In all other cases the sequence never converges, unless $A_0B_0C_0$, and hence every $A_nB_nC_n$, is equilateral.*

6. The second sequence

In this section we study another type of sequence, where the orientation of transformation triangles depends on the base triangle. More precisely, we consider two cases: when triangles built on the sides of the base triangle are oppositely or equally oriented to it. The main results of this section will be derived using the following important lemma.

Lemma 3. *Let ABC be any triangle, and $V = (A, B, C)$ the corresponding vector of points in the complex plane with the centroid of ABC at the origin. If S is defined as in (4), then ABC is positively (negatively) oriented when $|s_1| > |s_2|$ ($|s_1| < |s_2|$), and A, B, C are collinear if $|s_1| = |s_2|$.*

Proof. According to (7), Im$(z(A, B, C)) \sim s_1\overline{s_1} - s_2\overline{s_2} = |s_1|^2 - |s_2|^2$. □

Before proceeding, we extend notations. As the orientation of the transformation triangle may change throughout the sequence, $z(X, Y, Z)$ equals t or $\overline{t}$, depending on the orientation of the base triangle. So, for the transformation matrix we shall use the notation $T(t)$ or $T(\overline{t})$ accordingly. Note that if the eigenvalues of $T(t)$ are λ_0, λ_1, λ_2, then the eigenvalues of $T(\overline{t})$ are $\overline{\lambda_0}$, $\overline{\lambda_1}$, $\overline{\lambda_2}$. The first result concerning the case of the oppositely oriented triangles is as follows.

Theorem 4. *Let $A_0B_0C_0$ be the base triangle, and XYZ the transformation triangle. If the sequence of triangles $A_nB_nC_n$ is generated as described in §3 with every triangle $B_{n-1}C_{n-1}A'_n$ etc. oppositely oriented to $A_{n-1}B_{n-1}C_{n-1}$, then the sequence converges to the equilateral triangle that has the same orientation as $A_0B_0C_0$.*

Proof. Without loss of generality, we may assume $A_0B_0C_0$ positively oriented. It is sufficient to show that triangle $A_nB_nC_n$ is positively oriented for every n. Then, every triangle $B_{n-1}C_{n-1}A'_n$ etc. is negatively oriented, and the result follows immediately from Theorem 1.

We shall show this by induction. Assume triangles $A_0B_0C_0, \ldots, A_{n-1}B_{n-1}C_{n-1}$ are positively oriented, then they all are the base for the *negatively* oriented directly similar triangles to build their successors, so arg$(t) < 0$, and $|\lambda_1^n| > |\lambda_2^n|$. Also, $|s_1| > |s_2|$, and according to (6) and the above lemma, $A_nB_nC_n$ is positively oriented. □

We proceed with the case when triangles are constructed with the same orientation of the base triangle. In this case, the behavior of the sequence turns out to be much more complicated. Like in the first case, assume $A_0B_0C_0$ positively oriented. If $s_2 = 0$, which corresponds to the equilateral triangle, then all triangles $A_nB_nC_n$ are positively oriented and, of course, equilateral. Otherwise, because arg$(t) > 0$, and therefore $|\lambda_1| < |\lambda_2|$, it follows that $|s_1\lambda_1^n| - |s_2\lambda_2^n|$ eventually becomes negative, and the sequence of triangles changes the orientation. Specifically, it happens exactly after ℓ steps, where

$$\ell = \left\lceil \frac{\ln \frac{s_2}{s_1}}{\ln \frac{\lambda_1}{\lambda_2}} \right\rceil. \tag{10}$$

What happens next? We know that $A_\ell B_\ell C_\ell$ is the first negatively oriented triangle in the sequence, therefore triangles $B_\ell C_\ell A'_{\ell+1}$ etc. built on its sides are also negatively oriented. Thus, $z(B_\ell, C_\ell, A'_{\ell+1}) = \overline{t}$. Therefore, according to (3) and

(6),

$$V_{\ell+1} \sim T(t)^{\ell} T(\overline{t}) V_0 = s_1 \lambda_1^{\ell} \overline{\lambda_2} F_{3,1} + s_2 \lambda_2^{\ell} \overline{\lambda_1} F_{3,2}.$$

Since

$$|s_1 \lambda_1^{\ell} \overline{\lambda_2}| = |s_1 \lambda_1^{\ell-1}| |\lambda_1 \lambda_2| > |s_2 \lambda_2^{\ell-1}| |\lambda_1 \lambda_2| = |s_2 \lambda_2^{\ell} \overline{\lambda_1}|,$$

triangle $A_{\ell+1}B_{\ell+1}C_{\ell+1}$ is positively oriented. Analogously, we get that for $n \geq 0$, every triangle $A_{\ell+2n}B_{\ell+2n}C_{\ell+2n}$ is negatively oriented, while its successor $A_{\ell+2n+1}B_{\ell+2n+1}C_{\ell+2n+1}$ is positively oriented.

Consider now the sequence $(A_{\ell+2n}B_{\ell+2n}C_{\ell+2n})$ consisting of negatively oriented triangles. Clearly, the transformation matrix for this sequence is the product of $T(t)$ and $T(\overline{t})$, which is a circulant matrix

$$\begin{pmatrix} t+\overline{t}-2t\overline{t} & t\overline{t} & 1-t-\overline{t}+t\overline{t} \\ 1-t-\overline{t}+t\overline{t} & t+\overline{t}-2t\overline{t} & t\overline{t} \\ t\overline{t} & 1-t-\overline{t}+t\overline{t} & t+\overline{t}-2t\overline{t} \end{pmatrix}$$

with eigenvalues

$$\lambda'_j = t+\overline{t}-2t\overline{t} + t\overline{t}\omega^j + (1-t-\overline{t}+t\overline{t})\omega^{2j}, \qquad j = 0,1,2. \tag{11}$$

Since this matrix is real, the sequence $(A_{\ell+2n}B_{\ell+2n}C_{\ell+2n})$ of triangles does not converge. It follows at once that the sequence $(A_{\ell+2n+1}B_{\ell+2n+1}C_{\ell+2n+1})$ of successors does not converge either.

Finally, we consider the conditions when these two sequences are periodic. Clearly, only even periods $2m$ may exist. In this case, λ'_1 and λ'_2 must satisfy $\dfrac{\lambda'_1}{\lambda'_2} = \zeta_m^k$ for k relatively prime to m. Since λ'_1, λ'_2 are complex conjugates, this is equivalent to arg$(\lambda'_1) = \frac{k\pi}{m}$. Applying (11), we arrive at the following condition:

$$\tan \frac{k\pi}{m} = \frac{1}{\sqrt{3}} \cdot \frac{\mathrm{Re}(t) - \frac{1}{2}}{\mathrm{Re}(t) - |t|^2 - \frac{1}{6}}.$$

Several interesting properties about periodic sequences may be derived from this formula. First, for a given pair of numbers k, m, the locus of t is a *circle* centered at the point O on a real axis, and radius R defined as follows:

$$O(m) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \cot \frac{k\pi}{m}, \qquad R(m) = \frac{1}{2\sqrt{3}} \csc \frac{k\pi}{m}. \tag{12}$$

Furthermore, all the circles have the two points $\frac{1}{3}(1+\overline{\eta})$ and $\frac{1}{3}(1+\eta)$ in common. This is clear if we note that they correspond to the cases $\lambda'_1 = 0$ and $\lambda'_2 = 0$ respectively, *i.e.*, when the triangle becomes equilateral after the first iteration (see the discussion following Theorem 1 in §3).

Summarizing, we have the following theorem.

Theorem 5. *Let $A_0B_0C_0$ be the base triangle, and XYZ the transformation triangle. The sequence $(A_nB_nC_n)$ of triangles constructed by the transformation f ($B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ with the same orientation of $A_{n-1}B_{n-1}C_{n-1}$) converges only if $A_0B_0C_0$ is equilateral (and so is the whole sequence). Otherwise the orientation of $A_0B_0C_0$ is preserved for first $\ell - 1$ iterations, where ℓ is determined by (10); afterwards, it is reversed in each iteration.*

The sequence consists of similar triangles with an even period $2m$ *if and only if* $t = z(X, Y, Z)$ *lies on a circle* $O(R)$ *defined by (12) for some* k *relatively prime to* m*. In this case, triangles* $A_nB_nC_n$ *and* $A_{n+4m}B_{n+4m}C_{n+4m}$ *coincide for every* $n \geq \ell$*.*

We conclude with a demonstration of the last theorem's results. Setting $m = 1$ in (12), both O and R tend to infinity, and the circle degenerates into line $\mathrm{Re}(t) = \frac{1}{2}$, that corresponds to any isosceles triangle. Figures 5a through 5d illustrate this case when XYZ is the right isosceles triangle, and $A_0B_0C_0$ is also isosceles positively oriented with base angle $\frac{3\pi}{8}$. According to (10), $\ell = 2$. Indeed, $A_2B_2C_2$ is the first negatively oriented triangle in the sequence, $A_3B_3C_3$ is again positively oriented and similar to $A_1B_1C_1$. The next similar triangle $A_5B_5C_5$ will coincide with $A_1B_1C_1$.

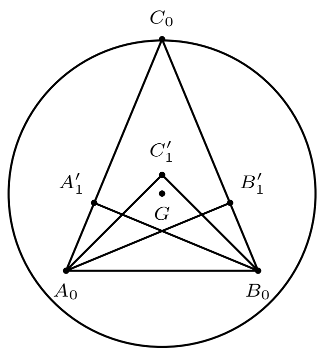

Figure 5a

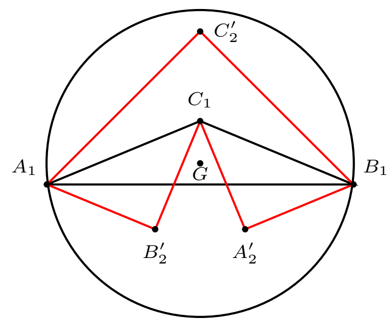

Figure 5b

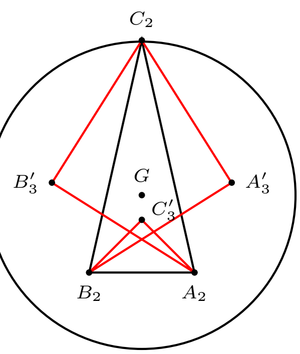

Figure 5c

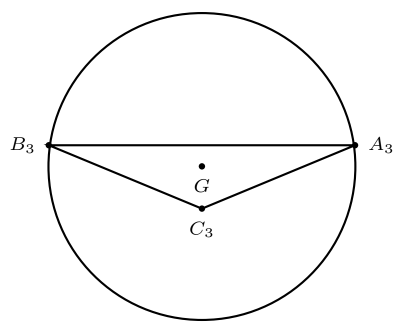

Figure 5d

7. Generalization to polygons

In this section, we generalize the results in §4 by replacing the sequences of triangles by sequences of polygons. The transformation performed at every iteration remains much the same as in §3, with triangles built on every side of the base polygon directly similar to a given transformation triangle. We seek the conditions under which the resulting sequence of polygons converges in shape.

Let the unit circle be divided into k equal parts by the points P_0, P_1, $\dots$, P_{k-1}. We call the polygon regular k-gon of q-type if it is similar to the polygon $P_0P_q\cdots P_{(k-1)q}$, where the indices are taken modulo k [5, p. 558]. The regular 1-type and $(k-1)$-type polygons are simply the convex regular polygons in an ordinary sense. Other regular k-gons may be further classified into

(i) star-shaped if q, k are co-prime, (for example, a pentagram is a 2-type regular pentagon, Figure 6a), and

(ii) multiply traversed polygons with fewer vertices if q, k have a common divisor, (for example, a regular hexagon of 2-type is an equilateral triangle traversed twice, Figure 6b).

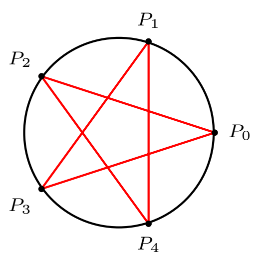

Figure 6a

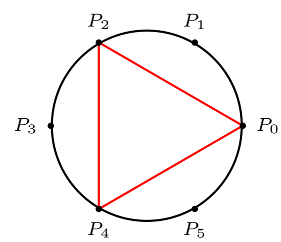

Figure 6b

In general, regular k-gons of q-type and $(k-q)$-type are equally shaped and oppositely oriented. It is also evident that $(-q)$-type and $(k-q)$-type k-gons are identical. We shall show that under certain conditions the sequence of polygons converges to regular polygons so defined.

Let $\Pi_0 = P_{0,0}P_{1,0}\cdots P_{k-1,0}$ be an arbitrary k-gon, and XYZ the non-degenerate transformation triangle, and let the sequence of k-gons $\Pi_n = P_{0,n}P_{1,n}\cdots P_{k-1,n}$ be generated as in §3, with triangles $P_{0,n-1}P_{1,n-1}P'_{0,n}, \dots P_{k-1,n-1}P_{0,n-1}P'_{k-1,n}$ built on the sides of Π_{n-1} directly similar to XYZ and then normalized. The transformation matrix T_k for such a sequence is a circulant $k\times k$ matrix with the first row

$$\begin{pmatrix}1-t & t & 0 & \cdots & 0\end{pmatrix},$$

whose eigenvectors are columns of Fourier matrix

$$F_k = \frac{1}{\sqrt{k}}(\zeta_k^{ij}), \quad i,j = 0,\dots,k-1,$$

and the eigenvalues:

$$\lambda_i = (1-t) + t\zeta_k^i, \quad i = 0,\dots,k-1. \tag{13}$$

Put Π_0 into the complex plane so that its centroid $G = \frac{1}{k}\sum_{i=0}^{k-1} P_{i,0}$ is at the origin, and let V_n be a vector of points corresponding to Π_n. If $S = (s_0, \ldots, s_{k-1})$ is a vector of points that is transformed into V_0 by Fourier matrix, *i.e.*, $S = \overline{F_k}V_0$, then similar to (6), we get:

$$V_n \sim \sum_{i=0}^{k-1} s_i \lambda_i^n F_{k,i}. \tag{14}$$

Noticing that the column vectors $F_{k,q}$ correspond to regular k-gons of q-type, we have the following theorem:

Theorem 6. *The sequence of k-gons Π_n converges to a regular k-gon of q-type, if and only if $|\lambda_q| > |\lambda_i|$ for every $i \neq q$ such that $s_i \neq 0$.*

As in the case of triangles, we proceed to the cases when the sequence converges after a finite number of iterations. As follows immediately from (14), we may distinguish between two possibilities:

(i) $s_q \neq 0$ and $s_i = 0$ for every $i \neq q$. This corresponds to Π_0 - and the whole sequence - being regular of q-type.

(ii) There are two integers q, r such that $\lambda_r = 0$, $s_q, s_r \neq 0$, and $s_i = 0$ for every $i \neq q, r$. In this case, Π_0 turns into regular k-gon of q-type after the first iteration. An example will be in order here. Let $k = 4$, $q = 1$, $\lambda_2 = 0$ and $S = (0, 1, 1, 0)$. Then, $t = \frac{1}{2}$ and Π_0 is a concave kite-shaped quadrilateral; the midpoints of its sides form a square, as depicted in Figure 7.

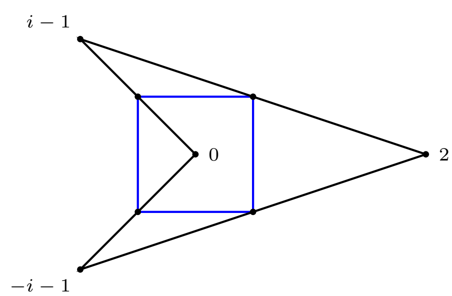

Figure 7

The last theorem shows that the convergence of the sequence of polygons depends on the shapes of both the transformation triangle and the original polygon Π_0. Let us now consider for what transformation triangles the sequence converges for any Π_0? Obviously, this will be the case if no two eigenvalues (13) have equal moduli. That is, for every pair of distinct integers q, r,

$$|(1-t) + t\zeta_k^q| \neq |(1-t) + t\zeta_k^r|.$$

Dividing both sides by $1 - t$, we conclude that $\frac{t}{1-t}\zeta_k^q$ and $\frac{t}{1-t}\zeta_k^r$ should not be complex conjugates, that is:

$$\arg\left(\frac{t}{1-t}\right) \neq -\frac{q+r}{k}\pi, \qquad 0 \leq q, r \leq k. \tag{15}$$

Solving for t and designating ℓ for $(q+r) \bmod k$, we get:

$$\frac{\mathrm{Im}(t)}{\mathrm{Re}(t)-|t|^2} \neq \frac{\ell}{k}\pi, \quad 0 \le \ell < k.$$

This last inequality is given a geometric interpretation in the following final theorem.

Theorem 7. *The sequence of k-gons converges to a regular k-gon for every Π_0 if and only if $t = z(X, Y, Z)$ does not lie on any circle $O(R)$ defined as follows:*

$$O = \left(\frac{1}{2},\ \frac{1}{2}\cot\frac{\ell}{k}\pi\right), \quad R = \frac{1}{2}\csc\frac{\ell}{k}\pi, \qquad 0 \le \ell < k.$$

We conclude with a curious application of the last result. Let $k = 5$, and XYZ be a negatively oriented equilateral triangle, *i.e.*, $t = \overline{\eta}$. It follows from (15) that the sequence of pentagons converges for any given Π_0. Let Π_0 be similar to

$$(1+\epsilon,\ \zeta_5,\ \zeta_5^2,\ \zeta_5^3,\ \zeta_5^4).$$

Taking $\epsilon \neq 0$ sufficiently small, Π_0 may be made as close to the regular convex pentagon as we please. The striking fact is that $q = 2$! Figures 8 depict this transforming of an "almost regular" convex pentagon into an "almost regular" pentagram in just 99 iterations.

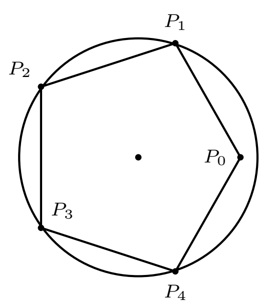

Figure 8a: $n = 0$

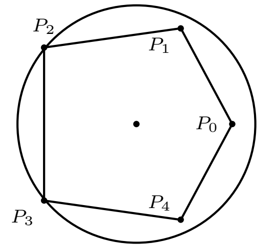

Figure 8b: $n = 20$

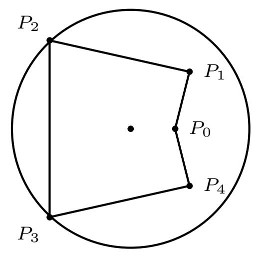

Figure 8c: $n = 40$

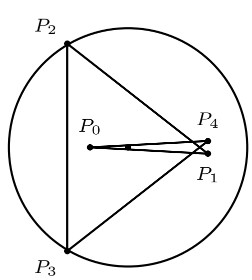

Figure 8d: $n = 60$

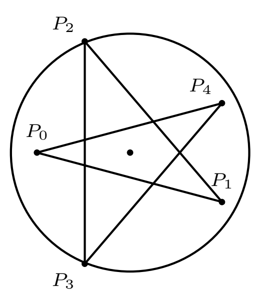

Figure 8e: $n = 80$

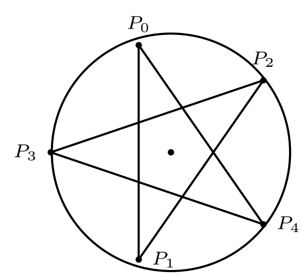

Figure 8f: $n = 99$

References

[1] E. R. Berlekamp, E. N. Gilbert, and F. W. Sinden, A polygon problem, *Amer. Math. Monthly*, 72 (1965) 233–241.
[2] J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer-Verlag, 1996.
[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, New Math. Library, vol. 19, Random House and L. W. Singer, NY, 1967; reprinted by Math. Assoc. America.
[4] P. J. Davis, *Circulant matrices*, John Wiley & Sons, NY, 1979.
[5] J. Douglass, On linear polygon transformation, *Bull. Amer. Math. Soc.*, 46 (1940), 551–560.
[6] L. Gerber, Napoleon's theorem and the parallelogram inequality for affine-regular polygons, *Amer. Math. Monthly*, 87 (1980) 644–648.
[7] J. G. Mauldon, Similar triangles, *Math. Magazine*, 39 (1966) 165–174.
[8] J. F. Rigby, Napoleon revisited. *Journal of Geometry*, 33 (1988) 129–146.
[9] I. J. Schoenberg, The finite Fourier series and elementary geometry, *Amer. Math. Monthly*, 57 (1950) 390–404.
[10] J. E. Wetzel, Converses of Napoleon's theorem, *Amer. Math. Monthly*, 99 (1992) 339–351.

Barukh Ziv: Havatzelet 4-5, Nesher 36730, Israel
E-mail address: `zbaruh@yahoo.co.uk`