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An Elementary Proof of the Isoperimetric Inequality

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Abstract. We give an elementary proof of the isoperimetric inequality for polygons, simplifying the proof given by T. Bonnesen.

We present an elementary proof of the known inequality $\mathcal{L}^2 \geq 4\pi\mathcal{A}$, where $\mathcal{L}$ and $\mathcal{A}$ are the perimeter and the area of a polygon. It simplifies the proof given by T. Bonnesen [1, 2].

Theorem. *In every polygon with perimeter $\mathcal{L}$ and area $\mathcal{A}$ we have $\mathcal{L}^2 \geq 4\pi\mathcal{A}$.*

Proof. It is sufficient to prove the inequality for a convex polygon $ABM \cdots Z$. From the vertex A of the polygon we can draw the segment AQ dividing the polygon in two polygons such that

(1) $AB + BM + \cdots + PQ = \frac{\mathcal{L}}{2}$, and
(2) the area $\mathcal{A}_1$ of the polygon $ABM \cdots PQA$ satisfies $\mathcal{A}_1 \geq \frac{\mathcal{A}}{2}$.

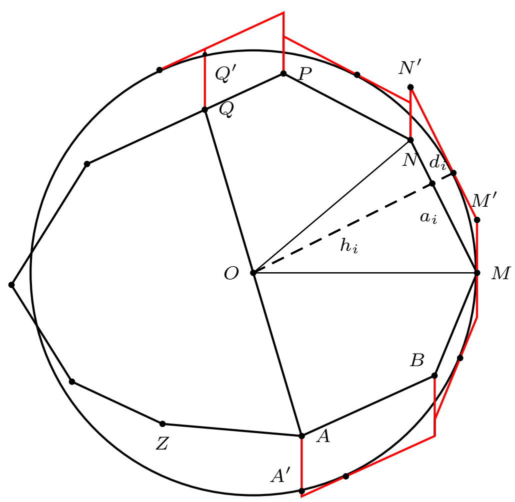

Figure 1

Let O be the mid-point of AQ, and let M be the vertex of $ABM \cdots PQA$ farthest from O, with $OM = \mathcal{R}$. Draw the circle $(O, \mathcal{R})$, and from the points A and Q draw perpendiculars to OM to meet the circle at A', Q' respectively. Because of symmetry, the part of the circle $AA'MQ'QA$ has area $\mathcal{S}$ equal to half of the area of the circle, *i.e.*, $\mathcal{S} = \frac{1}{2}\pi\mathcal{R}^2$. Outside the polygon $ABM \cdots PQ$ construct parallelograms touching the circle, with bases the sides such as $MN = a_i$ and

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other sides parallel to AA'. If h_i is the altitude of triangle OMN and d_i is the height of the parallelogram $MM'N'N$, then $h_i + d_i = \mathcal{R}$. Note that $\mathcal{A}_1$ is the sum of the areas of triangles $OAB, \ldots, OMN, \ldots, OPQ$, *i.e.*,

$$\mathcal{A}_1 = \frac{1}{2}\sum_i a_i h_i.$$

If we denote by $\mathcal{A}_2$ the sum of the areas of the parallelograms, we have

$$\mathcal{A}_2 = \sum_i a_i d_i = \sum_i a_i(\mathcal{R} - h_i) = \mathcal{R} \cdot \frac{\mathcal{L}}{2} - 2\mathcal{A}_1.$$

Since $\mathcal{A}_1 + \mathcal{A}_2 \geq \mathcal{S}$, we have $\mathcal{R} \cdot \frac{\mathcal{L}}{2} - \mathcal{A}_1 \geq \frac{1}{2}\pi\mathcal{R}^2$, and so $\pi\mathcal{R}^2 - \mathcal{L}\mathcal{R} + 2\mathcal{A}_1 \leq 0$. Rewriting this as

$$\pi\left(\mathcal{R} - \frac{\mathcal{L}}{2\pi}\right)^2 - \left(\frac{\mathcal{L}^2}{4\pi} - 2\mathcal{A}_1\right) \leq 0,$$

we conclude that $\mathcal{L}^2 \geq 4\pi \cdot 2\mathcal{A}_1 \geq 4\pi\mathcal{A}$. □

The above inequality, by means of limits can be extended to a closed curve. Since for the circle the inequality becomes equality, we conclude that of all closed curves with constant perimeter $\mathcal{L}$, the curve that contains the maximum area is the circle.

References

[1] T. Bonnesen, *Les Problèmes des Isopérimètres et des Isépiphanes*, Paris, Gauthier-Villars 1929; pp. 59-61.

[2] T. Bonnesen and W. Fenchel, *Theorie der Convexen Körper*, Chelsea Publishing, New York, 1948; S.111-112.

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