Forum Geometricorum
Volume 2 (2002) 131–134.

FORUM GEOM
ISSN 1534-1178

The Perimeter of a Cevian Triangle

Nikolaos Dergiades

Abstract. We show that the cevian triangles of certain triangle centers have perimeters not exceeding the semiperimeter of the reference triangle. These include the incenter, the centroid, the Gergonne point, and the orthocenter when the given triangle is acute angled.

1. Perimeter of an inscribed triangle

We begin by establishing an inequality for the perimeter of a triangle inscribed in a given triangle ABC.

Proposition 1. *Consider a triangle ABC with $a \leq b \leq c$. Denote by X, Y, Z the midpoints of the sides BC, CA, and AB respectively. Let D, E, F be points on the sides BC, CA, AB satisfying the following two conditions:*

(1.1) *D is between X and C, E is between Y and C, and F is between Z and B.*

(1.2) *$\angle CDE \leq \angle BDF$, $\angle CED \leq \angle AEF$, and $\angle BFD \leq \angle AFE$.*

Then the perimeter of triangle DEF does not exceed the semiperimeter of triangle ABC.

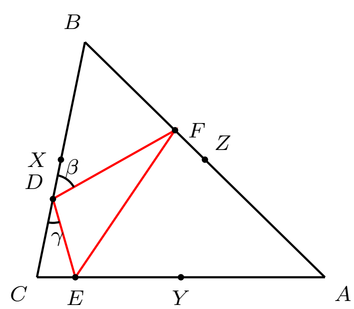

Figure 1

Proof. Denote by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along $\mathbf{EF}$, $\mathbf{FD}$, $\mathbf{DE}$. See Figure 1. Since $\angle BFD \leq \angle AFE$, we have $\mathbf{i} \cdot \mathbf{ZF} \leq \mathbf{j} \cdot \mathbf{ZF}$. Similarly, since $\angle CDE \leq \angle BDF$ and $\angle CED \leq \angle AEF$, we have $\mathbf{j} \cdot \mathbf{XD} \leq \mathbf{k} \cdot \mathbf{XD}$ and $\mathbf{i} \cdot \mathbf{EY} \leq \mathbf{k} \cdot \mathbf{EY}$. Now, we have

$$\begin{aligned} EF + FD + DE &= \mathbf{i} \cdot \mathbf{EF} + \mathbf{j} \cdot \mathbf{FD} + \mathbf{k} \cdot \mathbf{DE} \\ &= \mathbf{i} \cdot (\mathbf{EY} + \mathbf{YZ} + \mathbf{ZF}) + \mathbf{j} \cdot (\mathbf{FZ} + \mathbf{ZX} + \mathbf{XD}) + \mathbf{k} \cdot \mathbf{DE} \\ &\leq (\mathbf{k} \cdot \mathbf{EY} + \mathbf{i} \cdot \mathbf{YZ} + \mathbf{j} \cdot \mathbf{ZF}) \\ &\quad + (\mathbf{j} \cdot \mathbf{FZ} + \mathbf{j} \cdot \mathbf{ZX} + \mathbf{k} \cdot \mathbf{XD}) + \mathbf{k} \cdot \mathbf{DE} \end{aligned}$$

Publication Date: November 1, 2002. Communicating Editor: Paul Yiu.

$$\begin{aligned} &= \mathbf{i}\cdot\mathbf{YZ}+\mathbf{j}\cdot\mathbf{ZX}+\mathbf{k}\cdot\mathbf{XY} \\ &\le |\mathbf{i}||\mathbf{YZ}|+|\mathbf{j}||\mathbf{ZX}|+|\mathbf{k}||\mathbf{XY}| \qquad (1) \\ &= YZ+ZX+XY \\ &= \frac{1}{2}(AB+BC+CA). \end{aligned}$$

Equality holds in (1) only when the triangles DEF and XYZ have parallel sides, *i.e.*, when the points D, E, F coincide with the midpoints X, Y, Z respectively, as is easily seen. □

2. Cevian triangles

Proposition 2. *Suppose the side lengths of triangle ABC satisfy $a \le b \le c$. Let P be an interior point with (positive) homogeneous barycentric coordinates $(x:y:z)$ satisfying*

(2.1) $x \le y \le z$,
(2.2) $x\cot A \ge y\cot B \ge z\cot C$.

Then the perimeter of the cevian triangle of P does not exceed the perimeter of the medial triangle of ABC, i.e.*, the cevian triangle of the centroid.*

Proof. In Figure 1, $BD = \frac{az}{y+z}$, $DC = \frac{ay}{y+z}$, and $BF = \frac{cx}{x+y}$. Since $y \le z$, it is clear that $BD \ge DC$. Similarly, $AE \ge EC$, and $AF \ge FB$. Condition (1.1) is satisfied. Applying the law of sines to triangle BDF, we have $\frac{\sin(B+\beta)}{\sin\beta} = \frac{BD}{BF}$. It follows that

$$\frac{\sin(B+\beta)}{\sin B\sin\beta} = \frac{\sin(B+C)}{\sin B\sin C}\cdot\frac{z(x+y)}{x(y+z)}.$$

From this, $\cot\beta+\cot B = (\cot B+\cot C)\cdot\frac{z(x+y)}{x(y+z)}$. Similarly, $\cot\gamma+\cot C = (\cot B+\cot C)\cdot\frac{y(z+x)}{x(y+z)}$. Consequently,

$$\cot\gamma-\cot\beta = \frac{2(y\cot B - z\cot C)}{y+z},$$

so that $\beta \ge \gamma$ provided $y\cot B \ge z\cot C$. The other two inequalities in (1.2) can be similarly established. The result now follows from Proposition 1. □

This applies, for example, to the following triangle centers. For the case of the orthocenter, we require the triangle to be acute-angled. [1] It is easy to see that the barycentrics of each of these points satisfy condition (2.1).

P	$(x:y:z)$	$x\cot A \ge y\cot B \ge z\cot C$
Incenter	$(a:b:c)$	$\cos A \ge \cos B \ge \cos C$
Centroid	$(1:1:1)$	$\cot A \ge \cot B \ge \cot C$
Orthocenter	$(\tan A:\tan B:\tan C)$	$1 \ge 1 \ge 1$
Gergonne point	$\left(\tan\frac{A}{2}:\tan\frac{B}{2}:\tan\frac{C}{2}\right)$	$\frac{1}{2}(1-\tan^2\frac{A}{2}) \ge \frac{1}{2}(1-\tan^2\frac{B}{2})$ $\ge \frac{1}{2}(1-\tan^2\frac{C}{2})$

[1]For the homogeneous barycentric coordinates of triangle centers, see [1].

The perimeter of the cevian triangle of each of these points does not exceed the semiperimeter of ABC.[2] The case of the incenter can be found in [2].

3. Another example

The triangle center with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$ provides another example of a point P the perimeter of whose cevian triangle not exceeding the semiperimeter of ABC. It clearly satisfies (2.1). Since $\sin\frac{A}{2}\cot A = \cos\frac{A}{2} - \frac{1}{2\cos\frac{A}{2}}$, it also satisfies condition (2.2). In [1], this point appears as X_{174} and is called the Yff center of congruence. Here is another description of this triangle center [3]:

The tangents to the incircle at the intersections with the angle bisectors farther from the vertices intersect the corresponding sides at the traces of the point with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$.

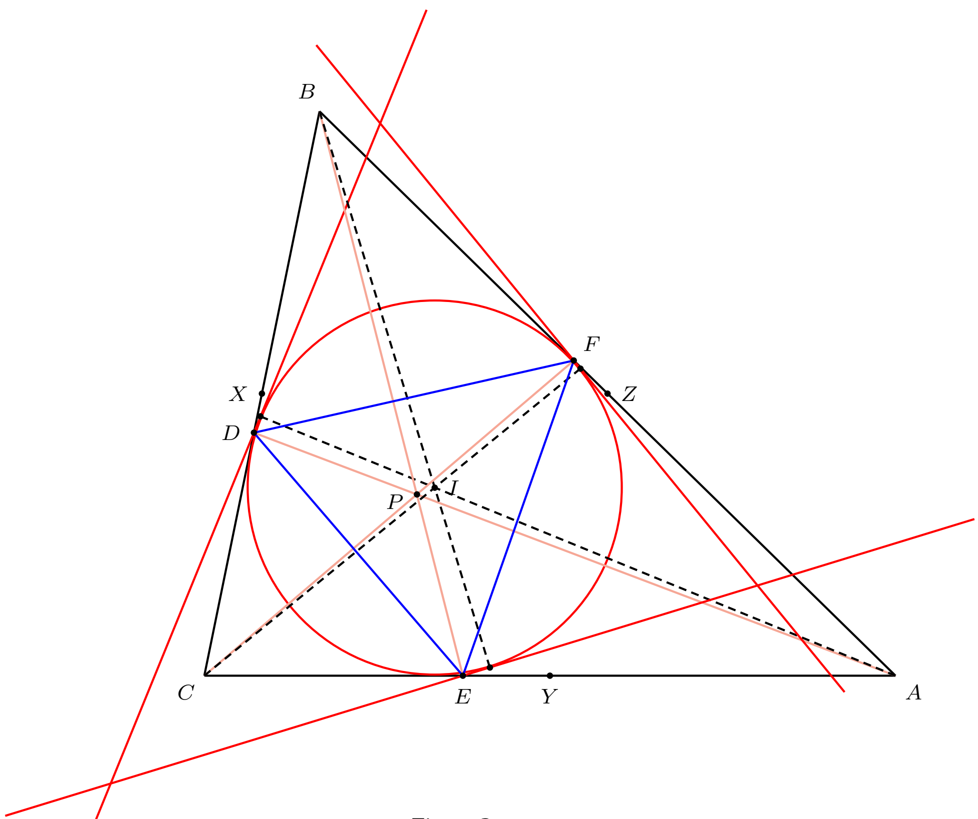

Figure 2

[2]The Nagel point, with homogeneous barycentric coordinates $(\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2})$, also satisfies (2.2). However, it does not satisfy (2.1) so that the conclusion of Proposition 2 does not apply. The same is true for the circumcenter.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.

[2] T. Seimiya and M. Bataille, Problem 2502 and solution, *Crux Math.*, 26 (2000) 45; 27 (2001) 53–54.

[3] P. Yiu, Hyacinthos message 2114, http://groups.yahoo.com/group/Hyacinthos/message/2114, December 18, 2000.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr