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On Some Remarkable Concurrences

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Abstract. In [2], Bruce Shawyer proved the following result : "At the midpoint of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. We show that the three lines obtained are always concurrent. Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the sides of the triangle at the midpoints of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle". Here, we extend these results in the projective plane and give a short synthethic proof.

We work in the complex or the real complexified projective plane $\mathcal{P}$. The conic through five points A, B, C, D, E is denoted by $\mathcal{C}(A, B, C, D, E)$ and $(XYZW)$ is the notation for the cross-ratio of four collinear points X, Y, Z, W.

Theorem 1. *Consider a triangle ABC and a line l, not through A, B or C, in $\mathcal{P}$. Put $AB \cap l = C'', BC \cap l = A''$, $CA \cap l = B''$ and construct the points A', B' and C' for which $(BCA'A'') = (CAB'B'') = (ABC'C'') = -1$. Then, take two different points I and I' on l (both different from A'', B'', C'') and consider the points A''', B''' and C''' such that $(II'A''A''') = (II'B''B''') = (II'C''C''') = -1$. Then the lines $A'A''', B'B'''$ and $C'C'''$ are concurrent at a point L.*

Proof. The line $A'A'''$ is clearly the polar line of A'' with respect to the conic $\mathcal{C}(A, B, C, I, I')$ and likewise for the line $B'B'''$ and B'', and for the line $C'C'''$ and C''. Thus, $A'A''', B'B'''$ and $C'C'''$ concur at the polar point L of l with respect to $\mathcal{C}(A, B, C, I, I')$. □

Theorem 2. *If I, I' are variable conjugate points in an involution Ω on the line l with double (or fixed) points D and D', then the locus of the point L is the conic $\mathcal{L} = \mathcal{C}(A', B', C', D, D')$. Moreover, putting $\mathcal{L} \cap AB = \{C', Z\}$, $\mathcal{L} \cap BC = \{A', X\}$ and $\mathcal{L} \cap CA = \{B', Y\}$, the triangles ABC and XYZ form a Ceva configuration. The point K of concurrency of the Cevians AX, BY, CZ is the fourth basis point (besides A, B, C) of the pencil of conics $\mathcal{C}(A, B, C, I, I')$.*

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Proof. Since the conics $\mathcal{C}(A,B,C,I,I')$ intersect the line l in the variable conjugate points I,I' of an involution on l, these conics must belong to a pencil with basis points A,B,C and a fourth point K : this follows from the Theorem of Desargues-Sturm (see [1], page 63). So, the locus $\mathcal{L}$ is the locus of the polar point L of the line l with respect to the conics of this pencil. Now, it is not difficult to prove (or even well known) that such locus is the conic through the points $A',B,'C',D,D'$ and through the points K',K'',K''' which are determined by $(AKK'K_1)=(BKK''K_2)=(CKK'''K_3)=-1$, where $K_1=l\cap KA, K_2=l\cap KB$ and $K_3=l\cap KC$, and finally, through the singular points $X=KA\cap BC$, $Y=KB\cap CA, Z=KC\cap AB$ of the degenerate conics of the pencil. This completes the proof. □

Next, let us consider a special case of the foregoing theorems in the Euclidean plane Π. Take a triangle ABC in Π and let $l=l_\infty$ be the line at infinity, while the points D and D' of theorem 2 are the points at infinity of the X-axis and the Y-axis of the rectangular coordinate system in Π, respectively.

Homogeneous coordinates in Π are (x,y,z) and $z=0$ is the line l_∞; the points D and D' have coordinates $(1,0,0)$ and $(0,1,0)$, respectively. A line with slope a has an equation $y=ax+bz$ and point at infinity $(1,a,0)$. Now, if (in Theorem 1) the product of the slopes of the lines BC and $A'A'''$, CA and $B'B'''$, AB and $C'C'''$ is a fixed constant $\lambda(\neq 0)$, then the points at infinity of these lines (i.e. A'' and A''', B'' and B''', C'' and C''') have coordinates of the form $(1,t,0)$ and $(1,t',0)$, with $tt'=\lambda$. This means that A'' and A''', B'' and B''', C'' and C''' are conjugate points in the involution on l_∞ with double points $I(1,-\sqrt{\lambda},0)$ and $I'(1,\sqrt{\lambda},0)$ and thus $(II'A''A''')=(II'B''B''')=(II'C''C''')=-1$. If we let λ be variable, the points I and I' are variable conjugate points in the involution on l_∞ with double points D and D', the latter occuring for $t=0$ and $t'=\infty$ respectively.

Now all the results of [2], given in the abstract, easily follow from Theorems 1 and 2. For instance, the locus $\mathcal{L}$ is the rectangular hyperbola $\mathcal{C}(A',B',C',D,D')$ (also) through the points K',K'',K''',X,Y,Z. Remark that the basis point K belongs to any conic $\mathcal{C}(A,B,C,I,I')$ and for $\lambda=-1$, we get that $I(1,i,0)$ and $I'(1,-i,0)$ are the cyclic points, so that $\mathcal{C}(A,B,C,I,I')$ becomes the circumcircle of ΔABC. For $\lambda=-1$, we have $A'A'''\perp BC, B'B'''\perp CA$ and $C'C'''\perp AB$, and $A'A''',B'B''',C'C'''$ concur at the center O of the circumcircle of ABC.

Remark also that O is the orthocenter of $\Delta A'B'C'$ and that any conic (like $\mathcal{L}$) through the vertices of a triangle and through its orthocenter is always a rectangular hyperbola.

At the end of his paper, B. Shawyer asks the following question : Does the Cevian intersection point K have any particular significance? It follows from the foregoing that K is a point of the parabolas through A,B,C and with centers $D(1,0,0)$ and $D'(0,1,0)$, the points at infinity of the X-axis and the Y-axis. And from this it follows that the circumcircle of any triangle ABC is the locus of the fourth common point of the two parabolas through A,B,C with variable orthogonal axes.

Next, we look for an (other) extension of the results of B. Shawyer : At the midpoint of each side of a triangle, construct the line such that the slope of this line and the slope of the side of the triangle satisfy the equation $ctt' - a(t+t') - b = 0$, with a, b and c constant and $a^2 + bc \neq 0$. Then these three lines are concurrent. This follows from Theorem 1, since the given equation determines a general non-singular involution. Shawyer's results correspond with $a = 0$ (and $\frac{b}{c} = \lambda$ and λ variable). Now, consider the special case where $c = 0$ and put $-\frac{b}{a} = \lambda$; the sum of the slopes is a constant λ or $t + t' = \lambda$. On the line l_∞ at infinity we get the corresponding points $(1, t, 0)$ and $(1, t', 0)$ and the fixed points of the involution on l_∞ determined by $t + t' = \lambda$ are $I(0, 1, 0)$ (or the point at infinity of the Y-axis) and $I'(1, \frac{\lambda}{2}, 0)$. In this case, the locus $\mathcal{L}$ of the point of concurrency L is the locus of the polar point L of the line l_∞ with respect to the conics of the pencil with basis points A, B, C and $I(0, 1, 0)$. A straightforward calculation shows that this locus $\mathcal{L}$ is the parabola through the midpoints A', B', C' of BC, CA, AB, respectively, and with center I. The second intersection points of this parabola $\mathcal{L}$ with the sides of the triangle are $X = IA \cap BC, Y = IB \cap CA$ and $Z = IC \cap AB$. Remark that IA, IB, IC are the lines parallel with the Y-axis through A, B, C, respectively.

References

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