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The Stammler Circles

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Abstract. We investigate circles intercepting chords of specified lengths on the sidelines of a triangle, a theme initiated by L. Stammler [6, 7]. We generalize his results, and concentrate specifically on the Stammler circles, for which the intercepts have lengths equal to the sidelengths of the given triangle.

1. Introduction

Ludwig Stammler [6, 7] has investigated, for a triangle with sidelengths a, b, c, circles that intercept chords of lengths μa, μb, μc ($\mu > 0$) on the sidelines BC, CA and AB respectively. He called these circles *proportionally cutting circles*,[1] and proved that their centers lie on the rectangular hyperbola through the circumcenter, the incenter, and the excenters. He also showed that, depending on μ, there are 2, 3 or 4 circles cutting chords of such lengths.

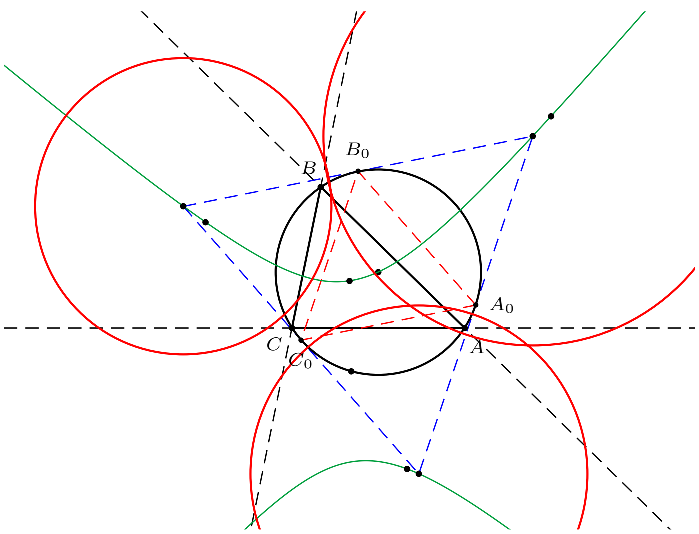

Figure 1. The three Stammler circles with the circumtangential triangle

As a special case Stammler investigated, for $\mu = 1$, the three proportionally cutting circles apart from the circumcircle. We call these the *Stammler circles*. Stammler proved that the centers of these circles form an equilateral triangle, circumscribed to the circumcircle and homothetic to Morley's (equilateral) trisector

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[1] Proportionalschnittkreise in [6].

triangle. In fact this triangle is tangent to the circumcircle at the vertices of the circumtangential triangle. [2] See Figure 1.

In this paper we investigate the circles that cut chords of specified lengths on the sidelines of ABC, and obtain generalizations of results in [6, 7], together with some further results on the Stammler circles.

2. The cutting circles

We define a (u, v, w)-cutting circle as one that cuts chords of lengths u, v, w on the sidelines BC, CA, AB of ABC respectively. This is to be distinguished from a $(u : v : w)$-cutting circle, which cuts out chords of lengths in the proportion $u : v : w$.

2.1. Consider a $(\mu u, \mu v, \mu w)$-cutting circle with center P, whose (signed) distances to the sidelines of ABC are respectively X, Y, Z. [3] It is clear that

$$Y^2 - Z^2 = \left(\frac{\mu}{2}\right)^2 (w^2 - v^2). \tag{1}$$

If $v \neq w$, this equation describes a rectangular hyperbola with center A and asymptotes the bisectors of angle A. In the same way, P also lies on the conics (generally rectangular hyperbolas)

$$Z^2 - X^2 = \left(\frac{\mu}{2}\right)^2 \left(u^2 - w^2\right) \tag{2}$$

and

$$X^2 - Y^2 = \left(\frac{\mu}{2}\right)^2 \left(v^2 - u^2\right). \tag{3}$$

These three hyperbolas generate a pencil which contains the conic with barycentric equation

$$\frac{(v^2 - w^2)x^2}{a^2} + \frac{(w^2 - u^2)y^2}{b^2} + \frac{(u^2 - v^2)z^2}{c^2} = 0. \tag{4}$$

This is a rectangular hyperbola through the incenter, excenters and the points $(\pm au : \pm bv : \pm cw)$.

Theorem 1. *The centers of the* $(u : v : w)$*-cutting circles lie on the rectangular hyperbola through the incenter and the excenters and the points with homogeneous barycentric coordinates* $(\pm au : \pm bv : \pm cw)$.

Remarks. 1. When $u = v = w$, the centers of $(u : v : w)$-cutting circles are the incenter and excenters themselves.

2. Triangle ABC is self polar with respect to the hyperbola (4).

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[2] The vertices of the circumtangential triangle are the triple of points X on the circumcircle for which the line through X and its isogonal conjugate is tangent to the circumcircle. These are the isogonal conjugates of the infinite points of the sidelines of the Morley trisector triangle. See [4] for more on the circumtangential triangle.

[3] We say that the point P has *absolute* normal coordinates (X, Y, Z) with respect to triangle ABC.

2.2. Since (1) and (2) represent two rectangular hyperbolas with distinct asymptote directions, these hyperbolas intersect in four points, of which at least two are real points. Such are the centers of $(\mu u, \mu v, \mu w)$-cutting circles. The limiting case $\mu = 0$ always yields four real intersections, the incenter and excenters. As μ increases, there is some $\mu = \mu_0$ for which the hyperbolas (1) and (2) are tangent, yielding a double point. For $\mu > \mu_0$, the hyperbolas (1, 2, 3) have only two real common points. When there are four real intersections, these form an orthocentric system. From (1), (2) and (3) we conclude that A, B, C must be on the nine point circle of this orthocentric system.

Theorem 2. *Given positive real numbers u, v, w, there are four (u, v, w)-cutting circles, at least two of which are real. When there are four distinct real circles, their centers form an orthocentric system, of which the circumcircle is the nine point circle. When two of these centers coincide, they form a right triangle with its right angle vertex on the circumcircle.*

2.3. Let (O_1) and (O_2) be two (u, v, w)-cutting circles with centers O_1 and O_2. Consider the midpoint M of O_1O_2. The orthogonal projection of M on BC clearly is the midpoint of the orthogonal projections of O_1 and O_2 on the same line. Hence, it has equal powers with respect to the circles (O_1) and (O_2), and lies on the radical axis of these circles. In the same way the orthogonal projections of M on AC and AB lie on this radical axis as well. It follows that M is on the circumcircle of ABC, its Simson-Wallace line being the radical axis of (O_1) and (O_2). See Figure 2.

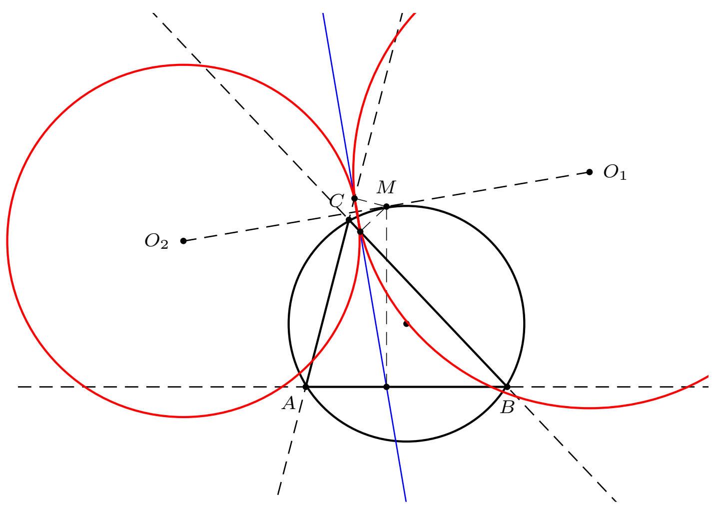

Figure 2. The radical axis of (O_1) and (O_2) is the Simson-Wallace line of M

2.4. Let Q be the reflection of the De Longchamps point L in M.[4] It lies on the circumcircle of the dilated (anticomplementary) triangle. The Simson-Wallace line

[4]The de Longchamps point L is the reflection of the orthocenter H in the circumcenter O. It is also the orthocenter of the dilated (anticomplementary) triangle.

of Q in the dilated triangle passes through M and is perpendicular to the Simson-Wallace line of M in ABC. It is therefore the line O_1O_2, which is also the same as MM^*, where M^* denotes the isogonal conjugate of M (in triangle ABC).

Theorem 3. *The lines connecting centers of (u, v, w)-cutting circles are Simson-Wallace lines of the dilated triangle. The radical axes of (u, v, w)-cutting circles are Simson-Wallace lines of ABC. When there are four real (u, v, w)-cutting circles, their radical axes form the sides of an orthocentric system perpendicular to the orthocentric system formed by the centers of the circles, and half of its size.*

2.5. For the special case of the centers O_1, O_2 and O_3 of the Stammler circles, we immediately see that they must lie on the circle $(O, 2R)$, where R is the circumradius. Since the medial triangle of $O_1O_2O_3$ must be circumscribed by the circumcircle, we see in fact that $O_1O_2O_3$ must be an equilateral triangle circumscribing the circumcircle. The sides of $O_1O_2O_3$ are thus Simson-Wallace lines of the dilated triangle, tangent to the nine point circle of the dilated triangle. See Figure 3.

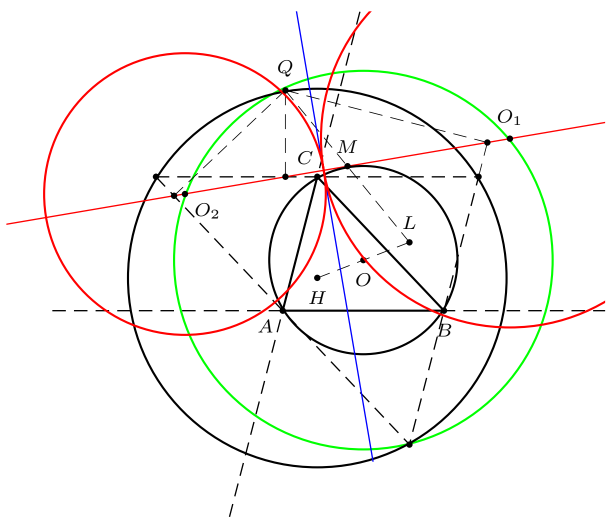

Figure 3. The line O_1O_2 is the dilated Simson-Wallace line of Q

Corollary 4. *The centers of the Stammler circles form an equilateral triangle circumscribing the circumcircle of ABC, and tangent to the circumcircle at the vertices $A_0B_0C_0$ of the circumtangential triangle. The radical axes of the Stammler circles among themselves are the Simson-Wallace lines of A_0, B_0, C_0.* [5] *The radical axes of the Stammler circles with the circumcircle are the sidelines of triangle $A_0B_0C_0$ translated by* $\mathbf{ON}$, *where N is the nine-point center of triangle ABC.*

[5] These are the three Simson-Wallace lines passing through N, *i.e.*, the cevian lines of N in the triangle which is the translation of $A_0B_0C_0$ by $\mathbf{ON}$. They are also the tangents to the Steiner deltoid at the cusps.

Remark. Since the nine-point circle of an equilateral triangle is also its incircle, we see that the centers of the Stammler circles are the only possible equilateral triangle of centers of (u, v, w)-cutting circles.

3. Constructions

3.1. Given a (u, v, w)-cutting circle with center P, let P' be the reflection of P in the circumcenter O. The centers of the other (u, v, w)-cutting circles can be found by intersecting the hyperbola (4) with the circle $P'(2R)$. One of the common points is the reflection of P in the center of the hyperbola.[6] The others are the required centers. This gives a *conic* construction. In general, the points of intersection are not constructible by ruler and compass. See Figure 4.

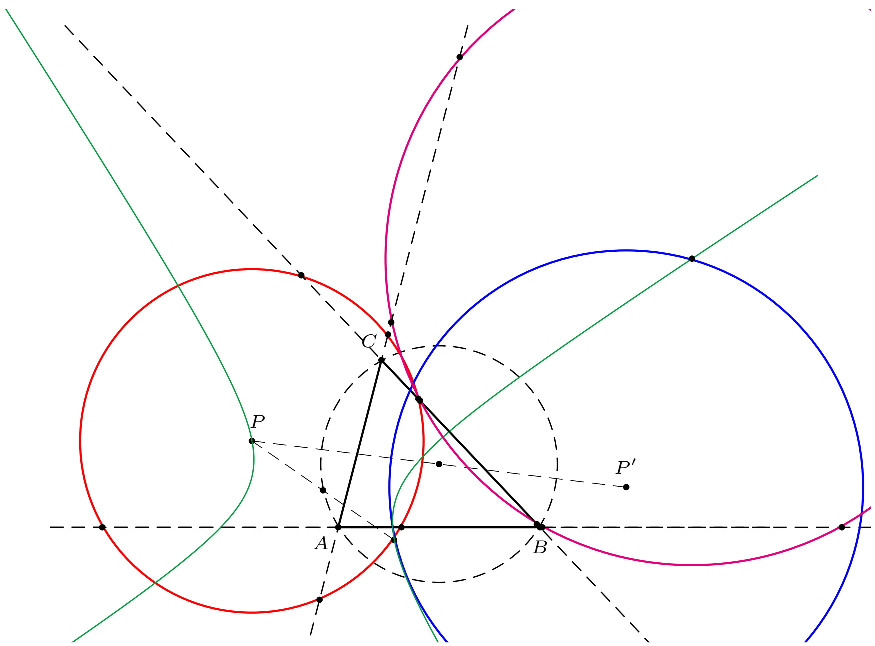

Figure 4. Construction of (u, v, w)-cutting circles

3.2. The same method applies when we are only given the magnitudes u, v, w. The centers of (u, v, w)-cutting circles can be constructed as the common points of the hyperbolas (1), (2), (3) with $\mu = 1$. If we consider two points T_A, T_B lying respectively on the lines CB, CA and such as $CT_A = u$, $CT_B = v$, the hyperbola (3) passes through the intersection M_0 of the perpendicular bisectors of CT_A and CT_B. Its asymptotes being the bisectors of angle C, a variable line through M_0 intersects these asymptotes at D, D'. The reflection of M_0 with respect to the midpoint of DD' lies on the hyperbola.

[6]The center of the hyperbola (4) is the point $\left(\frac{a^2}{v^2-w^2} : \frac{b^2}{w^2-u^2} : \frac{c^2}{u^2-v^2}\right)$ on the circumcircle.

3.3. When two distinct centers P and P' are given, then it is easy to construct the remaining two centers. Intersect the circumcircle and the circle with diameter PP', let the points of intersection be U and U'. Then the points $Q = PU \cap P'U'$ and $Q' = PU' \cap P'U$ are the points desired.

When one center P on the circumcircle is given, then P must in fact be a double point, and thus the right angle vertex of a right triangle containing the three $(u, v, w)-$intercepting circles. As the circumcircle of ABC is the nine point circle of the right triangle, the two remaining vertices must lie on the circle through P with P_r as center, where P_r is the reflection of P through O. By the last sentence before Theorem 3, we also know that the two remaining centers must lie on the line $P_rP_r^*$. Intersection of circle and line give the desired points.

3.4. Let three positive numbers u, v and w be given, and let P be a point on the hyperbola of centers of $(u : v : w)$-cutting circles. We can construct the circle with center P intercepting on the sidelines of ABC chords of lengths μu, μv and μw respectively for some μ.

We start from the point Q with barycentrics $(au : bv : cw)$. Let X, Y and Z be the distances from P to BC, AC and AB respectively. Since P satisfies (4) we have

$$(v^2 - w^2)X^2 + (w^2 - u^2)Y^2 + (u^2 - v^2)Z^2 = 0, \tag{5}$$

which is the equation in normal coordinates of the rectangular hyperbola through Q, the incenter and the excenters.

Now, the parallel through Q to AC (respectively AB) intersects AB (respectively AC) in Q_1 (respectively Q_2). The line perpendicular to Q_1Q_2 through P intersects AQ at U. The power p_a of P with respect to the circle with diameter AU is equal to $\dfrac{w^2Y^2 - v^2Z^2}{w^2 - v^2}$. Similarly we find powers p_b and p_c.

As P lies on the hyperbola given by (5), we have $p_a = p_b = p_c$. Define ρ by $\rho^2 = p_a$. Now, the circle (P, ρ) intercepts chords of with lengths L_a, L_b, L_c respectively on the sidelines of ABC, where

$$\left(\frac{L_a}{L_b}\right)^2 = \frac{\rho^2 - X^2}{\rho^2 - Y^2} = \frac{p_c - X^2}{p_c - Y^2} = \left(\frac{u}{v}\right)^2$$

and similarly

$$\left(\frac{L_b}{L_c}\right)^2 = \left(\frac{v}{w}\right)^2.$$

Hence this circle (P, ρ), if it exists and intersects the side lines, is the required circle. To construct this circle, note that if U' is the midpoint of AU, the circle goes through the common points of the circles with diameters AU and PU'.

4. The Stammler circles

For some particular results on the Stammler circles we use complex number coordinates. Each point is identified with a complex number $\rho \cdot e^{i\theta}$ called its *affix*. Here, (ρ, θ) are the polar coordinates with the circumcenter O as pole, scaled in such a way that points on the circumcircle are represented by unit complex numbers. Specifically, the vertices of the circumtangential triangle are represented by the cube roots of unity, namely,

$$A_0 = 1, \quad B_0 = \omega, \quad C_0 = \omega^2 = \overline{\omega},$$

where $\omega^3 = 1$. In this way, the vertices A, B, C have as affixes unit complex numbers $A = e^{i\theta}$, $B = e^{i\varphi}$, $C = e^{i\psi}$ satisfying $\theta + \varphi + \psi \equiv 0 \pmod{2\pi}$. In fact, we may take

$$\theta = \frac{2}{3}(\beta - \gamma), \quad \varphi = \frac{2}{3}(\beta + 2\gamma), \quad \psi = -\frac{2}{3}(2\beta + \gamma), \tag{6}$$

where α, β, γ are respectively the measures of angles A, B, C. In this setup the centers of the Stammler circles are the points

$$\Omega_A = -2, \qquad \Omega_B = -2\omega, \qquad \Omega_C = -2\overline{\omega}.$$

4.1. The intersections of the A-Stammler circle with the sidelines of ABC are

$$\begin{array}{ll} A_1 = B + \overline{A} - 1, & A_2 = C + \overline{A} - 1, \\ B_1 = C + \overline{B} - 1, & B_2 = A + \overline{B} - 1, \\ C_1 = A + \overline{C} - 1, & C_2 = B + \overline{C} - 1. \end{array}$$

The reflections of A, B, C in the line B_0C_0 are respectively

$$A' = -1 - \overline{A}, \qquad B' = -1 - \overline{B}, \qquad C' = -1 - \overline{C}.$$

The reflections of A', B', C' respectively in BC, CA, AB are

$$A'' = (1 + B)(1 + C), \qquad B'' = (1 + C)(1 + A), \qquad C'' = (1 + A)(1 + B).$$

Now,

$$\begin{aligned} B'' - A'' &= B_2 - A_1 = \frac{2}{\sqrt{3}} (\sin\theta - \sin\varphi) (C_0 - B_0), \\ C'' - B'' &= C_2 - B_1 = \frac{2}{\sqrt{3}} (\sin\varphi - \sin\psi)(C_0 - B_0), \\ A'' - C'' &= A_2 - C_1 = \frac{2}{\sqrt{3}} (\sin\psi - \sin\theta)(C_0 - B_0). \end{aligned}$$

Moreover, as the orthocenter $H = A + B + C$, the points $\Omega_B + H$ and $\Omega_C + H$ are collinear with $A''B''C''$.

4.2. Let R_A be the radius of the A-Stammler circle. It is easy to check that the twelve segments $A'B$, $A'C$, $A''B$, $A''C$, $B'C$, $B'A$, $B''C$, $B''A$, $C'A$, $C'B$, $C''A$, $C''B$ all have length equal to $R_A = \Omega_A A_1$. See Figure 6. Making use of the affixes, we easily obtain

$$R_A^2 = 3 + 2 (\cos\theta + \cos\varphi + \cos\psi). \tag{7}$$

Theorem 5. *From the points of intersection of each of the Stammler circles with the sidelines of ABC three chords can be formed, with the condition that each chord is parallel to the side of Morley's triangle corresponding to the Stammler circle. The smaller two of these chords together are as long as the greater one.*

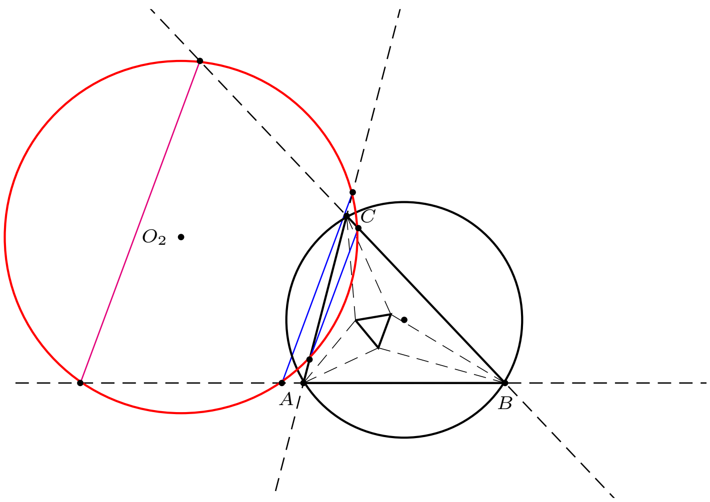

Figure 5. Three parallel chords on the $B-$Stammler circle

Remark. This is indeed true for any conic intercepting chords of lengths a, b, c on the sidelines.

4.3. We investigate the triangles $P_AP_BP_C$ with P_AB, P_AC, P_BA, P_BC, P_CA, P_CB all of length $\rho = \sqrt{\nu}$, which are perspective to ABC through P. Let P have homogeneous barycentric coordinates $(p : q : r)$. The line AP and the perpendicular bisector of BC meet in the point

$$P_A = \left(-(q-r)a^2 : q(b^2-c^2) : r(b^2-c^2)\right).$$

With the distance formula,[7] we have

$$|P_AB|^2 = a^2\frac{(a^2(c^2q^2+b^2r^2)+((b^2-c^2)^2-a^2(b^2+c^2))qr}{((a^2-b^2+c^2)q-(a^2+b^2-c^2)r)^2}$$

Similarly we find expressions for the squared distances $|P_BC|^2$ and $|P_CA|^2$.

Now let $|P_AB|^2 = |P_BC|^2 = |P_CA|^2 = \nu$. From these three equations we can eliminate q and r. When we simplify the equation assuming that ABC is nonisosceles and nondegenerate, this results in

$$p\nu(-16\triangle^2\nu+a^2b^2c^2)(-16\triangle^2\nu^3+a^2b^2c^2(9\nu^2-3(a^2+b^2+c^2)\nu+a^2b^2+b^2c^2+a^2c^2)) = 0. \tag{8}$$

Here, $\triangle$ is the area of triangle ABC. One real solution is clearly $\rho = \frac{a^2b^2c^2}{16\triangle^2} = R^2$. The other nonzero solutions are the roots of the cubic equation

$$\nu^3 - R^2(9\nu^2 - 3(a^2+b^2+c^2)\nu + a^2b^2+b^2c^2+a^2c^2) = 0. \tag{9}$$

[7] See for instance [5, Proposition 2].

As $A'B'C'$ is a particular solution of the problem, the roots of this cubic equation are the squares of the radii of the Stammler circles. A simple check of cases shows that the mentioned solutions are indeed the only ones.

Theorem 6. *Reflect the vertices of ABC through one of the sides of the circum-tangential triangle to A', B' and C'. Then $A'B'C'$ lie on the perpendicular bisectors. In particular, together with O as a triple point and the reflections of O through the sides of ABC these are the only triangles perspective to ABC with $A'B = A'C = B'A = B'C = C'A = C'B$, for nonisosceles (and nondegenerate) ABC.*

Remark. Theorem 6 answers a question posed by A. P. Hatzipolakis [3].

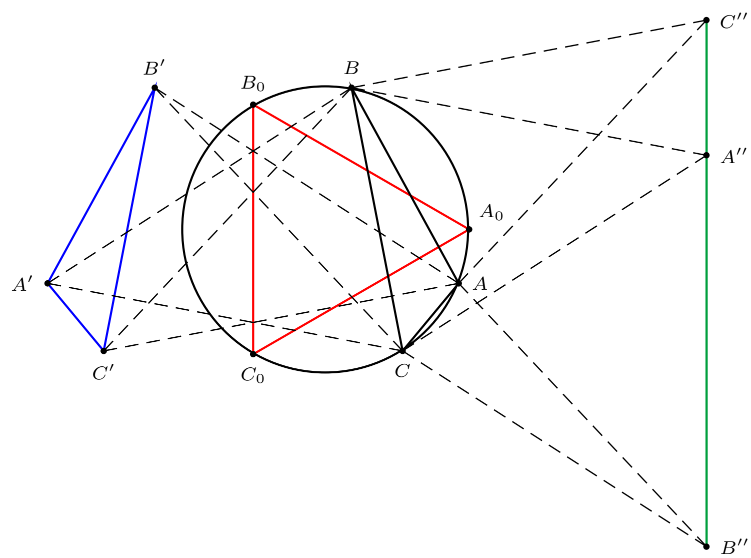

Figure 6. A perspective triangle $A'B'C'$ and the corresponding degenerate $A''B''C''$

4.4. Suppose that three points U, V, W lie on a same line ℓ and that $UB = UC = VC = VA = WA = WB = r \neq R$.

Let z_a the signed distance from A to ℓ. We have $\tan(\ell, BC) = 2 \cdot \dfrac{z_b - z_c}{\overline{VW}}$ and $z_a^2 = r^2 - \frac{1}{4}\overline{VW}^2$. It follows that

$$(\ell, BC) + (\ell, CA) + (\ell, AB) = 0,$$

and ℓ is parallel to a sideline of the Morley triangle of ABC. See [2, Proposition 5]. Now, U, V, W are the intersections of ℓ with the perpendicular bisectors of ABC and, for a fixed direction of ℓ, there is only one position of ℓ for which $VA = WA \neq R$. Hence the degenerate triangles $A''B''C''$, together with O as a triple point, are the only solutions in the collinear cases with $A''B = A''C = B''A = B''C = C''A = C''B$.

Theorem 7. *Reflect $A'B'C'$ through the sides of ABC respectively to A'', B'', C''. Then $A''B''C''$ are contained in the same line ℓ_i parallel to the side L_i of the circumtangential triangle. Together with O as a triple point these are the only degenerate triangles $A''B''C''$ satisfying the condition $A''B = A''C = B''A = B''C = C''A = C''B$. The lines ℓ_A, ℓ_B, ℓ_C bound the triangle which is the translation of $\Omega_A\Omega_B\Omega_C$ through the vector $\mathbf{OH}$.*

The three segments from $A''B''C''$ are congruent to the chords of Theorem 5. See Figure 6.

4.5. With θ, φ, ψ given by (6), we obtain from (7), after some simplifications,

$$\left(\frac{R_A}{R}\right)^2 = 1 + 8\cos\frac{\beta-\gamma}{3}\cos\frac{\beta+2\gamma}{3}\cos\frac{2\beta+\gamma}{3}.$$

Since $\left(\frac{OH}{R}\right)^2 = 1 - 8\sin\alpha\sin\beta\sin\gamma$, (see, for instance, [1, Chapter XI]), this shows that the radius R_A can be constructed, allowing angle trisection. R_A is the distance from O to the orthocenter of the triangle $AB'C'$, where B' is the image of B after rotation through $\frac{2(\beta-\gamma)}{3}$ about O, and C' is the image of A after rotation through $\frac{2(\gamma-\beta)}{3}$ about O.

The barycentric coordinates of Ω_A are

$$\left(a\left(\cos\alpha - 2\cos\frac{\beta-\gamma}{3}\right) : b\left(\cos\beta + 2\cos\frac{\beta+2\gamma}{3}\right) : c\left(\cos\gamma + 2\cos\frac{2\beta+\gamma}{3}\right)\right).$$

We find the distances

$$\begin{array}{ll} B_1C_2 = 2a\cos\frac{\beta-\gamma}{3}, & BA_1 = CA_2 = 2R\sin\frac{|\beta-\gamma|}{3}, \\ C_1A_2 = 2b\cos\frac{\beta+2\gamma}{3}, & CB_1 = AB_2 = 2R\sin\frac{\beta+2\gamma}{3}, \\ A_1B_2 = 2c\cos\frac{2\beta+\gamma}{3}, & AC_1 = BC_2 = 2R\sin\frac{2\beta+\gamma}{3}. \end{array}$$

Finally we mention the following relations of the Stammler radii. These follow easily from the fact that they are the roots of the cubic equation (9).

$$R_A^2 + R_B^2 + R_C^2 = 9R^2;$$
$$\frac{1}{R_A^2} + \frac{1}{R_B^2} + \frac{1}{R_C^2} = \frac{3(a^2+b^2+c^2)}{a^2b^2+a^2c^2+b^2c^2};$$
$$R_AR_BR_C = R\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

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