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Some Similarities Associated with Pedals

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Abstract. The pedals of a point divide the sides of a triangle into six segments. We build on these segments six squares and obtain some interesting similarities.

Given a triangle ABC, the pedals of a point P are its orthogonal projections A', B', C' on the sidelines BC, CA, AB of the triangle. We build on the segments AC', $C'B$, BA', $A'C$, CB' and $B'A$ squares with orientation opposite to that of ABC.

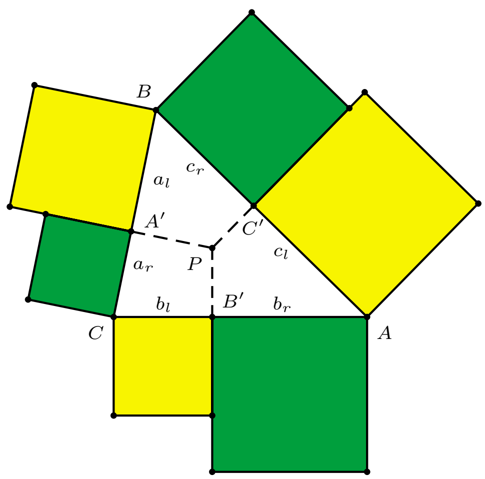

Figure 1

About this figure, O. Bottema [1, §77] showed that the sum of the areas of the squares on BA', CB' and AC' is equal to the sum of the areas of the squares on $A'C$, $B'A$ and $C'B$, namely,

$$a_l^2 + b_l^2 + c_l^2 = a_r^2 + b_r^2 + c_r^2.$$

See also [2, p.112]. Bottema showed conversely that when this equation holds, $A'B'C'$ is indeed a pedal triangle. While this can be easily established by applying the Pythagorean Theorem to the right triangles $AB'P$, $AC'P$, $BA'P$, $BC'P$, $CA'P$ and $CB'P$, we find a few more interesting properties of the figure. We adopt the following notations.

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O	circumcenter	
K	symmedian point	
$\triangle$	area of triangle ABC	
ω	Brocard angle	$\cot\omega = \frac{a^2+b^2+c^2}{4\triangle}$
Ω_1	Brocard point	$\angle BA\Omega_1 = \angle CB\Omega_1 = \angle AC\Omega_1 = \omega$
Ω_2	Brocard point	$\angle AB\Omega_2 = \angle BC\Omega_2 = \angle CA\Omega_2 = \omega$
$\mathsf{h}(P,r)$	homothety with center P and ratio r	
$\rho(P,\theta)$	rotation about P through an angle θ	

Let $A_1B_1C_1$ be the triangle bounded by the lines containing the sides of the squares opposite to BA', CB', AC' respectively. Similarly, let $A_2B_2C_2$ be the one bounded by the lines containing the sides of the squares opposite to $A'C$, $B'A$ and $C'B$.

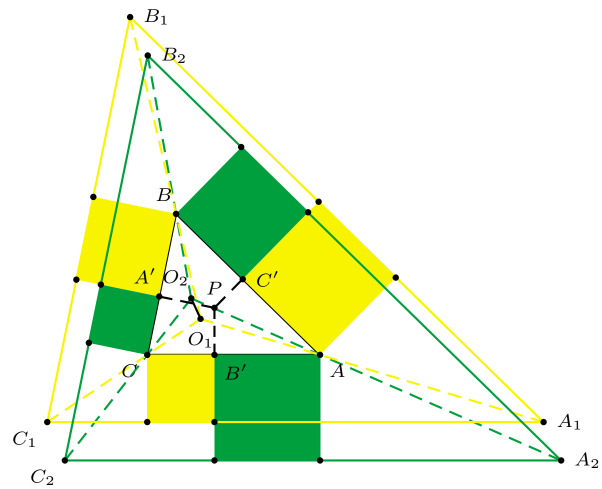

Figure 2

Theorem. *Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are each homothetic to ABC. Let O_1, and O_2 be the respective centers of homothety.*

(1) *The ratio of homothety in each case is $1 + \cot\omega$. Therefore, $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic and congruent.*

(2) *The mapping $P \mapsto O_1$ is the direct similarity which is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan\omega)$. Likewise, The mapping $P \mapsto O_2$ is the direct similarity which is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan\omega)$.*

(3) *The midpoint of the segment O_1O_2 is the symmedian point K.*

(4) *The vector of translation $A_1B_1C_1 \mapsto A_2B_2C_2$ is the image of $2\mathbf{OP}$ under the rotation $\rho(O, \frac{\pi}{2})$.*

Proof. We label the directed distances $a_l = BA'$, $a_r = A'C$, $b_l = CB'$, $b_r = B'A$, $c_l = AC'$ and $c_r = C'B$ as in Figure 1. Because ABC and $A_1B_1C_1$ are homothetic through O_1, the distances f, g, h of O_1 to the respective sides of ABC are in the same ratio as the distances between the corresponding sides of ABC and $A_1B_1C_1$. We have $f : g : h = a_l : b_l : c_l$. See Figure 2. Furthermore, the sum of the areas of triangles O_1BC, AO_1C and ABO_1 is equal to the area $\triangle$ of ABC, so that $af + bg + ch = 2\triangle$. But we also have

$$\begin{aligned} a_l^2 + b_l^2 + c_l^2 &= a_r^2 + b_r^2 + c_r^2 \\ &= (a - a_l)^2 + (b - b_l)^2 + (c - c_l)^2, \end{aligned}$$

from which we find

$$aa_l + bb_l + cc_l = \frac{a^2 + b^2 + c^2}{2} = 2\triangle \cot\omega.$$

This shows that $\frac{a_l}{f} = \frac{b_l}{g} = \frac{c_l}{h} = \cot\omega$, and thus that the ratio of homothety of $A_1B_1C_1$ to ABC is $1 + \cot\omega$. By symmetry, we find the same ratio of homothety of $A_2B_2C_2$ to ABC. This proves (1).

Now suppose that $P = O_1$. Then $\tan\angle CBO_1 = \frac{f}{a_l} = \tan\omega$. By symmetry this shows that P must be the Brocard point Ω_1.

To investigate the mapping $P \mapsto O_1$, we imagine that P moves through a line perpendicular to BC. For all points P on this line a_l is the same, so that for all images O_1 the distance f is the same. Therefore, O_1 traverses a line parallel to BC. Now imagine that P travels a distance d in the direction $A'P$. Then $AC' = c_l$ decreases with $d/\sin B$. The distance h of O_1 to AB thus decreases with $\frac{d\tan\omega}{\sin B}$, and O_1 must have travelled in the direction $\mathbf{CB}$ through $d\tan\omega$. Of course we can find similar results by letting P move through a line perpendicular to AC or AB.

Now any point P can be reached from Ω_1 by first going through a certain distance perpendicular to BC and then through another distance perpendicular to AC. Since Ω_1 is a fixed point of $P \mapsto O_1$, we can combine the results of the previous paragraph to conclude that $P \mapsto O_1$ is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan\omega)$.

In a similar fashion we see that $P \mapsto O_2$ is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan\omega)$. This proves (2).

Now note that the pedal triangle of O is the medial triangle, so that the images of O under both mappings are identical. This image must be the point for which the distances to the sides are proportional to the corresponding sides, well known to be the symmedian point K. Now the segment OP is mapped to KO_1 and KO_2 respectively under the above mappings, while the image segments are congruent and make an angle of π. This proves (3).

More precisely the ratio of lengths $|KO_1| : |OP| = \tan\omega : 1$, so that $|O_1O_2| : |OP| = 2\tan\omega : 1$. By (1), we also know that $|O_1O_2| : |A_1A_2| = \tan\omega : 1$. Together with the observation that O_1O_2 and A_1A_2 are oppositely parallel, this proves (4). □

We remark that (1) can be generalized to *inscribed* triangles $A'B'C'$. Since $BA' + A'C = BC$ it is clear that the line midway between B_1C_1 and B_2C_2 is at distance $\frac{a}{2}$ from BC, it is the line passing through the apex of the isosceles right triangle erected outwardly on BC. We conclude that the midpoints of A_1A_2, B_1B_2

and C_1C_2 form a triangle independent from $A'B'C'$, homothetic to ABC through K with ratio $1 + \cot\omega$. But then since $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic to each other, as well as to ABC, it follows that the sum of their homothety ratios is $2(1 + \cot\omega)$.

References

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