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An Application of Thébault's Theorem

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Abstract. We prove the "Japanese theorem" as a very simple corollary of Thébault's theorem.

Theorem 1 below is due to the French geometer Victor Thébault [8]. See Figure 1. It had been a long standing problem, but a number of proofs have appeared since the early 1980's. See, for example, [7, 6, 1], and also [5] for a list of proofs in Dutch published in the 1970's. A very natural and understandable proof based on Ptolemy's theorem can be found in [3].

Theorem 1 (Thébault)**.** *Let E be a point on the side of triangle ABC such that $\angle AEB = \theta$. Let $O_1(r_1)$ be a circle tangent to the circumcircle and to the segments EA, EB. Let $O_2(r_2)$ be also tangent to the circumcircle and to EA, EC. If $I(\rho)$ is the incircle of ABC, then*

(1.1) *I lies on the segment O_1O_2 and* $\dfrac{O_1I}{IO_2} = \tan^2 \frac{\theta}{2}$,

(1.2) $\rho = r_1 \cos^2 \frac{\theta}{2} + r_2 \sin^2 \frac{\theta}{2}$.

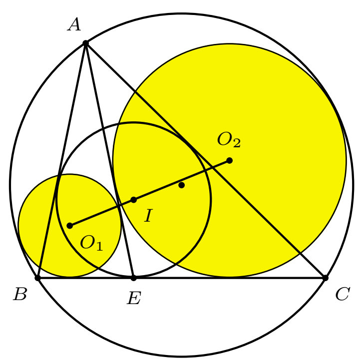

Figure 1

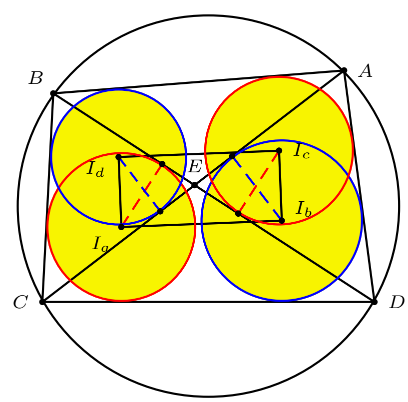

Figure 2

Theorem 2 below is called the "Japanese Theorem" in [4, p.193]. See Figure 2. A very long proof can be found in [2, pp.125–128]. In this note we deduce the Japanese Theorem as a very simple corollary of Thébault's Theorem.

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Theorem 2. *Let $ABCD$ be a convex quadrilateral inscribed in a circle. Denote by $I_a(\rho_a)$, $I_b(\rho_b)$, $I_c(\rho_c)$, $I_d(\rho_d)$ the incircles of the triangles BCD, CDA, DAB, and ABC.*

(2.1) *The incenters form a rectangle.*
(2.2) $\rho_a + \rho_c = \rho_b + \rho_d$.

Proof. In $ABCD$ we have the following circles: $O_{cd}(r_{cd})$, $O_{da}(r_{da})$, $O_{ab}(r_{ab})$, and $O_{bc}(r_{bc})$ inscribed respectively in angles AEB, BEC, CED, and DEA, each tangent internally to the circumcircle. Let $\angle AEB = \angle CED = \theta$ and $\angle BEC = \angle DEA = \pi - \theta$.

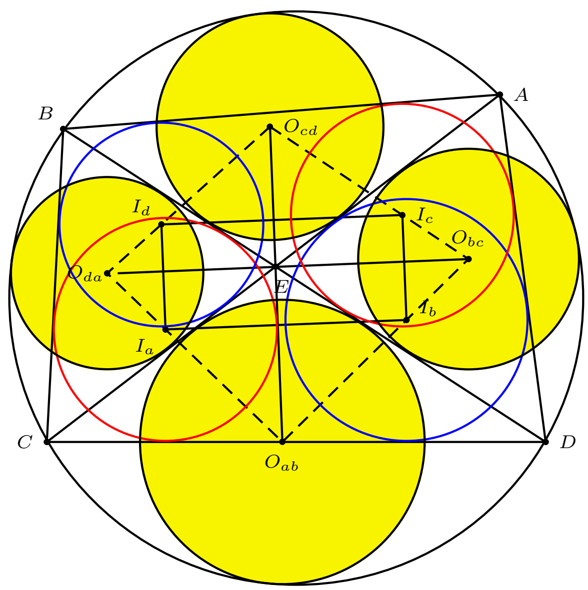

Figure 3

Now, by Theorem 1, the centers I_a, I_b, I_c, I_d lie on the lines $O_{da}O_{ab}$, $O_{ab}O_{bc}$, $O_{bc}O_{cd}$, $O_{cd}O_{da}$ respectively. Furthermore,

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_c}{I_cO_{cd}} = \tan^2\left(\frac{\pi-\theta}{2}\right) = \cot^2\frac{\theta}{2},$$
$$\frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_d}{I_dO_{da}} = \tan^2\frac{\theta}{2}.$$

From these, we have

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_b}{I_bO_{ab}}, \qquad \frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_c}{I_cO_{bc}},$$
$$\frac{O_{bc}I_c}{I_cO_{cd}} = \frac{O_{da}I_d}{I_dO_{cd}}, \qquad \frac{O_{cd}I_d}{I_dO_{da}} = \frac{O_{ab}I_a}{I_aO_{da}}.$$

These proportions imply the following parallelism:

$$I_aI_b//O_{da}O_{bc}, \quad I_bI_c//O_{ab}O_{cd}, \quad I_cI_d//O_{bc}O_{da}, \quad I_dI_a//O_{cd}O_{ab}.$$

As the segments $O_{cd}O_{ab}$ and $O_{da}O_{bc}$ are perpendicular because they are along the bisectors of the angles at E, $I_aI_bI_cI_d$ is an inscribed rectangle in $O_{ab}O_{bc}O_{cd}O_{da}$, and this proves (2.1).

Also, the following relation results from (1.2):

$$\rho_a + \rho_c = (r_{ab} + r_{cd})\cos^2\frac{\theta}{2} + (r_{da} + r_{bc})\sin^2\frac{\theta}{2}.$$

This same expression is readily seen to be equal to $\rho_b + \rho_d$ as well. This proves (2.2). □

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