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Orthocorrespondence and Orthopivotal Cubics

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Abstract. We define and study a transformation in the triangle plane called the orthocorrespondence. This transformation leads to the consideration of a family of circular circumcubics containing the Neuberg cubic and several hitherto unknown ones.

1. The orthocorrespondence

Let P be a point in the plane of triangle ABC with barycentric coordinates $(u : v : w)$. The perpendicular lines at P to AP, BP, CP intersect BC, CA, AB respectively at P_a, P_b, P_c, which we call the *orthotraces* of P. These orthotraces lie on a line $\mathcal{L}_P$, which we call the *orthotransversal* of P.[1] We denote the trilinear pole of $\mathcal{L}_P$ by $P^\perp$, and call it the *orthocorrespondent* of P.

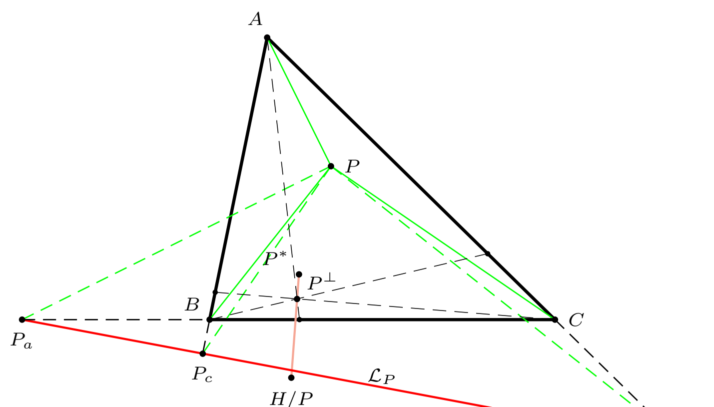

Figure 1. The orthotransversal and orthocorrespondent

In barycentric coordinates,[2]

$$P^\perp = (u(-uS_A + vS_B + wS_C) + a^2vw : \cdots : \cdots), \tag{1}$$

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[1]The homography on the pencil of lines through P which swaps a line and its perpendicular at P is an involution. According to a Desargues theorem, the points are collinear.

[2]All coordinates in this paper are homogeneous barycentric coordinates. Often for triangle centers, we list only the first coordinate. The remaining two can be easily obtained by cyclically permuting a, b, c, and corresponding quantities. Thus, for example, in (1), the second and third coordinates are $v(-vS_B + wS_C + uS_A) + b^2wu$ and $w(-wS_C + uS_A + vS_B) + c^2uv$ respectively.

where, a, b, c are respectively the lengths of the sides BC, CA, AB of triangle ABC, and, in J.H. Conway's notations,

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2),\ S_B = \frac{1}{2}(c^2 + a^2 - b^2),\ S_C = \frac{1}{2}(a^2 + b^2 - c^2). \qquad (2)$$

The mapping $\Phi : P \mapsto P^{\perp}$ is called the *orthocorrespondence* (with respect to triangle ABC).

Here are some examples. We adopt the notations of [5] for triangle centers, except for a few commonest ones. Triangle centers without an explicit identification as X_n are not in the current edition of [5].

(1) $I^{\perp} = X_{57}$, the isogonal conjugate of the Mittenpunkt X_9.
(2) $G^{\perp} = (b^2 + c^2 - 5a^2 : \cdots : \cdots)$ is the reflection of G about K, and the orthotransversal is perpendicular to GK.
(3) $H^{\perp} = G$.
(4) $O^{\perp} = (\cos 2A : \cos 2B : \cos 2C)$ on the line GK.
(5) More generally, the orthocorrespondent of the Euler line is the line GK. The orthotransversal envelopes the Kiepert parabola.
(6) $K^{\perp} = \left(a^2(b^4 + c^4 - a^4 - 4b^2c^2) : \cdots : \cdots\right)$ on the Euler line.
(7) $X_{15}^{\perp} = X_{62}$ and $X_{16}^{\perp} = X_{61}$.
(8) $X_{112}^{\perp} = X_{115}^{\perp} = X_{110}$.

See §2.3 for points on the circumcircle and the nine-point circle with orthocorrespondents having simple barycentric coordinates.

Remarks. (1) While the geometric definition above of $P^{\perp}$ is not valid when P is a vertex of triangle ABC, by (1) we extend the orthocorrespondence Φ to cover these points. Thus, $A^{\perp} = A$, $B^{\perp} = B$, and $C^{\perp} = C$.

(2) The orthocorrespondent of P is not defined if and only if the three coordinates of $P^{\perp}$ given in (1) are simultaneously zero. This is the case when P belongs to the three circles with diameters BC, CA, AB.[3] There are only two such points, namely, the circular points at infinity.

(3) We denote by P^* the isogonal conjugate of P and by H/P the cevian quotient of H and P.[4] It is known that

$$H/P = (u(-uS_A + vS_B + wS_C) : \cdots : \cdots).$$

This shows that $P^{\perp}$ lies on the line through P^* and H/P. In fact,

$$(H/P)P^{\perp} : (H/P)P^* = a^2vw + b^2wu + c^2uv : S_Au^2 + S_Bv^2 + S_Cw^2.$$

In [6], Jim Parish claimed that this line also contains the isogonal conjugate of P with respect to its anticevian triangle. We add that this point is in fact the harmonic conjugate of $P^{\perp}$ with respect to P^* and H/P. Note also that the line through P and H/P is perpendicular to the orthotransversal $\mathcal{L}_P$.

(4) The orthocorrespondent of any (real) point on the line at infinity $\mathcal{L}^{\infty}$ is G.

[3] See Proposition 2 below.

[4] H/P is the perspector of the cevian triangle of H (orthic triangle) and the anticevian triangle of P.

(5) A straightforward computation shows that the orthocorrespondence Φ has exactly five fixed points. These are the vertices A, B, C, and the two Fermat points X_{13}, X_{14}. Jim Parish [7] and Aad Goddijn [2] have given nice synthetic proofs of this in answering a question of Floor van Lamoen [3]. In other words, X_{13} and X_{14} are the only points whose orthotransversal and trilinear polar coincide.

Theorem 1. *The orthocorrespondent $P^\perp$ is a point at infinity if and only if P lies on the Monge (orthoptic) circle of the inscribed Steiner ellipse.*

Proof. From (1), $P^\perp$ is a point at infinity if and only if

$$\sum_{\text{cyclic}} S_A x^2 - 2a^2yz = 0. \tag{3}$$

This is a circle in the pencil generated by the circumcircle and the nine-point circle, and is readily identified as the Monge circle of the inscribed Steiner ellipse.[5] □

It is obvious that $P^\perp$ is at infinity if and only if $\mathcal{L}_P$ is tangent to the inscribed Steiner ellipse. [6]

Proposition 2. *The orthocorrespondent $P^\perp$ lies on the sideline BC if and only if P lies on the circle Γ_{BC} with diameter BC. The perpendicular at P to AP intersects BC at the harmonic conjugate of $P^\perp$ with respect to B and C.*

Proof. $P^\perp$ lies on BC if and only if its first barycentric coordinate is 0, *i.e.*, if and only if $u(-uS_A + vS_B + wS_C) + a^2vw = 0$ which shows that P must lie on Γ_{BC}. □

2. Orthoassociates and the critical conic

2.1. *Orthoassociates and antiorthocorrespondents.*

Theorem 3. *Let Q be a finite point. There are exactly two points P_1 and P_2 (not necessarily real nor distinct) such that $Q = P_1^\perp = P_2^\perp$.*

Proof. Let Q be a finite point. The trilinear polar ℓ_Q of Q intersects the sidelines of triangle ABC at Q_a, Q_b, Q_c. The circles Γ_a, Γ_b, Γ_c with diameters AQ_a, BQ_b, CQ_c are in the same pencil of circles since their centers O_a, O_b, O_c are collinear (on the Newton line of the quadrilateral formed by the sidelines of ABC and ℓ_Q), and since they are all orthogonal to the polar circle. Thus, they have two points P_1 and P_2 in common. These points, if real, satisfy $P_1^\perp = Q = P_2^\perp$. [7] □

We call P_1 and P_2 the *antiorthocorrespondents* of Q and write $Q^\top = \{P_1, P_2\}$. We also say that P_1 and P_2 are *orthoassociates*, since they share the same orthocorrespondent and the same orthotransversal. Note that P_1 and P_2 are homologous

[5]The Monge (orthoptic) circle of a conic is the locus of points whose two tangents to the conic are perpendicular to each other. It has the same center of the conic. For the inscribed Steiner ellipse, the radius of the Monge circle is $\frac{\sqrt{2}}{6}\sqrt{a^2+b^2+c^2}$.

[6]The trilinear polar of a point at infinity is tangent to the in-Steiner ellipse since it is the in-conic with perspector G.

[7]P_1 and P_2 are not always real when ABC is obtuse angled, see §2.2 below.

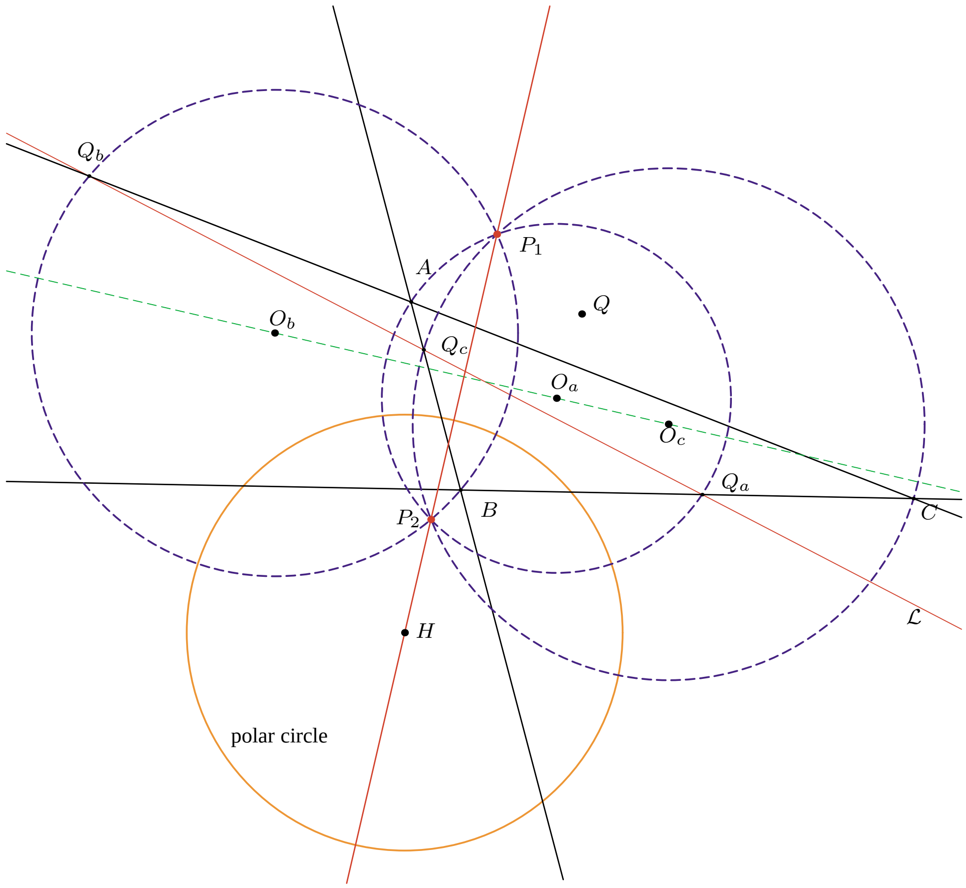

Figure 2. Antiorthocorrespondents

under the inversion ι_H with pole H which swaps the circumcircle and the nine-point circle.

Proposition 4. *The orthoassociate* $\overline{P}$ *of* $P(u:v:w)$ *has coordinates*

$$\left(\frac{S_Bv^2+S_Cw^2-S_Au(v+w)}{S_A}:\frac{S_Cw^2+S_Au^2-S_Bv(w+u)}{S_B}:\frac{S_Au^2+S_Bv^2-S_Cw(u+v)}{S_C}\right). \tag{4}$$

Let S denote *twice* of the area of triangle ABC. In terms of S_A, S_B, S_C in (2), we have

$$S^2 = S_AS_B + S_BS_C + S_CS_A.$$

Proposition 5. *Let*

$$K(u,v,w) = S^2(u+v+w)^2 - 4(a^2S_Avw + b^2S_Bwu + c^2S_Cuv).$$

The antiorthocorrespondents of $Q = (u : v : w)$ are the points with barycentric coordinates

$$((u-w)(u+v-w)S_B+(u-v)(u-v+w)S_C \pm \frac{\sqrt{K(u,v,w)}}{S}((u-w)S_B+(u-v)S_C) : \cdots : \cdots). \quad (5)$$

These are real points if and only if $K(u, v, w) \geq 0$.

2.2. *The critical conic $\mathcal{C}$.* Consider the *critical conic $\mathcal{C}$* with equation

$$S^2(x + y + z)^2 - 4 \sum_{\text{cyclic}} a^2 S_A yz = 0, \quad (6)$$

which is degenerate, real, imaginary according as triangle ABC is right-, obtuse-, or acute-angled. It has center the Lemoine point K, and the same infinite points as the circumconic

$$a^2 S_A yz + b^2 S_B zx + c^2 S_C xy = 0,$$

which is the isogonal conjugate of the orthic axis $S_A x + S_B y + S_C z = 0$, and has the same center K. This critical conic is a hyperbola when it is real. Clearly, if Q lies on the critical conic, its two real antiorthocorrespondents coincide.

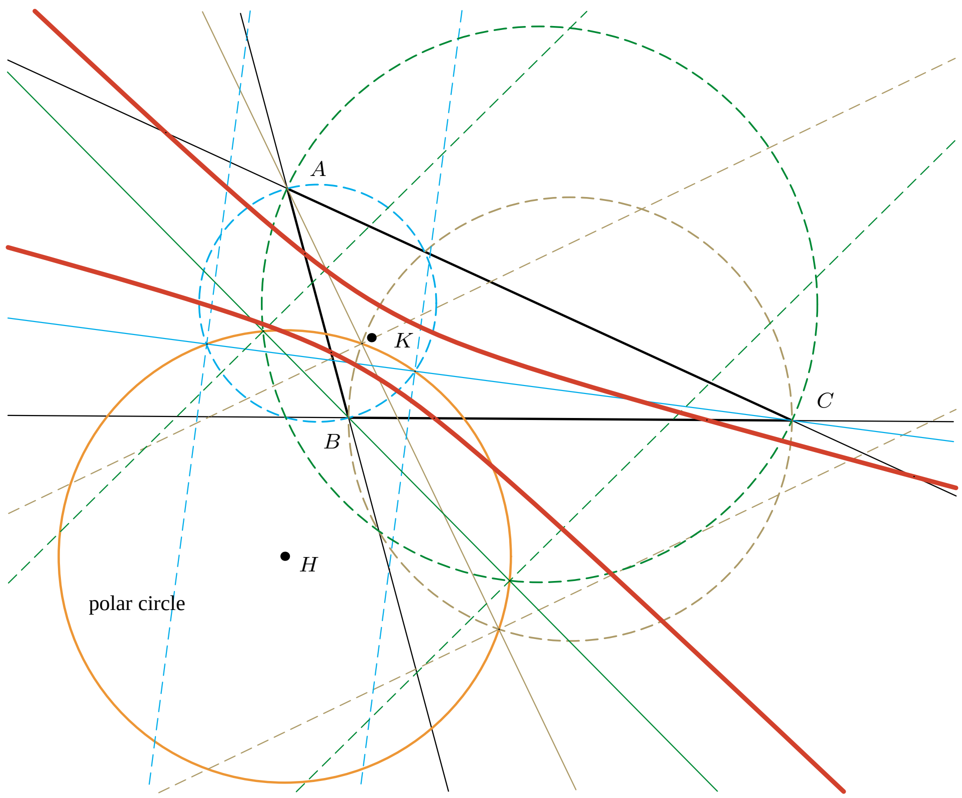

Figure 3. The critical conic

Proposition 6. *The antiorthocorrespondents of Q are real if and only if one of the following conditions holds.*
(1) Triangle ABC is acute-angled.
(2) Triangle ABC is obtuse-angled and Q lies in the component of the critical hyperbola not containing the center K.

Proposition 7. *The critical conic is the orthocorrespondent of the polar circle. When it is real, it intersects each sideline of ABC at two points symmetric about the corresponding midpoint. These points are the orthocorrespondents of the intersections of the polar circle and the circles Γ_{BC}, Γ_{CA}, Γ_{AB} with diameters BC, CA, AB.*

2.3. *Orthocorrespondent of the circumcircle.* Let P be a point on the circumcircle. Its orthotransversal passes through O, and $P^{\perp}$ lies on the circumconic centered at K. [8] The orthoassociate $\overline{P}$ lies on the nine-point circle. The table below shows several examples of such points. [9]

P	P^*	$\overline{P}$	$P^{\perp}$
X_{74}	X_{30}	X_{133}	$a^2S_A/((b^2-c^2)^2+a^2(2S_A-a^2))$
X_{98}	X_{511}	X_{132}	X_{287}
X_{99}	X_{512}	$(b^2-c^2)^2(S_A-a^2)/S_A$	$S_A/(b^2-c^2)$
X_{100}	X_{513}		$aS_A/(b-c)$
X_{101}	X_{514}		$a^2S_A/(b-c)$
X_{105}	X_{518}		$aS_A/(b^2+c^2-ab-ac)$
X_{106}	X_{519}		$a^2S_A/(b+c-2a)$
X_{107}	X_{520}	X_{125}	$X_{648}=X^*_{647}$
X_{108}	X_{521}	X_{11}	$X_{651}=X^*_{650}$
X_{109}	X_{522}		$a^2S_A/((b-c)(b+c-a))$
X_{110}	X_{523}	X_{136}	$a^2S_A/(b^2-c^2)$
X_{111}	X_{524}		$a^2S_A/(b^2+c^2-2a^2)=X^*_{468}$
X_{112}	X_{525}	X_{115}	$X_{110}=X^*_{523}$
X_{675}	X_{674}		$S_A/(b^3+c^3-a(b^2+c^2))$
X_{689}	X_{688}		$S_A/(a^2(b^4-c^4))$
X_{691}	X_{690}		$a^2S_A/((b^2-c^2)(b^2+c^2-2a^2))$
P_1	P_1^*	X_{114}	X^*_{230}

Remark. The coordinates of P_1 can be obtained from those of X_{230} by making use of the fact that X^*_{230} is the barycentric product of P_1 and X_{69}. Thus,

$$P_1=\left(\frac{a^2}{S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))}:\dots:\dots\right).$$

[8] If $P=(u:v:w)$ lies on the circumcircle, then $P^{\perp}=(uS_A:vS_B:wS_C)$ is the barycentric product of P and X_{69}. See [9]. The orthotransversal is the line $\frac{x}{uS_A}+\frac{y}{vS_B}+\frac{z}{wS_C}=0$ which contains O.

[9] The isogonal conjugates are trivially infinite points.

2.4. *The orthocorrespondent of a line.* The orthocorrespondent of a sideline, say BC, is the circumconic through G and its projection on the corresponding altitude. The orthoassociate is the circle with the segment AH as diameter.

Consider a line ℓ intersecting BC, CA, AB at X, Y, Z respectively. The orthocorrespondent $\ell^{\perp}$ of ℓ is a conic containing the centroid G (the orthocorrespondent of the infinite point of ℓ) and the points $X^{\perp}$, $Y^{\perp}$, $Z^{\perp}$. [10] A fifth point can be constructed as $P^{\perp}$, where P is the pedal of G on ℓ. [11] These five points entirely determine the conic. According to Proposition 2, $\ell^{\perp}$ meets BC at the orthocorrespondents of the points where ℓ intersects the circle Γ_{BC}. [12] It is also the orthocorrespondent of the circle through H which is the orthoassociate of ℓ.

If the line ℓ contains H, the conic $\ell^{\perp}$ degenerates into a double line containing G. If ℓ also contains $P = (u : v : w)$ other than H, then this line has equation

$$(S_B v - S_C w)x + (S_C w - S_A u)y + (S_A u - S_B v)z = 0.$$

This double line passes through the second intersection of ℓ with the Kiepert hyperbola. [13] It also contains the point $(uS_A : vS_B : wS_C)$. The two lines intersect at the point

$$\left(\frac{S_B - S_C}{S_B v - S_C w} : \frac{S_C - S_A}{S_C w - S_A u} : \frac{S_A - S_B}{S_A u - S_B v}\right).$$

The orthotransversals of points on ℓ envelope the inscribed parabola with directrix ℓ and focus the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of ℓ.

2.5. *The antiorthocorrespondent of a line.* Let ℓ be the line with equation $lx + my + nz = 0$.

When ABC is acute angled, the antiorthocorrespondent $\ell^{\top}$ of ℓ is the circle centered at $\Omega_\ell = (m + n : n + l : l + m)$[14] and orthogonal to the polar circle. It has square radius

$$\frac{S_A(m+n)^2 + S_B(n+l)^2 + S_C(l+m)^2}{4(l+m+n)^2}$$

and equation

$$(x + y + z)\left(\sum_{\text{cyclic}} S_A lx\right) - (l + m + n)\left(\sum_{\text{cyclic}} a^2 yz\right) = 0.$$

When ABC is obtuse angled, $\ell^{\top}$ is only a part of this circle according to its position with respect to the critical hyperbola $\mathcal{C}$. This circle clearly degenerates

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[10]These points can be easily constructed. For example, $X^{\perp}$ is the trilinear pole of the perpendicular at X to BC.

[11]$P^{\perp}$ is the antipode of G on the conic.

[12]These points can be real or imaginary, distinct or equal.

[13]In particular, the orthocorrespondent of the tangent at H to the Kiepert hyperbola, *i.e.*, the line HK, is the Euler line.

[14]Ω_ℓ is the complement of the isotomic conjugate of the trilinear pole of ℓ.

into the union of $\mathcal{L}^\infty$and a line through H when G lies on ℓ. This line is the directrix of the inscribed conic which is now a parabola.

Conversely, any circle centered at Ω (proper or degenerate) orthogonal to the polar circle is the orthoptic circle of the inscribed conic whose perspector P is the isotomic conjugate of the anticomplement of the center of the circle. The ortho-correspondent of this circle is the trilinear polar ℓ_P of P. The table below shows a selection of usual lines and inscribed conics.[15]

P	Ω	ℓ	inscribed conic
X_1	X_{37}	antiorthic axis	ellipse, center I
X_2	X_2	$\mathcal{L}^\infty$	Steiner in-ellipse
X_4	X_6	orthic axis	ellipse, center K
X_6	X_{39}	Lemoine axis	Brocard ellipse
X_7	X_1	Gergonne axis	incircle
X_8	X_9		Mandart ellipse
X_{13}	X_{396}		Simmons conic
X_{76}	X_{141}	de Longchamps axis	
X_{110}	X_{647}	Brocard axis	
X_{598}	X_{597}		Lemoine ellipse

2.6. *Orthocorrespondent and antiorthocorrespondent of a circle.* In general, the orthocorrespondent of a circle is a conic. More precisely, two orthoassociate circles share the same orthocorrespondent conic, or the part of it outside the critical conic $\mathcal{C}$ when ABC is obtuse-angled. For example, the circumcircle and the nine-point circle have the same orthocorrespondent which is the circumconic centered at K. The orthocorrespondent of each circle (and its orthoassociate) of the pencil generated by circumcircle and the nine-point circle is another conic also centered at K and homothetic of the previous one. The axis of these conics are the parallels at K to the asymptotes of the Kiepert hyperbola. The critical conic is one of them since the polar circle belongs to the pencil.

This conic degenerates into a double line (or part of it) if and only if the circle is orthogonal to the polar circle. If the radical axis of the circumcircle and this circle is $lx + my + nz = 0$, this double line has equation $\frac{l}{S_A}x + \frac{m}{S_B}y + \frac{n}{S_C}z = 0$. This is the trilinear polar of the barycentric product X_{69} and the trilinear pole of the radical axis.

The antiorthocorrespondent of a circle is in general a bicircular quartic.

[15]The conics in this table are entirely defined either by their center or their perspector in the table. See [1]. In fact, there are two Simmons conics (and not ellipses as Brocard and Lemoyne wrote) with perspectors (and foci) X_{13} and X_{14}.

3. Orthopivotal cubics

For a given a point P with barycentric coordinates $(u : v : w)$, the locus of point M such that P, M, $M^\perp$ are collinear is the cubic curve $\mathcal{O}(P)$:

$$\sum_{\text{cyclic}} x\left((c^2u - 2S_Bw)y^2 - (b^2u - 2S_Cv)z^2\right) = 0. \tag{7}$$

Equivalently, $\mathcal{O}(P)$ is the locus of the intersections of a line through P with the circle which is its antiorthocorrespondent. See §2.5. We shall say that $\mathcal{O}(P)$ is an *orthopivotal* cubic, and call P its *orthopivot*.

Equation (7) can be rewritten as

$$\sum_{\text{cyclic}} u\left(x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz)\right) = 0. \tag{8}$$

Accordingly, we consider the cubic curves

$$\begin{array}{ll} \Sigma_a : & x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz) = 0, \\ \Sigma_b : & y(a^2z^2 - c^2x^2) + 2zx(S_Cz - S_Ax) = 0, \\ \Sigma_c : & z(b^2x^2 - a^2y^2) + 2xy(S_Ax - S_By) = 0, \end{array} \tag{9}$$

and very loosely write (8) in the form

$$u\Sigma_a + v\Sigma_b + w\Sigma_c = 0. \tag{10}$$

We study the cubics Σ_a, Σ_b, Σ_c in §6.5 below, where we shall see that they are strophoids. We list some basic properties of the $\mathcal{O}(P)$.

Proposition 8. (1) *The orthopivotal cubic $\mathcal{O}(P)$ is a circular circumcubic*[16] *passing through the Fermat points, P, the infinite point of the line GP, and*

$$P' = \left(\frac{b^2 - c^2}{v - w} : \frac{c^2 - a^2}{w - u} : \frac{a^2 - b^2}{u - v}\right), \tag{11}$$

which is the second intersection of the line GP and the Kiepert hyperbola.[17]

(2) *The "third" intersection of $\mathcal{O}(P)$ and the Fermat line $X_{13}X_{14}$ is on the line PX_{110}.*

(3) *The tangent to $\mathcal{O}(P)$ at P is the line $PP^\perp$.*

(4) *$\mathcal{O}(P)$ intersects the sidelines BC, CA, AB at U, V, W respectively given by*

$$\begin{aligned} U =&(0 : 2S_Cu - a^2v : 2S_Bu - a^2w), \\ V =&(2S_Cv - b^2u : 0 : 2S_Av - b^2w), \\ W =&(2S_Bw - c^2u : 2S_Aw - c^2v : 0). \end{aligned}$$

(5) *$\mathcal{O}(P)$ also contains the (not always real) antiorthocorrespondents P_1 and P_2 of P.*

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[16]This means that the cubic passes through the two circular points at infinity common to all circles, and the three vertices of the reference triangle.

[17]This is therefore the sixth intersection of $\mathcal{O}(P)$ with the Kiepert hyperbola.

Here is a simple construction of the intersection U in (4) above. If the parallel at G to BC intersects the altitude AH at H_a, then U is the intersection of PH_a and BC. [18]

4. Construction of $\mathcal{O}(P)$ and other points

Let the trilinear polar of P intersect the sidelines BC, CA, AB at X, Y, Z respectively. Denote by Γ_a, Γ_b, Γ_c the circles with diameters AX, BY, CZ and centers O_a, O_b, O_c. They are in the same pencil $\mathbb{F}$ whose radical axis is the perpendicular at H to the line $\mathcal{L}$ passing through O_a, O_b, O_c, and the points P_1 and P_2 seen above. [19]

For an arbitrary point M on $\mathcal{L}$, let Γ be the circle of $\mathbb{F}$ passing through M. The line $PM^{\perp}$ intersects Γ at two points N_1 and N_2 on $\mathcal{O}(P)$. From these we note the following.

(1) $\mathcal{O}(P)$ contains the second intersections A_2, B_2, C_2 of the lines AP, BP, CP with the circles Γ_a, Γ_b, Γ_c.
(2) The point P' in (11) lies on the radical axis of $\mathbb{F}$.
(3) The circle of $\mathbb{F}$ passing through P meets the line $PP^{\perp}$ at $\widetilde{P}$, tangential of P.
(4) The perpendicular bisector of N_1N_2 envelopes the parabola with focus F_P (see §5 below) and directrix the line GP. This parabola is tangent to $\mathcal{L}$ and to the two axes of the inscribed Steiner ellipse.

This yields another construction of $\mathcal{O}(P)$: a tangent to the parabola meets $\mathcal{L}$ at ω. The perpendicular at P to this tangent intersects the circle of $\mathbb{F}$ centered at ω at two points on $\mathcal{O}(P)$.

5. Singular focus and an involutive transformation

The singular focus of a circular cubic is the intersection of the two tangents to the curve at the circular points at infinity. When this singular focus lies on the curve, the cubic is said to be a focal cubic. The singular focus of $\mathcal{O}(P)$ is the point

$$F_P = \left(a^2(v^2+w^2-u^2-vw)+b^2u(u+v-2w)+c^2u(u+w-2v)\ :\cdots:\cdots\right).$$

If we denote by F_1 and F_2 the foci of the inscribed Steiner ellipse, then F_P is the inverse of the reflection of P in the line F_1F_2 with respect to the circle with diameter F_1F_2.

Consider the mapping $\Psi : P \mapsto F_P$ in the affine plane (without the centroid G) which transforms a point P into the singular focus F_P of $\mathcal{O}(P)$. This is clearly an involution: F_P is the singular focus of $\mathcal{O}(P)$ if and only if P is the singular focus of $\mathcal{O}(F_P)$. It has exactly two fixed points, *i.e.*, F_1 and F_2. [20]

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[18] H_a is the "third" intersection of AH with the Napoleon cubic, the isogonal cubic with pivot X_5.

[19] This line $\mathcal{L}$ is the trilinear polar of the isotomic conjugate of the anticomplement of P.

[20] The two cubics $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are central focals with centers at F_1 and F_2 respectively, with inflexional tangents through K, sharing the same real asymptote F_1F_2.

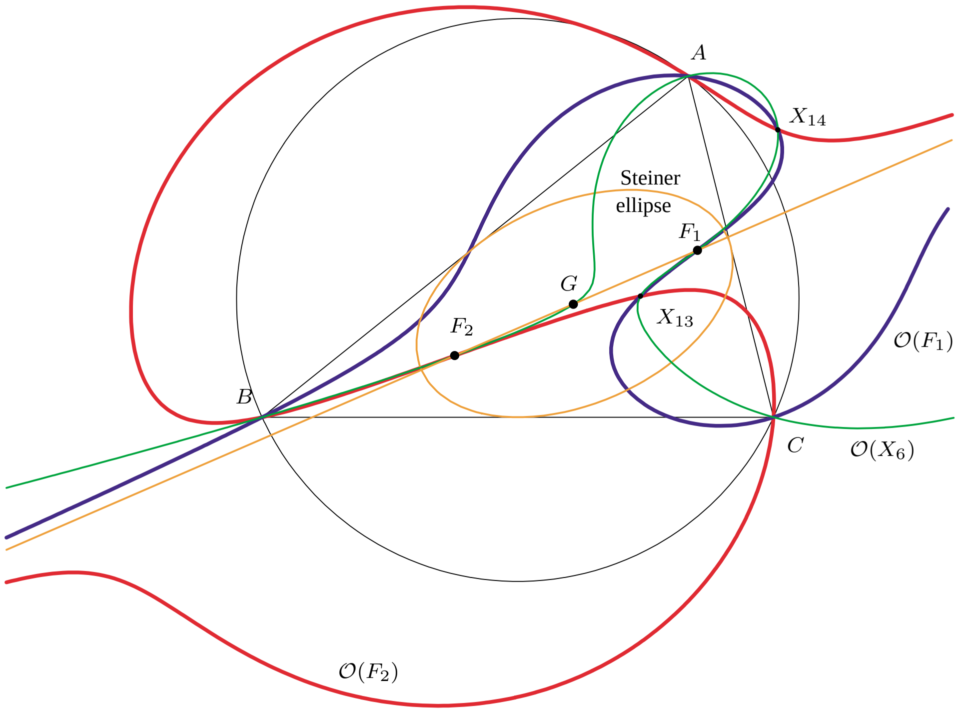

Figure 4. $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$

The table below shows a selection of homologous points under Ψ, most of which we shall meet in the sequel. When P is at infinity, $F_P = G$, *i.e.*, all $\mathcal{O}(P)$ with orthopivot at infinity have G as singular focus.

P	X_1	X_3	X_4	X_6	X_{13}	X_{15}	X_{23}	X_{69}
F_P	X_{1054}	X_{110}	X_{125}	X_{111}	X_{14}	X_{16}	X_{182}	X_{216}

P	X_{100}	X_{184}	X_{187}	X_{352}	X_{616}	X_{617}	X_{621}	X_{622}
F_P	X_{1083}	X_{186}	X_{353}	X_{574}	X_{619}	X_{618}	X_{624}	X_{623}

The involutive transformation Ψ swaps

(1) the Euler line and the line through GX_{110}, [21]
(2) more generally, any line GP and its reflection in F_1F_2,
(3) the Brocard axis OK and the Parry circle.
(4) more generally, any line OP (which is not the Euler line) and the circle through G, X_{110}, and F_P,
(5) the circumcircle and the Brocard circle,
(6) more generally, any circle not through G and another circle not through G.

[21] The nine-point center is swapped into the anticomplement of X_{110}.

The involutive transformation Ψ leaves the second Brocard cubic $\mathcal{B}_2$ [22]

$$\sum_{\text{cyclic}} (b^2 - c^2)x(c^2y^2 + b^2z^2) = 0$$

globally invariant. See §6.4 below. More generally, Ψ leaves invariant the pencil of circular circumcubics through the vertices of the second Brocard triangle (they all pass through G). [23] There is another cubic from this pencil which is also globally invariant, namely,

$$(a^2b^2c^2 - 8S_AS_BS_C)xyz + \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x(c^2S_Cy^2 + b^2S_Bz^2) = 0.$$

We call this cubic $\mathcal{B}_6$. It passes through X_3, X_{110}, and X_{525}.

If $\mathcal{O}(P)$ is nondegenerate, then its real asymptote is the homothetic image of the line GP under the homothety $\mathsf{h}(F_P, 2)$.

6. Special orthopivotal cubics

6.1. *Degenerate orthopivotal cubics.* There are only two situations where we find a degenerate $\mathcal{O}(P)$. A cubic can only degenerate into the union of a line and a conic. If the line is $\mathcal{L}^\infty$, we find only one such cubic. It is $\mathcal{O}(G)$, the union of $\mathcal{L}^\infty$ and the Kiepert hyperbola. If the line is not $\mathcal{L}^\infty$, there are ten different possibilities depending of the number of vertices of triangle ABC lying on the conic above which now must be a circle.

(1) $\mathcal{O}(X_{110})$ is the union of the circumcircle and the Fermat line. [24]
(2) $\mathcal{O}(P)$ is the union of one sideline of triangle ABC and the circle through the remaining vertex and the two Fermat points when P is the "third" intersection of an altitude of ABC with the Napoleon cubic. [25]
(3) $\mathcal{O}(P)$ is the union of a circle through two vertices of ABC and one Fermat point and a line through the remaining vertex and Fermat point when P is a vertex of one of the two Napoleon triangles. See [4, §6.31].

6.2. *Isocubics $\mathcal{O}(P)$.* We denote by $p\mathcal{K}$ a *pivotal* isocubic and by $n\mathcal{K}$ a *non-pivotal* isocubic. Consider an orthopivotal circumcubic $\mathcal{O}(P)$ intersecting the sidelines of triangle ABC at U, V, W respectively. The cubic $\mathcal{O}(P)$ is an isocubic in the two following cases.

---

[22] The second Brocard cubic $\mathcal{B}_2$ is the locus of foci of inscribed conics centered on the line GK. It is also the locus of M for which the line $MM^\perp$ contains the Lemoine point K.

[23] The inversive image of a circular cubic with respect to one of its points is another circular cubic through the same point. Here, Ψ swaps ABC and the second Brocard triangle $A_2B_2C_2$. Hence, each circular cubic through A, B, C, A_2, B_2, C_2 and G has an inversive image through the same points.

[24] X_{110} is the focus of the Kiepert parabola.

[25] The Napoleon cubic is the isogonal cubic with pivot X_5. These third intersections are the intersections of the altitudes with the parallel through G to the corresponding sidelines.

6.2.1. *Pivotal* $\mathcal{O}(P)$.

Proposition 9. *An orthopivotal cubic* $\mathcal{O}(P)$ *is a pivotal circumcubic* $p\mathcal{K}$ *if and only if the triangles* ABC *and* UVW *are perspective,* i.e., *if and only if* P *lies on the Napoleon cubic (isogonal* $p\mathcal{K}$ *with pivot* X_5*). In this case,*

(1) *the* pivot Q *of* $\mathcal{O}(P)$ *lies on the cubic* $\mathcal{K}_n$*:* [26] *it is the perspector of* ABC *and the* (-2)*-pedal triangle of* P*,*[27] *and lies on the line* PX_5*;*
(2) *the* pole Ω *of the isoconjugation lies on the cubic*

$$\mathcal{C}_o: \qquad \sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(b^2z - c^2y) = 0.$$

The Ω-isoconjugate Q^* of Q lies on the Neuberg cubic and is the inverse in the circumcircle of the isogonal conjugate of Q. The Ω-isoconjugate P^* of P lies on $\mathcal{K}_n$ and is the third intersection with the line QX_5.

Here are several examples of such cubics.

(1) $\mathcal{O}(O) = \mathcal{O}(X_3)$ is the Neuberg cubic.
(2) $\mathcal{O}(X_5)$ is $\mathcal{K}_n$.
(3) $\mathcal{O}(I) = \mathcal{O}(X_1)$ has pivot $X_{80} = ((2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, pole $(a(2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, and singular focus

$$(a(2S_A + ab + ac - 3bc) : \cdots : \cdots).$$

(4) $\mathcal{O}(H) = \mathcal{O}(X_4)$ has pivot H, pole M_o the intersection of HK and the orthic axis, with coordinates

$$\left(\frac{a^2(b^2+c^2-2a^2)+(b^2-c^2)^2}{S_A} : \cdots : \cdots\right),$$

and singular focus X_{125}, center of the Jerabek hyperbola.

$\mathcal{O}(H)$ is a very remarkable cubic since every point on it has orthocorrespondent on the Kiepert hyperbola. It is invariant under the inversion with respect to the conjugated polar circle and is also invariant under the isogonal transformation with respect to the orthic triangle. It is an isogonal $p\mathcal{K}$ with pivot X_{30} with respect to this triangle.

6.2.2. *Non-pivotal* $\mathcal{O}(P)$.

Proposition 10. *An orthopivotal cubic* $\mathcal{O}(P)$ *is a non-pivotal circumcubic* $n\mathcal{K}$ *if and only if its "third" intersections with the sidelines*[28] *are collinear,* i.e., *if and only if* P *lies on the isogonal* $n\mathcal{K}$ *with root* X_{30}*:* [29]

$$\sum_{\text{cyclic}} \left((b^2-c^2)^2 + a^2(b^2+c^2-2a^2)\right) x(c^2y^2+b^2z^2) + 2(8S_AS_BS_C - a^2b^2c^2)xyz = 0.$$

We give two examples of such cubics.

[26] $\mathcal{K}_n$ is the 2-cevian cubic associated with the Neuberg and the Napoleon cubics. See [8].

[27] For any non-zero real number t, the t-pedal triangle of P is the image of its pedal triangle under the homothety $\mathsf{h}(P,t)$.

[28] These are the points U, V, W in Proposition 8(4).

[29] This passes through G, K, X_{110}, and X_{523}.

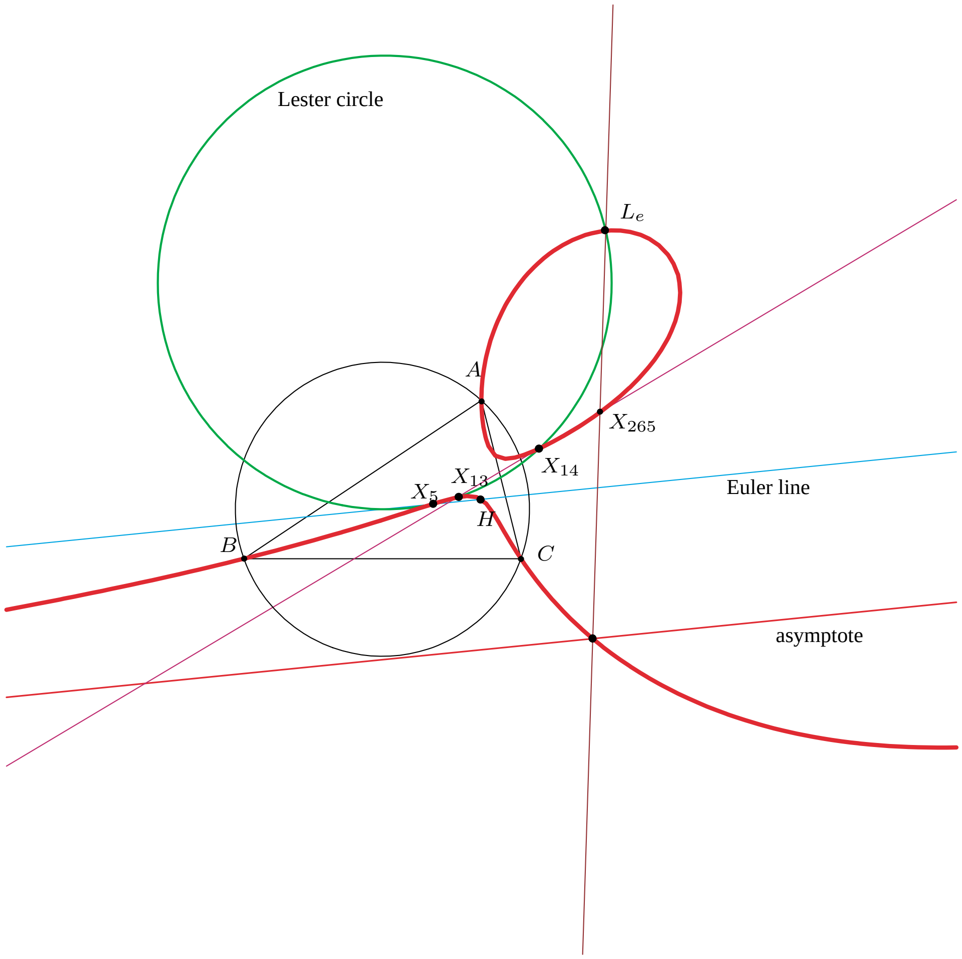

Figure 5. $\mathcal{K}_n$

(1) $\mathcal{O}(K) = \mathcal{O}(X_6)$ is the second Brocard cubic $\mathcal{B}_2$.
(2) $\mathcal{O}(X_{523})$ is a $n\mathcal{K}$ with pole and root both at the isogonal conjugate of X_{323}, and singular focus G: [30]

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(y+z) = 0$$

6.3. *Isogonal* $\mathcal{O}(P)$. There are only two)$\mathcal{O}(P)$ which are isogonal cubics, one pivotal and one non-pivotal:

(i) $\mathcal{O}(X_3)$ is the Neuberg cubic (pivotal),
(ii) $\mathcal{O}(X_6)$ is $\mathcal{B}_2$ (nonpivotal).

[30] $\mathcal{O}(X_{523})$ meets the circumcircle at the Tixier point X_{476}.

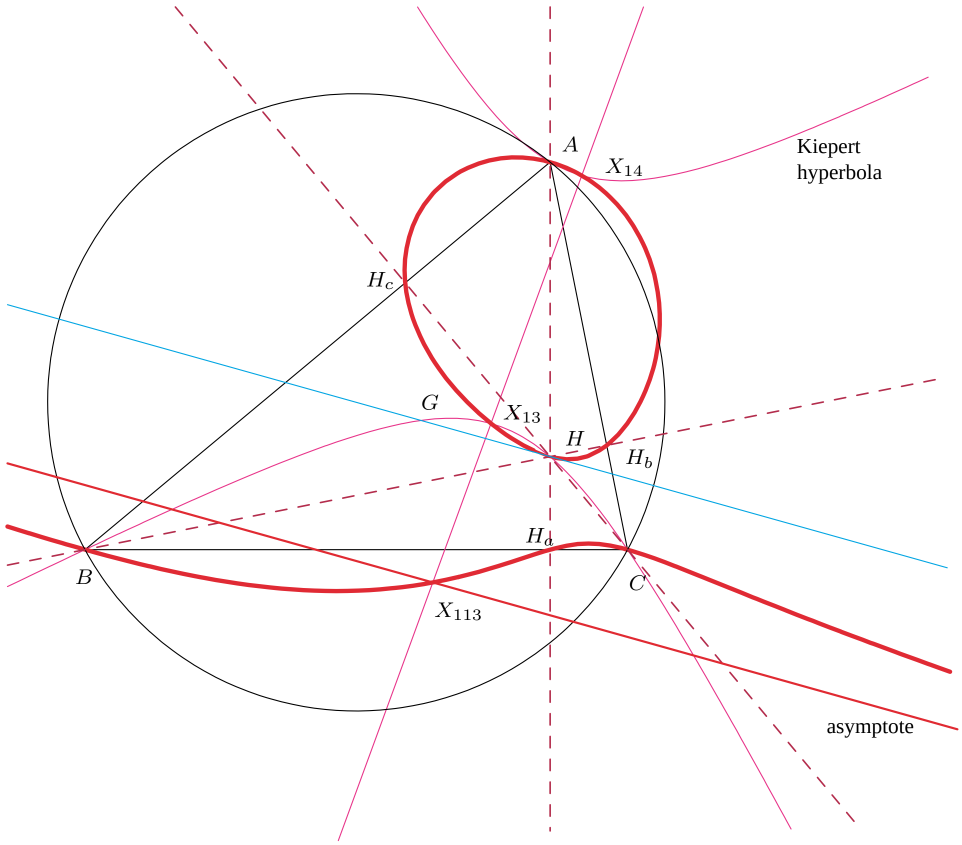

Figure 6. $\mathcal{O}(X_4)$

6.4. *Orthopivotal focals.* Recall that a focal is a circular cubic containing its own singular focus. [31]

Proposition 11. *An orthopivotal cubic $\mathcal{O}(P)$ is a focal if and only if P lies on $\mathcal{B}_2$.*

This is the case of $\mathcal{B}_2$ itself, which is an isogonal focal cubic passing through the following points: A, B, C, G, K, X_{13}, X_{14}, X_{15}, X_{16}, X_{111} (the singular focus), X_{368}, X_{524}, the vertices of the second Brocard triangle and their isogonal conjugates. All those points are orthopivots of orthopivotal focals. When the orthopivot is a fixed point of the orthocorrespondence, we shall see in §6.5 below that $\mathcal{O}(P)$ is a strophoid.

We have seen in §5 that F_1 and F_2 are invariant under Ψ. These two points lie on $\mathcal{B}_2$ (and also on the Thomson cubic). The singular focus of an orthopivotal focal $\mathcal{O}(P)$ always lies on $\mathcal{B}_2$; it is the "third" point of $\mathcal{B}_2$ and the line KP.

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[31]Typically, a focal is the locus of foci of conics inscribed in a quadrilateral. The only focals having double points (nodes) are the strophoids.

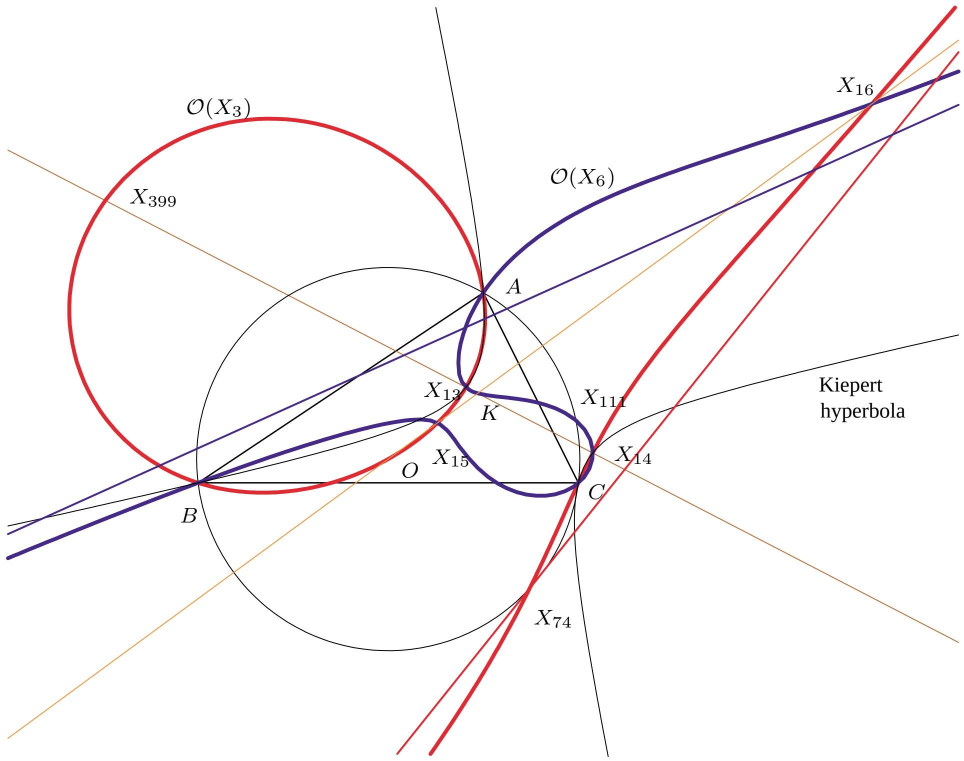

Figure 7. $\mathcal{O}(X_3)$ and $\mathcal{O}(X_6)$

One remarkable cubic is $\mathcal{O}(X_{524})$: it is another central cubic with center and singular focus at G and the line GK as real asymptote. This cubic passes through X_{67} and obviously the symmetrics of A, B, C, X_{13}, X_{14}, X_{67} about G. Its equation is

$$\sum_{\text{cyclic}} x\left(\left(b^2+c^4-a^4-c^2(a^2+2b^2-2c^2)\right)y^2-\left(b^4+c^4-a^4-b^2(a^2-2b^2+2c^2)\right)z^2\right)=0.$$

Another interesting cubic is $\mathcal{O}(X_{111})$ with K as singular focus. Its equation is

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2)x^2\left(c^2(a^4-b^2c^2+3b^4-c^4-2a^2b^2)y-b^2(a^4-b^2c^2+3c^4-b^4-2a^2c^2)z\right)=0.$$

The sixth intersection with the Kiepert hyperbola is X_{671}, a point on the Steiner circumellipse and on the line through X_{99} and X_{111}.

6.5. *Orthopivotal strophoids.* It is easy to see that $\mathcal{O}(P)$ is a strophoid if and only if P is one of the five real fixed points of the orthocorrespondence, namely, A, B, C, X_{13}, X_{14}, the fixed point being the double point of the curve. This means that the mesh of orthopivotal cubics contains five strophoids denoted by $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$, $\mathcal{O}(X_{13})$, $\mathcal{O}(X_{14})$.

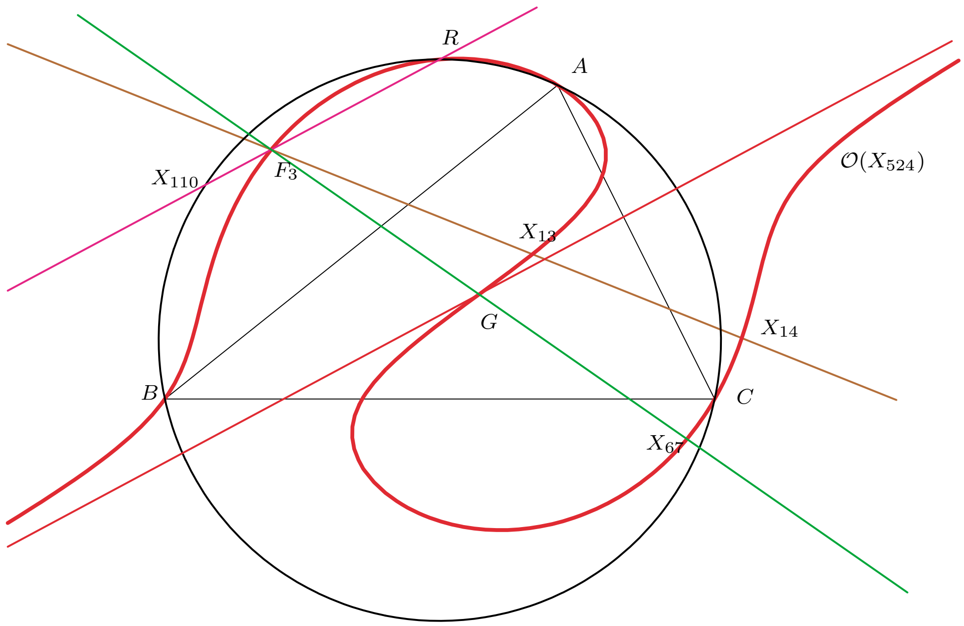

Figure 8. $\mathcal{O}(X_{524})$

6.5.1. *The strophoids* $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$. These are the cubics Σ_a, Σ_b, Σ_c with equations given in (9). It is enough to consider $\mathcal{O}(A) = \Sigma_a$. The bisectors of angle A are the tangents at the double point A. The singular focus is the corresponding vertex of the second Brocard triangle, namely, the point $A_2 = (2S_A : b^2 : c^2)$. [32] The real asymptote is parallel to the median AG, being the homothetic image of AG under $\mathsf{h}(A_2, 2)$.

Here are some interesting properties of $\mathcal{O}(A) = \Sigma_a$.

(1) Σ_a is the isogonal conjugate of the Apollonian A-circle

$$\mathcal{C}_A : \qquad a^2(b^2z^2 - c^2y^2) + 2x(b^2S_Bz - c^2S_Cy) = 0, \qquad (12)$$

which passes through A and the two isodynamic points X_{15} and X_{16}.

(2) The isogonal conjugate of A_2 is the point $A_4 = (a^2 : 2S_A : 2S_A)$ on the Apollonian circle $\mathcal{C}_A$, which is the projection of H on AG. The isogonal conjugate of the antipode of A_4 on $\mathcal{C}_A$ is the intersection of Σ_a with its real asymptote. [33]

(3) $\mathcal{O}(A) = \Sigma_a$ is the pedal curve with respect to A of the parabola with focus at the second intersection of $\mathcal{C}_A$ and the circumcircle and with directrix the median AG.

[32]This is the projection of O on the symmedian AK, the tangent at A_2 being the reflection about OA_2 of the parallel at A_2 to AG.

[33]This isogonal conjugate is on the perpendicular at A to AK, and on the tangent at A_2 to Σ_a.

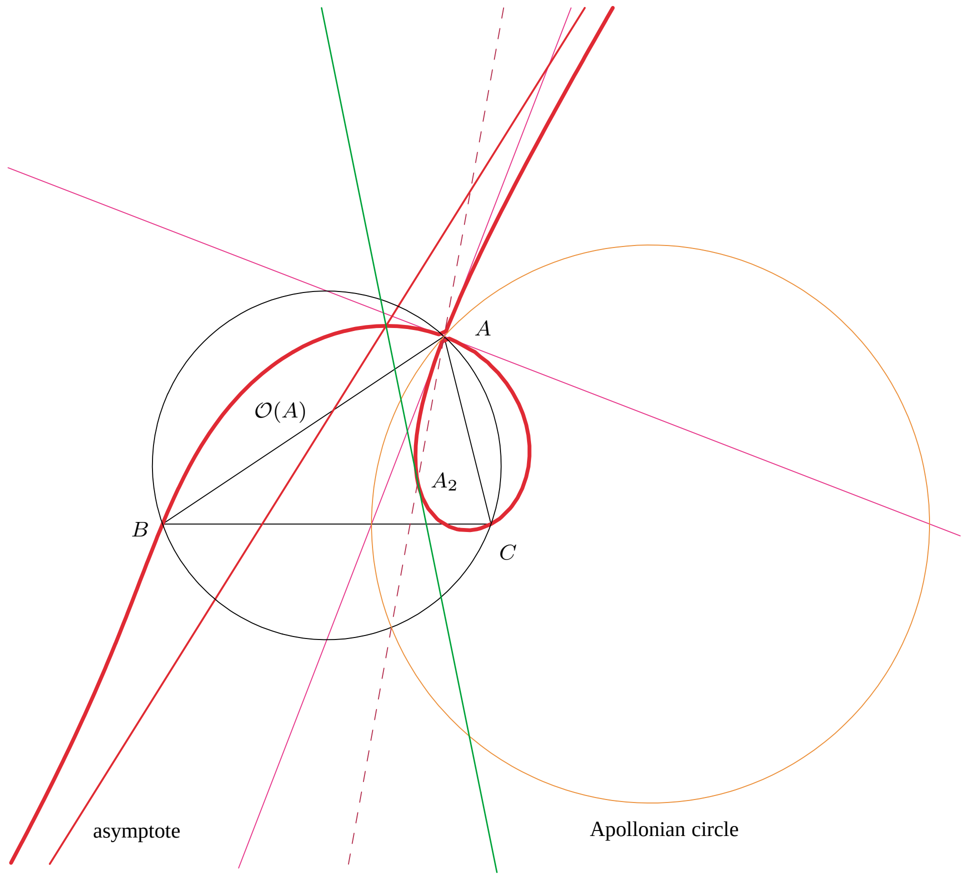

Figure 9. The strophoid $\mathcal{O}(A)$

6.5.2. *The strophoids* $\mathcal{O}(X_{13})$ *and* $\mathcal{O}(X_{14})$. The strophoid $\mathcal{O}(X_{13})$ has singular focus X_{14}, real asymptote the parallel at X_{99} to the line GX_{13}, [34] The circle centered at X_{14} passing through X_{13} intersects the parallel at X_{14} to GX_{13} at D_1 and D_2 which lie on the nodal tangents. The perpendicular at X_{14} to the Fermat line meets the bisectors of the nodal tangents at E_1 and E_2 which are the points where the tangents are parallel to the asymptote and therefore the centers of anallagmaty of the curve. [35]

$\mathcal{O}(X_{13})$ is the pedal curve with respect to X_{13} of the parabola with directrix the line GX_{13} and focus X'_{13}, the symmetric of X_{13} about X_{14}.

[34]The "third intersection" of this asymptote with the cubic lies on the perpendicular at X_{13} to the Fermat line. The intersection of the perpendicular at X_{13} to GX_{13} and the parallel at X_{14} to GX_{13} is another point on the curve.

[35]This means that E_1 and E_2 are the centers of two circles through X_{13} and the two inversions with respect to those circles leave $\mathcal{O}(X_{13})$ unchanged.

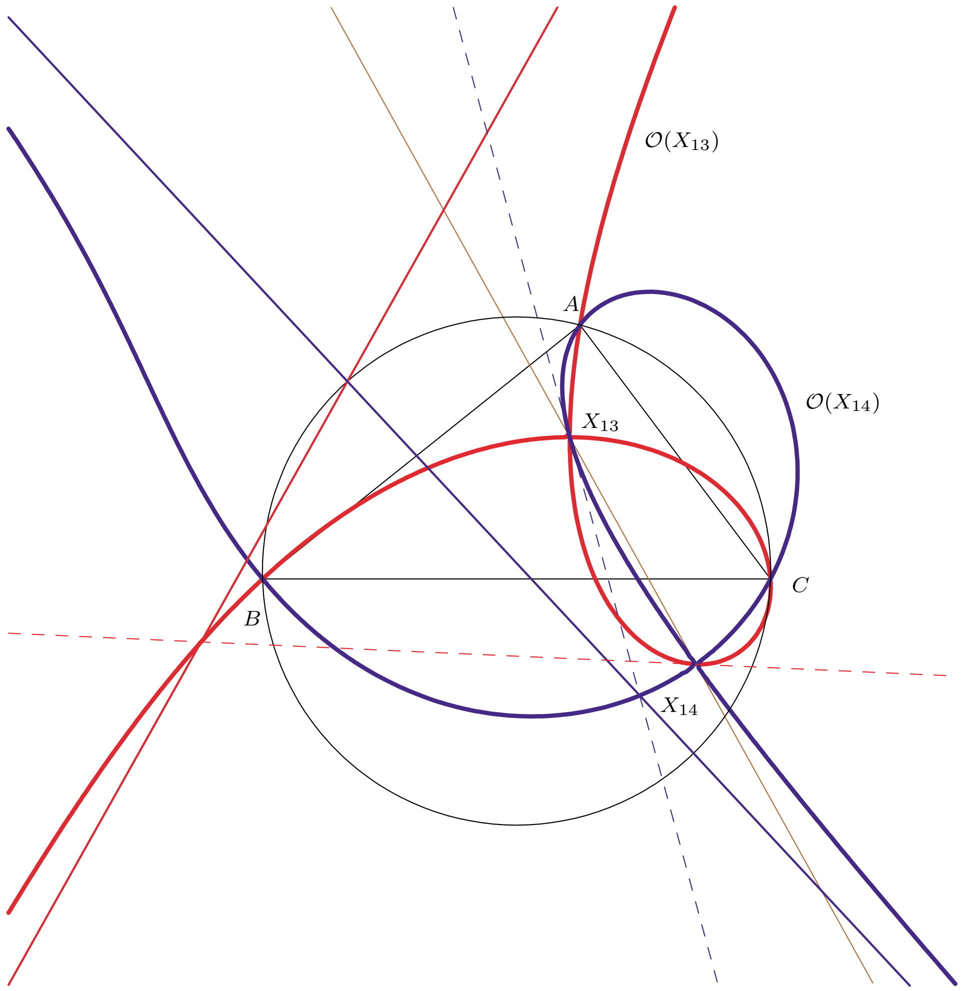

Figure 10. $\mathcal{O}(X_{13})$ and $\mathcal{O}(X_{14})$

The construction of $\mathcal{O}(X_{13})$ is easy to realize. Draw the parallel ℓ at X_{14} to GX_{13} and take a variable point M on it. The perpendicular at M to MX'_{13} and the parallel at X_{13} to MX'_{13} intersect at a point on the strophoid.

We can easily adapt all these to $\mathcal{O}(X_{14})$.

6.6. *Other remarkable* $\mathcal{O}(P)$. The following table gives a list triangle centers P with $\mathcal{O}(P)$ passing through the Fermat points X_{13}, X_{14}, and at least four more triangle centers of [5]. Some of them are already known and some others will be detailed in the next section. The very frequent appearance of X_{15}, X_{16} is explained in §7.3 below.

P	centers	P	centers
X_1	$X_{10,80,484,519,759}$	X_{182}	$X_{15,16,98,542}$
X_3	Neuberg cubic	X_{187}	$X_{15,16,598,843}$
X_5	$X_{4,30,79,80,265,621,622}$	X_{354}	$X_{1,105,484,518}$
X_6	$X_{2,15,16,111,368,524}$	X_{386}	$X_{10,15,16,519}$
X_{32}	$X_{15,16,83,729,754}$	X_{511}	$X_{15,16,262,842}$
X_{39}	$X_{15,16,76,538,755}$	X_{569}	$X_{15,16,96,539}$
X_{51}	$X_{61,62,250,262,511}$	X_{574}	$X_{15,16,543,671}$
X_{54}	$X_{3,96,265,539}$	X_{579}	$X_{15,16,226,527}$
X_{57}	$X_{1,226,484,527}$	X_{627}	$X_{17,532,617,618,622}$
X_{58}	$X_{15,16,106,540}$	X_{628}	$X_{18,533,616,619,621}$
X_{61}	$X_{15,16,18,533,618}$	X_{633}	$X_{18,533,617,623}$
X_{62}	$X_{15,16,17,532,619}$	X_{634}	$X_{17,532,616,624}$

7. Pencils of $\mathcal{O}(P)$

7.1. *Generalities.* The orthopivotal cubics with orthopivots on a given line ℓ form a pencil $\mathbb{F}_\ell$ generated by any two of them. Apart from the vertices, the Fermat points, and two circular points at infinity, all the cubics in the pencil pass through two fixed points depending on the line ℓ. Consequently, all the orthopivotal cubics passing through a given point Q have their orthopivots on the tangent at Q to $\mathcal{O}(Q)$, namely, the line $QQ^\perp$. They all pass through another point Q' on this line which is its second intersection with the circle which is its antiorthocorrespondent. For example, $\mathcal{O}(Q)$ passes through G, O, or H if and only if Q lies on GK, OX_{54}, or the Euler line respectively.

7.2. *Pencils with orthopivot on a line passing through G.* If ℓ contains the centroid G, every orthopivotal cubic in the pencil $\mathbb{F}_\ell$ passes through its infinite point and second intersection with the Kiepert hyperbola. As P traverses ℓ, the singular focus of $\mathcal{O}(P)$ traverses its reflection about F_1F_2 (see §5).

The most remarkable pencil is the one with ℓ the Euler line. In this case, the two fixed points are the infinite point X_{30} and the orthocenter H. In other words, all the cubics in this pencil have their asymptote parallel to the Euler line. In this pencil, we find the Neuberg cubic and $\mathcal{K}_n$. The singular focus traverses the line GX_{98}, X_{98} being the Tarry point.

Another worth noticing pencil is obtained when ℓ is the line GX_{98}. In this case, the two fixed points are the infinite point X_{542} and X_{98}. The singular focus traverses the Euler line. This pencil contains the two degenerate cubics $\mathcal{O}(G)$ and $\mathcal{O}(X_{110})$ seen in §6.1.

When ℓ is the line GK, the two fixed points are the infinite point X_{524} and the centroid G. The singular focus lies on the line GX_{99}, X_{99} being the Steiner point. This pencil contains $\mathcal{B}_2$ and the central cubic seen in §6.4.

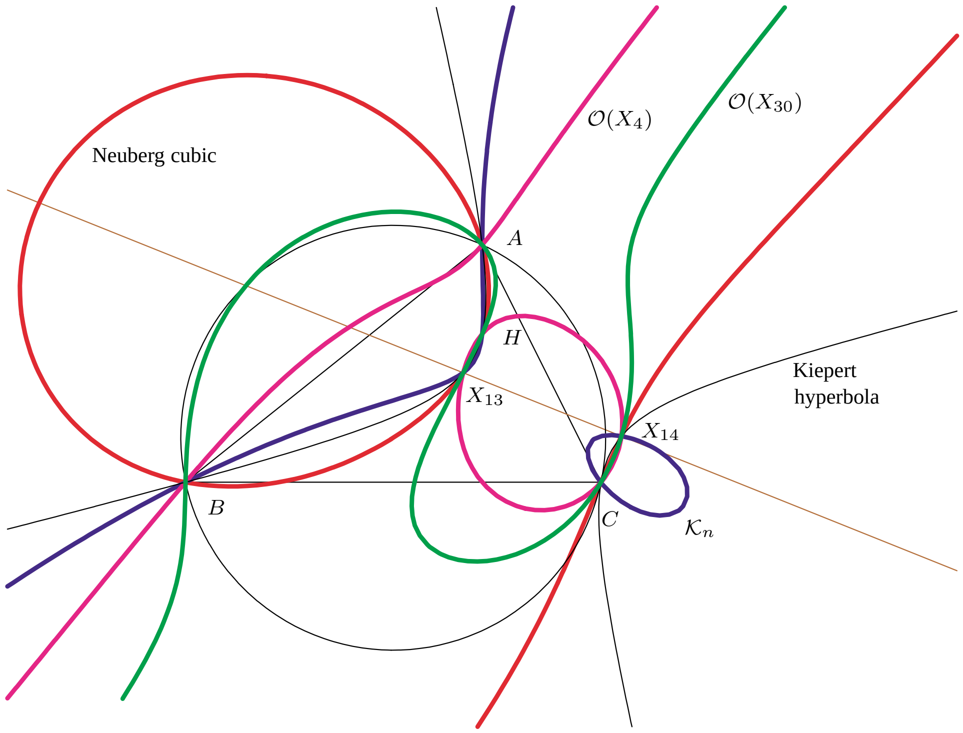

Figure 11. The Euler pencil

7.3. *Pencils with orthopivots on a line not passing through G.* If ℓ is a line not through G, the orthopivotal cubics in the pencil $\mathbb{F}_\ell$ pass through the two (not necessarily real nor distinct) intersections of ℓ with the circle which is its antiorthocorrespondent of. See §2.5 and §3. The singular focus lies on a circle through G, and the real asymptote envelopes a deltoid tangent to the line F_1F_2 and tritangent to the reflection of this circle about G.

According to §6.2.1, §6.2.2, §6.4, this pencil contains at least one, at most three $p\mathcal{K}$, $n\mathcal{K}$, focal(s) depending of the number of intersections of ℓ with the cubics met in those paragraphs respectively.

Consider, for example, the Brocard axis OK. We have seen in §6.3 that there are two and only two isogonal $\mathcal{O}(P)$, the Neuberg cubic and the second Brocard cubic $\mathcal{B}_2$ obtained when the orthopivots are O and K respectively. The two fixed points of the pencil are the isodynamic points.[36]

The singular focus lies on the Parry circle (see §5) and the asymptote envelopes a deltoid tritangent to the reflection of the Parry circle about G.

The pencil $\mathbb{F}_{OK}$ is invariant under isogonal conjugation, the isogonal conjugate of $\mathcal{O}(P)$ being $\mathcal{O}(Q)$, where Q is the harmonic conjugate of P with respect to

[36]The antiorthocorrespondent of the Brocard axis is a circle centered at X_{647}, the isogonal conjugate of the trilinear pole of the Euler line.

O and K. It is obvious that the Neuberg cubic and $\mathcal{B}_2$ are the only cubic which are "self-isogonal" and all the others correspond two by two. Since OK intersects the Napoleon cubic at O, X_{61} and X_{62}, there are only three $p\mathcal{K}$ in this pencil, the Neuberg cubic and $\mathcal{O}(X_{61})$, $\mathcal{O}(X_{62})$. [37]

$\mathcal{O}(X_{61})$ passes though X_{18}, X_{533}, X_{618}, and the isogonal conjugates of X_{532} and X_{619}.

$\mathcal{O}(X_{62})$ passes though X_{17}, X_{532}, X_{619}, and the isogonal conjugates of X_{533} and X_{618}. There are only three focals in the pencil $\mathbb{F}_{OK}$, namely, $\mathcal{B}_2$ and $\mathcal{O}(X_{15})$, $\mathcal{O}(X_{16})$ (with singular foci X_{16}, X_{15} respectively).

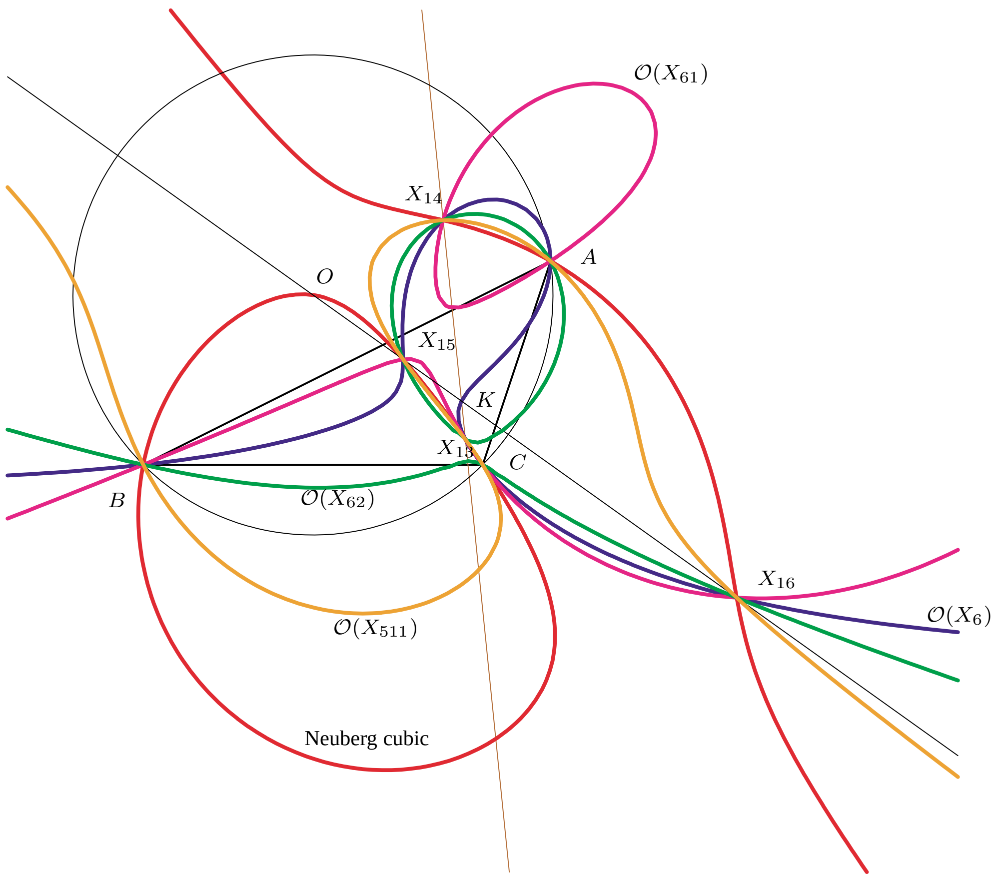

Figure 12. The Brocard pencil

An interesting situation is found when $P = X_{182}$, the midpoint of OK. Its harmonic conjugate with respect to OK is the infinite point $Q = X_{511}$. $\mathcal{O}(X_{511})$ passes through X_{262} which is its intersection with its real asymptote parallel at G

[37] $\mathcal{O}(X_{61})$ and $\mathcal{O}(X_{62})$ are isogonal conjugates of each other. Their pivots are X_{14} and X_{13} respectively and their poles are quite complicated and unknown in [5].

to OK. Its singular focus is G. The third intersection with the Fermat line is U_1 on $X_{23}X_{110}$ and the last intersection with the circumcircle is $X_{842} = X^*_{542}$.[38]

$\mathcal{O}(X_{182})$ is the isogonal conjugate of $\mathcal{O}(X_{511})$ and passes through X_{98}, X_{182}. Its singular focus is X_{23}, inverse of G in the circumcircle. Its real asymptote is parallel to the Fermat line at X_{323} and the intersection is the isogonal conjugate of U_1.

The following table gives several pairs of harmonic conjugates P and Q on OK. Each column gives two cubics $\mathcal{O}(P)$ and $\mathcal{O}(Q)$, each one being the isogonal conjugate of the other.

P	X_{32}	X_{50}	X_{52}	X_{58}	X_{187}	X_{216}	X_{284}	X_{371}	X_{389}	X_{500}
Q	X_{39}	X_{566}	X_{569}	X_{386}	X_{574}	X_{577}	X_{579}	X_{372}	X_{578}	X_{582}

8. A quintic and a quartic

We present a pair of interesting higher degree curves associated with the orthocorrespondence.

Theorem 12. *The locus of point P whose orthotransversal $\mathcal{L}_P$ and trilinear polar ℓ_P are parallel is the circular quintic*

$$\mathcal{Q}_1 : \qquad \sum_{\text{cyclic}} a^2y^2z^2(S_By - S_Cz) = 0.$$

Equivalently, $\mathcal{Q}_1$ is the locus of point P for which

(1) *the lines PP^* and ℓ_P (or $\mathcal{L}_P$) are perpendicular,*
(2) *P lies on the Euler line of the pedal triangle of P^*,*
(3) *P, P^*, H/P (and $P^\perp$) are collinear,*
(4) *P lies on $\mathcal{O}(P^*)$.*

Note that $\mathcal{L}_P$ and ℓ_P coincide when P is one of the Fermat points.[39]

Theorem 13. *The isogonal transform of the quintic $\mathcal{Q}_1$ is the circular quartic*

$$\mathcal{Q}_2 : \qquad \sum_{\text{cyclic}} a^4S_Ayz(c^2y^2 - b^2z^2) = 0,$$

which is also the locus of point P such that

(1) *the lines PP^* and ℓ_{P^*} (or $\mathcal{L}_{P^*}$) are perpendicular,*
(2) *P lies on the Euler line of its pedal triangle,*
(3) *P, P^*, H/P^* are collinear,*
(4) *P^* lies on $\mathcal{O}(P)$.*

These two curves $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contain a large number of interesting points, which we enumerate below.

Proposition 14. *The quintic $\mathcal{Q}_1$ contains the 58 following points:*

---

[38]This is on $X_{23}X_{110}$ too. It is the reflection of the Tarry point X_{98} about the Euler line and the reflection of X_{74} about the Brocard line.

[39]See §1, Remark (5).

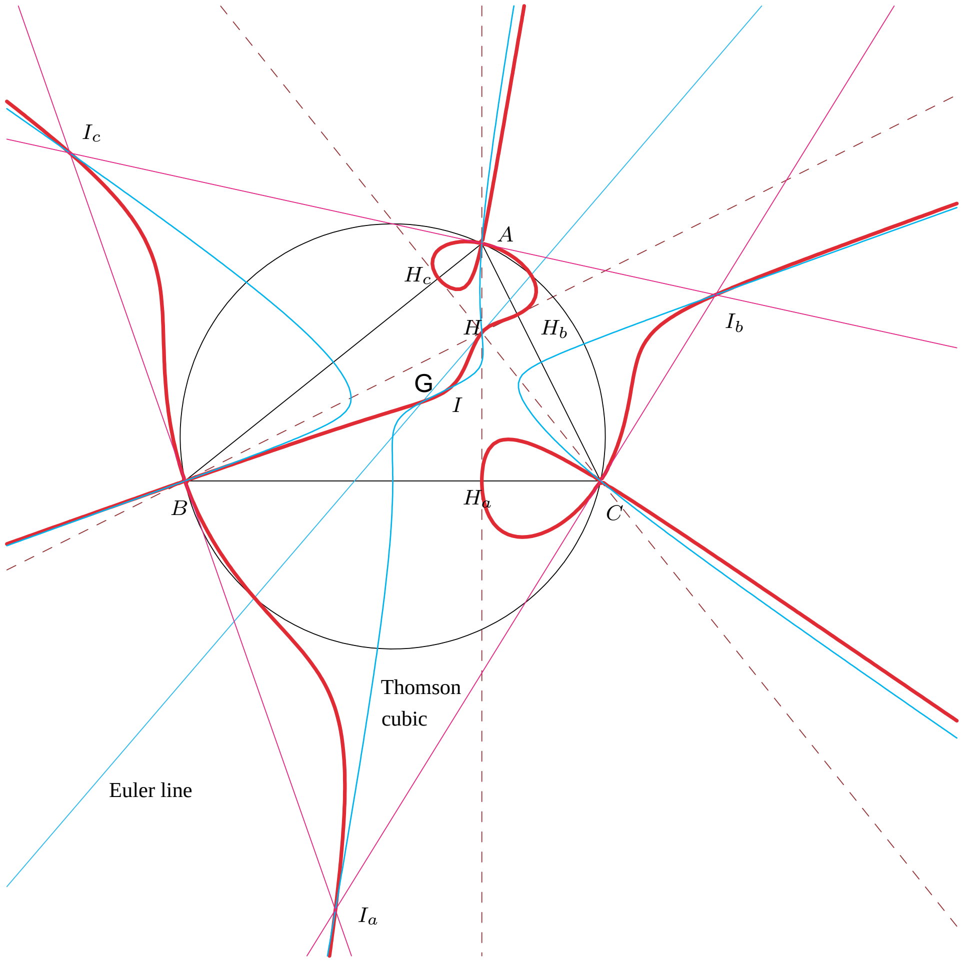

Figure 13. The quintic $\mathcal{Q}_1$

(1) *the vertices A, B, C, which are singular points with the bisectors as tangents,*
(2) *the circular points at infinity and the singular focus G,*[40]
(3) *the three infinite points of the Thomson cubic,*[41]
(4) *the in/excenters I, I_a, I_b, I_c, with tangents passing through O, and the isogonal conjugates of the intersections of these tangents with the trilinear polars of the corrresponding in/excenters,*
(5) *H, with tangent the Euler line,*

[40]The tangent at G passes through the isotomic conjugate of $G^{\perp}$, the point with coordinates $\left(\frac{1}{b^2+c^2-5a^2} : \cdots : \cdots\right)$.
[41]In other words, $\mathcal{Q}_1$ has three real asymptotes parallel to those of the Thomson cubic.

(6) *the six points where a circle with diameter a side of ABC intersects the corresponding median,* [42]
(7) *the feet of the altitudes, the tangents being the altitudes,*
(8) *the Fermat points X_{13} and X_{14},*
(9) *the points X_{1113} and X_{1114} where the Euler line meets the circumcircle,*
(10) *the perspectors of the 27 Morley triangles and ABC.* [43]

Proposition 15. *The quartic $\mathcal{Q}_2$ contains the 61 following points:*

(1) *the vertices A, B, C,* [44]
(2) *the circular points at infinity,* [45]
(3) *the three points where the Thomson cubic meets the circumcircle again,*
(4) *the in/excenters I, I_a, I_b, I_c, with tangents all passing through O, and the intersections of these tangents OI_x with the trilinear polars of the corresponding in/excenters,*
(5) *O and K,* [46]
(6) *the six points where a symmedian intersects a circle centered at the corresponding vertex of the tangential triangle passing through the remaining two vertices of ABC,* [47]
(7) *the six feet of bisectors,*
(8) *the isodynamic points X_{15} and X_{16}, with tangents passing through X_{23},*
(9) *the two infinite points of the Jerabek hyperbola,* [48]
(10) *the isogonal conjugates of the perspectors of the 27 Morley triangles and ABC.* [49]

We give a proof of (10). Let k_1, k_2, $k_3 = 0, \pm 1$, and consider

$$\varphi_1 = \frac{A + 2k_1\pi}{3}, \quad \varphi_2 = \frac{B + 2k_2\pi}{3}, \quad \varphi_3 = \frac{C + 2k_3\pi}{3}.$$

Denote by M one of the 27 points with barycentric coordinates

$$(a\cos\varphi_1 : b\cos\varphi_2 : c\cos\varphi_3).$$

---

[42]The two points on the median AG have coordinates

$$(2a : -a \pm \sqrt{2b^2 + 2c^2 - a^2} : -a \pm \sqrt{2b^2 + 2c^2 - a^2}).$$

[43]The existence of the these points was brought to my attention by Edward Brisse. In particular, X_{357}, the perspector of ABC and first Morley triangle.

[44]These are inflection points, with tangents passing through O.

[45]The singular focus is the inverse X_{23} of G in the circumcircle. This point is not on the curve $\mathcal{Q}_2$.

[46] Both tangents at O and K pass through the point $Z = (a^2S_A(b^2 + c^2 - 2a^2) : \cdots : \cdots)$, the intersection of the trilinear polar of O with the orthotransversal of X_{110}. The tangent at O is also tangent to the Jerabek hyperbola and the orthocubic.

[47]The two points on the symmedian AK have coordinates $(-a^2 \pm a\sqrt{2b^2 + 2c^2 - a^2} : 2b^2 : 2c^2)$.

[48]The two real asymptotes of $\mathcal{Q}_2$ are parallel to those of the Jerabek hyperbola and meet at Z in footnote 46 above.

[49]In particular, the Morley-Yff center X_{358}.

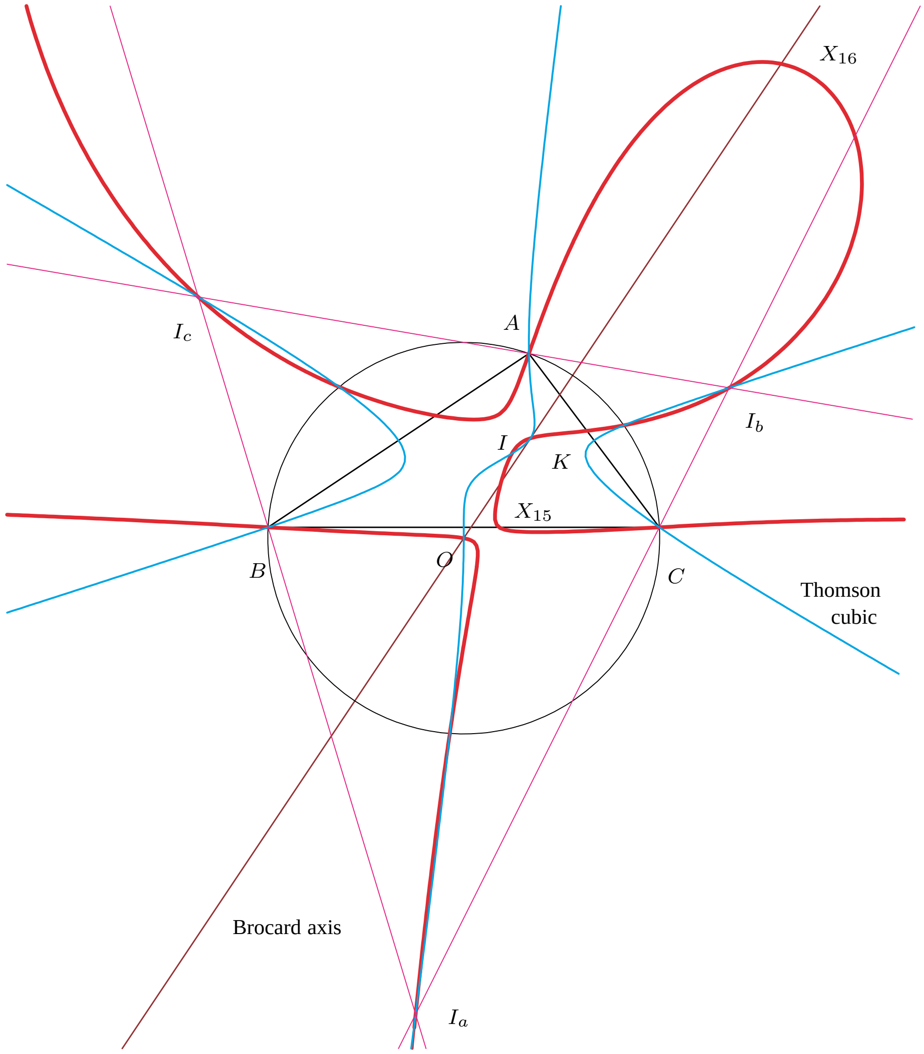

Figure 14. The quartic $\mathcal{Q}_2$

The isogonal conjugate of M is the perspector of ABC and one of the 27 Morley triangles. [50] We show that M lies on the quartic $\mathcal{Q}_2$. [51] Since $\cos A = \cos 3\varphi_1 = 4\cos^3\varphi_1 - 3\cos\varphi_1$, we have $\cos^3\varphi_1 = \frac{1}{4}(\cos A + 3\cos\varphi_1)$ and similar identities for $\cos^3\varphi_2$ and $\cos^3\varphi_3$. From this and the equation of $\mathcal{Q}_2$, we obtain

$$\sum_{\text{cyclic}} a^4 S_A b\cos\varphi_2\; c\cos\varphi_3\; (c^2b^2\cos^2\varphi_2 - b^2c^2\cos^2\varphi_3)$$

[50]For example, with $k_1 = k_2 = k_3 = 0$, $M^* = X_{357}$ and $M = X_{358}$.

[51]Consequently, M^* lies on the quintic $\mathcal{Q}_1$. See Proposition 14(10).

$$\begin{aligned}
&= \sum_{\text{cyclic}} a^4b^3c^3S_A(\cos\varphi_3\ \cos^3\varphi_2 - \cos\varphi_2\ \cos^3\varphi_3) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A(\cos\varphi_3\cos B - \cos\varphi_2\cos C) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A\left(\frac{S_B}{ac}\cos\varphi_3 - \frac{S_C}{ab}\cos\varphi_2\right) \\
&= \frac{1}{4}\, a^3b^3c^3S_AS_BS_C\sum_{\text{cyclic}}\left(\frac{\cos\varphi_3}{c\,S_C} - \frac{\cos\varphi_2}{b\,S_B}\right) \\
&= 0.
\end{aligned}$$

This completes the proof of (10).

Remark. $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are *strong* curves in the sense that they are invariant under extraversions: any point lying on one of them has its three extraversions also on the curve. [52]

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[52] The extraversions of a point are obtained by replacing one of a, b, c by its opposite. For example, the extraversions of the incenter I are the three excenters and I is said to be a *weak* point. On the contrary, K is said to be a ”strong” point.