

Bicentric Pairs of Points and Related Triangle Centers

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Abstract. Bicentric pairs of points in the plane of triangle ABC occur in connection with three configurations: (1) cevian traces of a triangle center; (2) points of intersection of a central line and central circumconic; and (3) vertex-products of bicentric triangles. These bicentric pairs are formulated using trilinear coordinates. Various binary operations, when applied to bicentric pairs, yield triangle centers.

1. Introduction

Much of modern triangle geometry is carried out in in one or the other of two homogeneous coordinate systems: barycentric and trilinear. Definitions of triangle center, central line, and bicentric pair, given in [2] in terms of trilinears, carry over readily to barycentric definitions and representations. In this paper, we choose to work in trilinears, except as otherwise noted.

Definitions of *triangle center* (or simply *center*) and *bicentric pair* will now be briefly summarized. A triangle center is a point (as defined in [2] as a function of variables a, b, c that are sidelengths of a triangle) of the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b),$$

where f is homogeneous in a, b, c , and

$$|f(a, c, b)| = |f(a, b, c)|. \quad (1)$$

If a point satisfies the other defining conditions but (1) fails, then the points

$$\begin{aligned} F_{ab} &:= f(a, b, c) : f(b, c, a) : f(c, a, b), \\ F_{ac} &:= f(a, c, b) : f(b, a, c) : f(c, b, a) \end{aligned} \quad (2)$$

are a *bicentric pair*. An example is the pair of Brocard points,

$$c/b : a/c : b/a \quad \text{and} \quad b/c : b/a : c/b.$$

Seven binary operations that carry bicentric pairs to centers are discussed in §§2, 3, along with three bicentric pairs associated with a center. In §4, bicentric pairs associated with cevian traces on the sidelines BC, CA, AB will be examined. §§6–10 examine points of intersection of a central line and central circumconic; these points are sometimes centers and sometimes bicentric pairs. §11 considers

bicentric pairs associated with bicentric triangles. §5 supports §6, and §12 revisits two operations discussed in §3.

2. Products: trilinear and barycentric

Suppose $U = u : v : w$ and $X = x : y : z$ are points expressed in general homogeneous coordinates. Their product is defined by

$$U \cdot X = ux : vy : wz.$$

Thus, when coordinates are specified as trilinear or barycentric, we have here two distinct product operations, corresponding to constructions of barycentric product [8] and trilinear product [6]. Because we have chosen trilinears as the primary means of representation in this paper, it is desirable to write, for future reference, a formula for barycentric product in terms of trilinear coordinates. To that end, suppose $u : v : w$ and $x : y : z$ are trilinear representations, so that in barycentrics,

$$U = au : bv : cw \quad \text{and} \quad X = ax : by : cz.$$

Then the barycentric product is $a^2ux : b^2vy : c^2wz$, and we conclude as follows: the trilinear representation for the barycentric product of $U = u : v : w$ and $X = x : y : z$, these being trilinear representations, is given by

$$U \cdot_b X = aux : bvy : cwz.$$

3. Other centralizing operations

Given a bicentric pair, aside from their trilinear and barycentric products, various other binary operations applied to the pair yield a center. Consider the bicentric pair (2). In [2, p. 48], the points

$$F_{ab} \oplus F_{ac} := f_{ab} + f_{ac} : f_{bc} + f_{ba} : f_{ca} + f_{cb} \quad (3)$$

and

$$F_{ab} \ominus F_{ac} := f_{ab} - f_{ac} : f_{bc} - f_{ba} : f_{ca} - f_{cb} \quad (4)$$

are observed to be triangle centers. See §8 for a geometric discussion.

Next, suppose that the points F_{ab} and F_{ac} do not lie on the line at infinity, \mathcal{L}^∞ , and consider normalized trilinears, represented thus:

$$F'_{ab} = (k_{ab}f_{ab}, k_{ab}f_{bc}, k_{ab}f_{ca}), \quad F'_{ac} = (k_{ac}f_{ac}, k_{ac}f_{ba}, k_{ac}f_{cb}), \quad (5)$$

where

$$k_{ab} := \frac{2\sigma}{af_{ab} + bf_{bc} + cf_{ca}}, \quad k_{ac} := \frac{2\sigma}{af_{ac} + bf_{ba} + cf_{cb}}, \quad \sigma := \text{area}(\triangle ABC).$$

These representations give

$$F'_{ab} \oplus F'_{ac} = k_{ab}f_{ab} + k_{ac}f_{ac} : k_{ab}f_{bc} + k_{ac}f_{ba} : k_{ab}f_{ca} + k_{ac}f_{cb}, \quad (6)$$

which for many choices of $f(a, b, c)$ differs from (3). In any case, (6) gives the the midpoint of the bicentric pair (2), and the harmonic conjugate of this midpoint with respect to F_{ab} and F_{ac} is the point in which the line $F_{ab}F_{ac}$ meets \mathcal{L}^∞ .

We turn now to another centralizing operation on the pair (2). Their line is given by the equation

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f_{ab} & f_{bc} & f_{ca} \\ f_{ac} & f_{ba} & f_{cb} \end{vmatrix} = 0$$

and is a central line. Its trilinear pole, P , and the isogonal conjugate of P , given by

$$f_{bc}f_{cb} - f_{ca}f_{ba} : f_{ca}f_{ac} - f_{ab}f_{cb} : f_{ab}f_{ba} - f_{bc}f_{ac},$$

are triangle centers.

If

$$X := x : y : z = f(a, b, c) : f(b, c, a) : f(c, a, b)$$

is a triangle center other than X_1 , then the points

$$Y := y : z : x \quad \text{and} \quad Z := z : x : y$$

are clearly bicentric. The operations discussed in §§2,3, applied to $\{Y, Z\}$, yield the following centers:

- trilinear product = X_1/X (the indexing of centers as X_i follows [3]);
- barycentric product = X_6/X (here, “/” signifies trilinear division);
- $Y \oplus Z = y + z : z + x : x + y$;
- $Y \ominus Z = y - z : z - x : x - y$;
- midpoint = $m(a, b, c) : m(b, c, a) : m(c, a, b)$, where

$$m(a, b, c) = cy^2 + bz^2 + 2ayz + x(by + cz);$$

- $YZ \cap \mathcal{L}^\infty = n(a, b, c) : n(b, c, a) : n(c, a, b)$, where

$$n(a, b, c) = cy^2 - bz^2 + x(by - cz);$$

- (isogonal conjugate of trilinear pole of YZ)

$$= x^2 - yz : y^2 - zx : z^2 - xy$$

$$= (X_1\text{-Hirst inverse of } X).$$

The points Z/Y and Y/Z are bicentric and readily yield the centers with first coordinates $x(y^2 + z^2)$, $x(y^2 - z^2)$, and $x^3 - y^2z^2/x$. One more way to make bicentric pairs from triangle centers will be mentioned: if $U = r : s : t$ and $X := x : y : z$ are centers, then ([2, p.49])

$$U \otimes X := sz : tx : ry, \quad X \otimes U := ty : rz : sx$$

are a bicentric pair. For example, $(U \otimes X) \ominus (X \otimes U)$ has for trilinears the coefficients for line UX .

4. Bicentric pairs associated with cevian traces

Suppose P is a point in the plane of $\triangle ABC$ but not on one of the sidelines BC , CA , AB and not on \mathcal{L}^∞ . Let A' , B' , C' be the points in which the lines AP , BP , CP meet the sidelines BC , CA , AB , respectively. The points A' , B' , C' are the *cevian traces* of P . Letting $|XY|$ denote the directed length of a segment from a point X to a point Y , we recall a fundamental theorem of triangle geometry (often called Ceva's Theorem, but Hogendijk [1] concludes that it was stated and proved by an ancient king) as follows:

$$|BA'| \cdot |CB'| \cdot |AC'| = |A'C| \cdot |B'A| \cdot |C'B|.$$

(The theorem will not be invoked in the sequel.) We shall soon see that if P is a center, then the points

$$P_{BC} := |BA'| : |CB'| : |AC'| \quad \text{and} \quad P_{CB} := |A'C| : |B'A| : |C'B|$$

comprise a bicentric pair, except for $P = \textit{centroid}$, in which case both points are the incenter. Let σ denote the area of $\triangle ABC$, and write $P = x : y : z$. Then the actual trilinear distances are given by

$$B = \left(0, \frac{2\sigma}{b}, 0\right) \quad \text{and} \quad A' = \left(0, \frac{2\sigma y}{by + cz}, \frac{2\sigma z}{by + cz}\right).$$

Substituting these into a distance formula (e.g. [2, p. 31]) and simplifying give

$$|BA'| = \frac{z}{b(by + cz)};$$

$$P_{BC} = \frac{z}{b(by + cz)} : \frac{x}{c(cz + ax)} : \frac{y}{a(ax + by)}; \quad (7)$$

$$P_{CB} = \frac{y}{c(by + cz)} : \frac{z}{a(cz + ax)} : \frac{x}{b(ax + by)}. \quad (8)$$

So represented, it is clear that P_{BC} and P_{CB} comprise a bicentric pair if P is a center other than the centroid. Next, let

$$P'_{BC} = \frac{|BA'|}{|CA'|} : \frac{|CB'|}{|AB'|} : \frac{|AC'|}{|BC'|} \quad \text{and} \quad P'_{CB} = \frac{|CA'|}{|BA'|} : \frac{|AB'|}{|CB'|} : \frac{|BC'|}{|AC'|}.$$

Equation (7) implies

$$P'_{BC} = \frac{cz}{by} : \frac{ax}{cz} : \frac{by}{ax} \quad \text{and} \quad P'_{CB} = \frac{by}{cz} : \frac{cz}{ax} : \frac{ax}{by}. \quad (9)$$

Thus, using "r" for trilinear quotient, or for barycentric quotient in case the coordinates in (7) and (8) are barycentrics, we have $P'_{BC} = P_{BC}/P_{CB}$ and $P'_{CB} = P_{CB}/P_{BC}$. The pair of isogonal conjugates in (9) generalize the previously mentioned Brocard points, represented by (9) when $P = X_1$.

As has been noted elsewhere, the trilinear (and hence barycentric) product of a bicentric pair is a triangle center. For both kinds of product, the representation is given by

$$P_{BC} \cdot P_{CB} = \frac{a}{x(by + cz)^2} : \frac{b}{y(cz + ax)^2} : \frac{c}{z(ax + by)^2}.$$

P	X_2	X_1	X_{75}	X_4	X_{69}	X_7	X_8
$P_{BC} \cdot P_{CB}$	X_{31}	X_{593}	X_{593}	X_{92}	X_{92}	X_{57}	X_{57}
$P_{BC} \cdot P_{CB}$	X_{32}	X_{849}	X_{849}	X_4	X_4	X_{56}	X_{56}

Table 1. Examples of trilinear and barycentric products

The line of a bicentric pair is clearly a central line. In particular, the line $P'_{BC}P'_{CB}$ is given by the equation

$$\left(\frac{a^2x^2}{bcyz} - \frac{bcyz}{a^2x^2}\right)\alpha + \left(\frac{b^2y^2}{cazx} - \frac{cazx}{b^2y^2}\right)\beta + \left(\frac{c^2z^2}{abxy} - \frac{abxy}{c^2z^2}\right)\gamma = 0.$$

This is the trilinear polar of the isogonal conjugate of the E -Hirst inverse of F , where $E = ax : by : cz$, and F is the isogonal conjugate of E . In other words, the point whose trilinears are the coefficients for the line $P'_{BC}P'_{CB}$ is the E -Hirst inverse of F .

The line $P_{BC}P_{CB}$ is given by $x'\alpha + y'\beta + z'\gamma = 0$, where

$$x' = bc(by + cz)(a^2x^2 - bcyz),$$

so that $P_{BC}P_{CB}$ is seen to be a certain product of centers if P is a center.

Regarding a euclidean construction for P_{BC} , it is easy to transfer distances for this purpose. Informally, we may describe P_{BC} and P'_{BC} as points constructed “by rotating through 90° the corresponding relative distances of the cevian traces from the vertices A, B, C ”.

5. The square of a line

Although this section does not involve bicentric pairs directly, the main result will make an appearance in §7, and it may also be of interest *per se*.

Suppose that $U_1 = u_1 : v_1 : w_1$ and $U_2 = u_2 : v_2 : w_2$ are distinct points on an arbitrary line L , represented in general homogeneous coordinates relative to $\triangle ABC$. For each point

$$X := u_1 + u_2t : v_1 + v_2t : w_1 + w_2t,$$

let

$$X^2 := (u_1 + u_2t)^2 : (v_1 + v_2t)^2 : (w_1 + w_2t)^2.$$

The locus of X^2 as t traverses the real number line is a conic section. Following the method in [4], we find an equation for this locus:

$$l^4\alpha^2 + m^4\beta^2 + n^4\gamma^2 - 2m^2n^2\beta\gamma - 2n^2l^2\gamma\alpha - 2l^2m^2\alpha\beta = 0, \quad (10)$$

where l, m, n are coefficients for the line U_1U_2 ; that is,

$$l : m : n = v_1w_2 - w_1v_2 : w_1u_2 - u_1w_2 : u_1v_2 - v_1u_2.$$

Equation (10) represents an inscribed ellipse, which we denote by L^2 . If the coordinates are trilinears, then the center of the ellipse is the point

$$bn^2 + cm^2 : cl^2 + an^2 : am^2 + bl^2.$$

6. (Line L) \cap (Circumconic Γ), two methods

Returning to general homogeneous coordinates, suppose that line L , given by $l\alpha + m\beta + n\gamma = 0$, meets circumconic Γ , given by $u/\alpha + v/\beta + w/\gamma = 0$. Let R and S denote the points of intersection, where $R = S$ if L is tangent to Γ . Substituting $-(m\beta + n\gamma)/l$ for α yields

$$mw\beta^2 + (mv + nw - lu)\beta\gamma + nv\gamma^2 = 0, \quad (11)$$

with discriminant

$$D := l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv, \quad (12)$$

so that solutions of (11) are given by

$$\frac{\beta}{\gamma} = \frac{lu - mv - nw \pm \sqrt{D}}{2mw}. \quad (13)$$

Putting β and γ equal to the numerator and denominator, respectively, of the right-hand side (13), putting $\alpha = -(m\beta + n\gamma)/l$, and simplifying give for R and S the representation

$$x_1 : y_1 : z_1 = m(mv - lu - nw \mp \sqrt{D}) : l(lu - mv - nw \pm \sqrt{D}) : 2lmw. \quad (14)$$

Cyclically, we obtain two more representations for R and S :

$$x_2 : y_2 : z_2 = 2mnu : n(nw - mv - lu \mp \sqrt{D}) : m(mv - nw - lu \pm \sqrt{D}) \quad (15)$$

and

$$x_3 : y_3 : z_3 = n(nw - lu - mv \pm \sqrt{D}) : 2nlv : l(lu - nw - mv \mp \sqrt{D}). \quad (16)$$

Multiplying the equal points in (14)-(16) gives R^3 and S^3 as

$$x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3$$

in cyclic form. The first coordinates in this form are

$$2m^2n^2u(mv - lu - nw \mp \sqrt{D})(nw - lu - mv \pm \sqrt{D}),$$

and these yield

$$(\text{1st coordinate of } R^3) = m^2n^2u[l^2u^2 - (mv - nw - \sqrt{D})^2] \quad (17)$$

$$(\text{1st coordinate of } S^3) = m^2n^2u[l^2u^2 - (mv - nw + \sqrt{D})^2]. \quad (18)$$

The 2nd and 3rd coordinates are determined cyclically.

In general, products (as in §2) of points on Γ intercepted by a line are notable: multiplying the first coordinates shown in (17) and (18) gives

$$(\text{1st coordinate of } R^3 \cdot S^3) = l^2m^5n^5u^4vw,$$

so that

$$R \cdot S = mnu : nlv : lmw.$$

Thus, on writing $L = l : m : n$ and $U = u : v : w$, we have $R \cdot S = U/L$.

The above method for finding coordinates of R and S in symmetric form could be called the multiplicative method. There is also an additive method.¹ Adding the coordinates of the points in (14) gives

$$m(mv - lu - nw) : l(lu - mv - nw) : 2lmw.$$

Do the same using (15) and (16), then add coordinates of all three resulting points, obtaining the point $U = u_1 : u_2 : u_3$, where

$$\begin{aligned} u_1 &= (lm + ln - 2mn)u + (m - n)(nw - mv) \\ u_2 &= (mn + ml - 2nl)v + (n - l)(lu - nw) \\ u_3 &= (nl + nm - 2lm)w + (l - m)(mv - lu). \end{aligned}$$

Obviously, the point

$$V = v_1 : v_2 : v_3 = m - n : n - l : l - m$$

also lies on L , so that L is given parametrically by

$$u_1 + tv_1 : u_2 + tv_2 : u_3 + tv_3. \quad (19)$$

Substituting into the equation for Γ gives

$$u(u_2 + tv_2)(u_3 + tv_3) + v(u_3 + tv_3)(u_1 + tv_1) + w(u_1 + tv_1)(u_2 + tv_2) = 0.$$

The expression of the left side factors as

$$(t^2 - D)F = 0, \quad (20)$$

where

$$F = u(n - l)(l - m) + v(l - m)(m - n) + w(m - n)(n - l).$$

Equation (20) indicates two cases:

Case 1: $F = 0$. Here, V lies on both L and Γ , and it is then easy to check that the point

$$W = mnu(n - l)(l - m) : nlv(l - m)(m - n) : lmw(m - n)(n - l)$$

also lies on both.

Case 2: $F \neq 0$. By (20), the points of intersection are

$$u_1 \pm v_1\sqrt{D} : u_2 \pm v_2\sqrt{D} : u_3 \pm v_3\sqrt{D}. \quad (21)$$

As an example to illustrate Case 1, take $u(a, b, c) = (b - c)^2$ and $l(a, b, c) = a$. Then $D = (b - c)^2(c - a)^2(a - b)^2$, and the points of intersection are $b - c : c - a : a - b$ and $(b - c)/a : (c - a)/b : (a - b)/c$.

¹I thank the Jean-Pierre Ehrmann for describing this method and its application.

7. $L \cap \Gamma$ when $D = 0$

The points R and S are identical if and only if $D = 0$. In this case, if in equation (12) we regard either $l : m : n$ or $u : v : w$ as a variable $\alpha : \beta : \gamma$, then the resulting equation is that of a conic inscribed in $\triangle ABC$. In view of equation (10), we may also describe this locus in terms of squares of lines; to wit, if $u : v : w$ is the variable $\alpha : \beta : \gamma$, then the locus is the set of squares of points on the four lines indicated by the equations

$$\sqrt{|l|}\alpha \pm \sqrt{|m|}\beta \pm \sqrt{|n|}\gamma = 0.$$

Taking the coordinates to be trilinears, examples of centers $X_i = l : m : n$ and $X_j = u : v : w$ for which $D = 0$ are given in Table 2. It suffices to show results for $i \leq j$, since L and U are interchangeable in (12).

i	j
1	244, 678
2	1015, 1017
3	125
6	115
11	55, 56, 181, 202, 203, 215
31	244, 1099, 1109, 1111
44	44

Table 2. Examples for $D = 0$

8. $L \cap \Gamma$ when $D \neq 0$ and $l : m : n = u : v : w$

Returning to general homogeneous coordinates, suppose now that $l : m : n$ and $u : v : w$ are triangle centers for which $D \neq 0$. Then, sometimes, R and S are centers, and sometimes, a bicentric pair. We begin with the case $l : m : n = u : v : w$, for which (12) gives

$$D := (u + v + w)(u - v + w)(u + v - w)(u - v - w).$$

This factorization shows that if $u + v + w = 0$, then $D = 0$. We shall prove that converse also holds. Suppose $D = 0$ but $u + v + w \neq 0$. Then one of the other three factors must be 0, and by symmetry, they must all be 0, so that $u = v + w$, so that

$$\begin{aligned} u(a, b, c) &= v(a, b, c) + w(a, b, c) \\ u(a, b, c) &= u(b, c, a) + u(c, a, b) \\ u(b, c, a) &= u(c, a, b) + u(a, b, c). \end{aligned}$$

Applying the third equation to the second gives $u(a, b, c) = u(c, a, b) + u(a, b, c) + u(c, a, b)$, so that $u(a, b, c) = 0$, contrary to the hypothesis that U is a triangle center.

Writing the roots of (11) as r_2/r_3 and s_2/s_3 , we find

$$\frac{r_2 s_2}{r_3 s_3} = \frac{(u^2 - v^2 - w^2 + \sqrt{D})(u^2 - v^2 - w^2 - \sqrt{D})}{4v^2 w^2} = 1,$$

which proves that R and S are a conjugate pair (isogonal conjugates in case the coordinates are trilinears). Of particular interest are cases for which these points are polynomial centers, as listed in Table 3, where, for convenience, we put

$$E := (b^2 - c^2)(c^2 - a^2)(a^2 - b^2).$$

u	\sqrt{D}	r_1	s_1
$a(b^2 - c^2)$	E	a	bc
$a(b^2 - c^2)(b^2 + c^2 - a^2)$	$16\sigma^2 E$	$\sec A$	$\cos A$
$a(b - c)(b + c - a)$	$4abc(b - c)(c - a)(a - b)$	$\cot(A/2)$	$\tan(A/2)$
$a^2(b^2 - c^2)(b^2 + c^2 - a^2)$	$4a^2 b^2 c^2 E$	$\tan A$	$\cot A$
$bc(a^4 - b^2 c^2)$	$(a^4 - b^2 c^2)(b^4 - c^2 a^2)(c^4 - a^2 b^2)$	b/c	c/b

Table 3. Points $R = r_1 : r_2 : r_3$ and $S = s_1 : s_2 : s_3$ of intersection

In Table 3, the penultimate row indicates that for $u : v : w = X_{647}$, the Euler line meets the circumconic $u/\alpha + v/\beta + w/\gamma = 0$ in the points X_4 and X_3 . The final row shows that R and S can be a bicentric pair.

9. $L \cap \Gamma$: Starting with Intersection Points

It is easy to check that a point R lies on Γ if and only if there exists a point $x : y : z$ for which

$$R = \frac{u}{by - cz} : \frac{v}{cz - ax} : \frac{w}{ax - by}.$$

From this representation, it follows that every line that meets Γ in distinct points

$$\frac{u}{by_i - cz_i} : \frac{v}{cz_i - ax_i} : \frac{w}{ax_i - by_i}, \quad i = 1, 2,$$

has the form

$$\frac{(by_1 - cz_1)(by_2 - cz_2)\alpha}{u} + \frac{(cz_1 - ax_1)(cz_2 - ax_2)\beta}{v} + \frac{(ax_1 - by_1)(ax_2 - by_2)\gamma}{w} = 0. \quad (22)$$

and conversely. In this case,

$$D = u^2 v^2 w^2 \begin{vmatrix} bc & ca & ab \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2,$$

indicating that $D = 0$ if and only if the points $x_i : y_i : z_i$ are collinear with the $bc : ca : ab$, which, in case the coordinates are trilinears, is the centroid of $\triangle ABC$.

Example 1. Let

$$x_1 : y_1 : z_1 = c/b : a/c : b/a \quad \text{and} \quad x_2 : y_2 : z_2 = b/c : c/a : a/b.$$

These are the 1st and 2nd Brocard points in case the coordinates are trilinears, but in any case, (22) represents the central line

$$\frac{\alpha}{ua^2(b^2 - c^2)} + \frac{\beta}{vb^2(c^2 - a^2)} + \frac{\gamma}{wc^2(a^2 - b^2)} = 0,$$

meeting Γ in the bicentric pair

$$\frac{u}{b^2(a^2 - c^2)} : \frac{v}{c^2(b^2 - a^2)} : \frac{w}{a^2(c^2 - b^2)}, \quad \frac{u}{c^2(a^2 - b^2)} : \frac{v}{a^2(b^2 - c^2)} : \frac{w}{b^2(c^2 - a^2)}.$$

Example 2. Let $X = x : y : z$ be a triangle center other than X_1 , so that $y : z : x$ and $z : x : y$ are a bicentric pair. The points

$$\frac{u}{bz - cx} : \frac{v}{cx - ay} : \frac{w}{ay - bz}, \quad \text{and} \quad \frac{u}{cy - bx} : \frac{v}{az - cy} : \frac{w}{bx - az}$$

are the bicentric pair in which the central line

$$vw(bx - cy)(cx - bz)\alpha + wu(cy - az)(ay - cx)\beta + uv(az - bx)(bz - ay)\gamma = 0$$

meets Γ .

10. $L \cap \Gamma$: Euler Line and Circumcircle

Example 3. Using trilinears, the circumcircle is given by $u(a, b, c) = a$ and the Euler line by

$$l(a, b, c) = a(b^2 - c^2)(b^2 + c^2 - a^2).$$

The discriminant $D = 4a^2b^2c^2d^2$, where

$$d = \sqrt{a^6 + b^6 + c^6 + 3a^2b^2c^2 - b^2c^2(b^2 + c^2) - c^2a^2(c^2 + a^2) - a^2b^2(a^2 + b^2)}.$$

Substitutions into (17) and (18) and simplification give the points of intersection, centers R and S , represented by 1st coordinates

$$\left\{ \frac{[ca(a^2 - c^2) \pm bd][ba(a^2 - b^2) \pm cd]}{(b^2 - c^2)^2(b^2 + c^2 - a^2)^2} \right\}^{1/3}.$$

11. Vertex-products of bicentric triangles

Suppose that $f(a, b, c) : g(b, c, a) : h(c, a, b)$ is a point, as defined in [2] We abbreviate this point as $f_{ab} : g_{bc} : h_{ca}$ and recall from [5, 7] that bicentric triangles are defined by the forms

$$\begin{pmatrix} f_{ab} & g_{bc} & h_{ca} \\ h_{ab} & f_{bc} & g_{ca} \\ g_{ab} & h_{bc} & f_{ca} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_{ac} & h_{ba} & g_{cb} \\ g_{ac} & f_{ba} & h_{cb} \\ g_{ac} & h_{ba} & f_{cb} \end{pmatrix}.$$

The vertices of the first of these two triangles are the rows of the first matrix, etc. We assume that $f_{ab}g_{ab}h_{ab} \neq 0$. Then the product of the three vertices, namely

$$f_{ab}g_{ab}h_{ab} : f_{bc}g_{bc}h_{bc} : f_{ca}g_{ca}h_{ca} \quad (23)$$

and the product of the vertices of the second triangle, namely

$$f_{ac}g_{ac}h_{ac} : f_{ba}g_{ba}h_{ba} : f_{cb}g_{cb}h_{cb} \quad (24)$$

clearly comprise a bicentric pair if they are distinct, and a triangle center otherwise.

Examples of bicentric pairs thus obtained will now be presented. An inductive method [6] of generating the non-circle-dependent objects of triangle geometry enumerates such objects in sets formally of size six. When the actual size is six, which means that no two of the six objects are identical, the objects form a pair

of bicentric triangles. The least such pair for which $f_{ab}g_{ab}h_{ab} \neq 0$ are given by Objects 31-36:

$$\begin{pmatrix} b & c \cos B & -b \cos B \\ -c \cos C & c & a \cos C \\ b \cos A & -a \cos A & a \end{pmatrix} \text{ and } \begin{pmatrix} c & -c \cos C & b \cos C \\ c \cos A & a & -a \cos A \\ -b \cos B & a \cos B & b \end{pmatrix}.$$

In this example, the bicentric pair of points (23) and (24) are

$$\frac{b}{a \cos B} : \frac{c}{b \cos C} : \frac{a}{c \cos A} \quad \text{and} \quad \frac{c}{a \cos C} : \frac{a}{b \cos A} : \frac{b}{c \cos B},$$

and the product of these is the center $\cos A \csc^3 A : \cos B \csc^3 B : \cos C \csc^3 C$.

This example and others obtained successively from Generation 2 of the aforementioned enumeration are presented in Table 4. Column 1 tells the Object numbers in [5]; column 2, the A -vertex of the least Object; column 3, the first coordinate of point (23) after canceling a symmetric function of (a, b, c) ; and column 4, the first coordinate of the product of points (23) and (24) after canceling a symmetric function of (a, b, c) . In Table 4, $\cos A, \cos B, \cos C$ are abbreviated as a_1, b_1, c_1 , respectively.

Objects	$f_{ab} : g_{ab} : h_{ab}$	$[f_{ab}g_{ab}h_{ab}]$	$[f_{ab}f_{ac}g_{ab}g_{ac}h_{ab}h_{ac}]$
31-36	$b : cb_1 : -bb_1$	b/ab_1	a_1/a^3
37-42	$bc_1 : -ca_1 : ba_1$	bc_1/aa_1	$(aa_1)^{-3}$
43-48	$bb_1 : c : -b$	bb_1/a	$(a_1a^3)^{-1}$
49-54	$ab : -c^2 : bc$	b/c	1
58-63	$c + ba_1 : cc_1 : -bc_1$	$(ba_1 + c)/ac_1$	$a_1(ba_1 + c)(ca_1 + b)a^{-2}$
71-76	$-b_1^2 : c_1 : b_1$	b_1^2/a_1	a_1^{-4}
86-91	$c_1 - a_1b_1 : c_1^2 : b_1c_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
92-97	$a_1b_1 : 1 : -a_1$	b_1/c_1	1
98-103	$1 : -c_1 : c_1a_1$	b_1/c_1	1
104-109	$aa_1 : -c : ca_1$	a/cc_1	a^3a_1
110-115	$a : b : -ba_1$	ab_1/b	a^3/a_1
116-121	$c_1 - a_1b_1 : 1 : -a_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
122-127	$1 + a_1^2 : c_1 : -c_1a_1$	$b_1(1 + a_1^2)/c_1$	$(1 + a_1^2)^2$
128-133	$2a_1 : -b_1 : a_1b_1$	a_1	a_1^2

Table 4. Bicentric triangles, bicentric points, and central vertex-products

Table 4 includes examples of interest: (i) bicentric triangles for which (23) and (24) are identical and therefore represent a center; (ii) distinct pairs of bicentric triangles that yield the identical bicentric pairs of points; and (iii) cases in which the pair (23) and (24) are isogonal conjugates. Note that Objects 49-54 yield for (23) and (24) the 2nd Brocard point, $\Omega_2 = b/c : c/a : a/b$ and the 1st Brocard point, $\Omega_1 = c/b : a/c : b/a$.

12. Geometric discussion: \oplus and \ominus

Equations (3) and (4) define operations \oplus and \ominus on pairs of bicentric points. Here, we shall consider the geometric meaning of these operations. First, note that one of the points in (2) lies on \mathcal{L}^∞ if and only if the other lies on \mathcal{L}^∞ , since the transformation $(a, b, c) \rightarrow (a, c, b)$ carries each of the equations

$$af_{ab} + bf_{bc} + cf_{ca} = 0, \quad af_{ac} + bf_{ba} + cf_{cb} = 0$$

to the other. Accordingly, the discussion breaks into two cases.

Case 1: F_{ab} not on \mathcal{L}^∞ . Let k_{ab} and k_{ac} be the normalization factors given in §3. Then the actual directed trilinear distances of F_{ab} and F_{ac} (to the sidelines BC, CA, AB) are given by (5). The point F that separates the segment $F_{ab}F_{ac}$ into segments satisfying

$$\frac{|F_{ab}F|}{|FF_{ac}|} = \frac{k_{ab}}{k_{ac}},$$

where $||$ denotes directed length, is then

$$\frac{k_{ac}}{k_{ab} + k_{ac}}F'_{ab} + \frac{k_{ab}}{k_{ab} + k_{ac}}F'_{ac} = \frac{k_{ac}k_{ab}}{k_{ab} + k_{ac}}F_{ab} + \frac{k_{ab}k_{ac}}{k_{ab} + k_{ac}}F_{ac},$$

which, by homogeneity, equals $F_{ab} \oplus F_{ac}$. Similarly, the point “constructed” as

$$\frac{k_{ac}}{k_{ab} + k_{ac}}F'_{ab} - \frac{k_{ab}}{k_{ab} + k_{ac}}F'_{ac}$$

equals $F_{ab} \ominus F_{ac}$. These representations show that $F_{ab} \oplus F_{ac}$ and $F_{ab} \ominus F_{ac}$ are a harmonic conjugate pair with respect to F_{ab} and F_{ac} .

Case 2: F_{ab} on \mathcal{L}^∞ . In this case, the isogonal conjugates F_{ab}^{-1} and F_{ac}^{-1} lie on the circumcircle, so that Case 1 applies:

$$F_{ab}^{-1} \oplus F_{ac}^{-1} = \frac{f_{ab} + f_{ac}}{f_{ab}f_{ac}} : \frac{f_{bc} + f_{ba}}{f_{bc}f_{ba}} : \frac{f_{ca} + f_{cb}}{f_{ca}f_{cb}}.$$

Trilinear multiplication [6] by the center $F_{ab} \cdot F_{ac}$ gives

$$F_{ab} \oplus F_{ac} = (F_{ab}^{-1} \oplus F_{ac}^{-1}) \cdot F_{ab} \cdot F_{ac}.$$

In like manner, $F_{ab} \ominus F_{ac}$ is “constructed”.

It is easy to prove that a pair P_{ab} and P_{ac} of bicentric points on \mathcal{L}^∞ are necessarily given by

$$P_{ab} = bf_{ca} - cf_{bc} : cf_{ab} - af_{ca} : af_{bc} - bf_{ab}$$

for some bicentric pair as in (2). Consequently,

$$\begin{aligned} P_{ab} \oplus P_{ac} &= g(a, b, c) : g(b, c, a) : g(c, a, b), \\ P_{ab} \ominus P_{ac} &= h(a, b, c) : h(b, c, a) : h(c, a, b), \end{aligned}$$

where

$$\begin{aligned} g(a, b, c) &= b(f_{ca} + f_{cb}) - c(f_{bc} + f_{ba}), \\ h(a, b, c) &= b(f_{ca} - f_{cb}) + c(f_{ba} - f_{bc}). \end{aligned}$$

Example 4. We start with $f_{ab} = c/b$, so that F_{ab} and F_{ac} are the Brocard points, and P_{ab} and P_{ac} are given by 1st coordinates $a - c^2/a$ and $a - b^2/a$, respectively, yielding 1st coordinates $(2a^2 - b^2 - c^2)/a$ and $(b^2 - c^2)/a$ for $P_{ab} \oplus P_{ac}$ and $P_{ab} \ominus P_{ac}$. These points are the isogonal conjugates of X_{111} (the Parry point) and X_{110} (focus of the Kiepert parabola), respectively.

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