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Some Configurations of Triangle Centers

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Abstract. Many collections of triangle centers and symmetrically defined triangle points are vertices of configurations. This illustrates a high level of organization among the points and their collinearities. Some of the configurations illustrated are inscriptable in Neuberg's cubic curve and others arise from Monge's theorem.

1. Introduction

By a configuration $\mathcal{K}$ we shall mean a collection of p points and g lines with r points on each line and q lines meeting at each point. This implies the relationship $pq = gr$. We then say that $\mathcal{K}$ is a (p_q, g_r) configuration. The simplest configuration is a point with a line through it. Another example is the triangle configuration, $(3_2, 3_2)$ with $p = g = 3$ and $q = r = 2$. When $p = g$, $\mathcal{K}$ is called *self-dual*, and then we must also have $q = r$. The symbol for the configuration is now simplified to read (p_q). The smallest (n_3) self-dual configurations exist combinatorially, when the "lines" are considered as suitable triples of points (vertices), but they cannot be realized with lines in the Euclidean plane. Usually when configurations are presented graphically, the lines appear as segments to make the figure compact and easy to interpret. Only one (7_3) configuration exists, the Fano plane of projective geometry, and only one (8_3) configuration exists, the Möbius-Kantor configuration. Neither of these can be realized with straight line segments. For larger n, the symbol may not determine a configuration uniquely. The smallest (n_3) configurations consisting of line segments in the Euclidean plane are (9_3), and there are three of them, one of which is the familiar Pappus configuration [4, pp.94–170]. The number of distinct (n_3) configurations grows rapidly with n. For example, there are 228 different (12_3) configurations [11, p.40]. In the discussion here, we shall only be concerned with configurations lying in a plane.

While configurations have long been studied as combinatorial objects, it does not appear that in any examples the vertices have been identified with triangle-derived points. In recent years there has been a resurgence of interest in triangle geometry along with the recognition of many new special points defined in different very ways. Since each point is defined from original principles, it is somewhat surprising that so many of them are collinear in small sets. An even higher level of relationship among special points is seen when they can be incorporated into

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certain configurations of moderate size. Then the collinearities and their incidences are summarized in a tidy, symmetrical, and graphic way. Here we exhibit several configurations whose vertices are naturally defined by triangles and whose lines are collinearities among them. It happens that the general theory for the first three examples was worked out long ago, but then the configurations were not identified as consisting of familiar triangle points and their collinearities.

2. Some configurations inscriptable in a cubic

First let us set the notation for several triangles. Given a triangle $\mathbf{T}$ with vertices A, B, and C, let A^* be the reflection of vertex A in side BC, A_+ the apex of an equilateral triangle erected outward on BC, and A_- the apex of an equilateral triangle erected inward on BC. Similarly define the corresponding points for B and C. Denote the triangle with vertices A^*, B^*, C^* as $\mathbf{T}^*$ and similarly define the triangles $\mathbf{T}_+$ and $\mathbf{T}_-$. Using trilinear coordinates it is straightforward to verify that the four triangles above are pairwise in perspective to one another. The points of perspective are as follows.

	$\mathbf{T}$	$\mathbf{T}^*$	$\mathbf{T}_+$	$\mathbf{T}_-$
$\mathbf{T}$		H	F_+	F_-
$\mathbf{T}^*$	H		J_-	J_+
$\mathbf{T}_+$	F_+	J_-		O
$\mathbf{T}_-$	F_-	J_+	O	

Here, O and H are respectively the circumcenter and orthocenter, $F_\pm$ the isogonic (Fermat) points, and $J_\pm$ the isodynamic points. They are triangle centers as defined by Kimberling [5, 6, 7, 8], who gives their trilinear coordinates and discusses their geometric significance. See also the in §5. For a simple simultaneous construction of all these points, see Evans [2].

To assemble the configurations, we first need to identify certain sets of collinear points. Now it is advantageous to introduce a notation for collinearity. Write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The key to identifying configurations among all the previously mentioned points depends on the observation that A^* , B_+, and C_- are always collinear, so we may write $\mathcal{L}(A^*, B_+, C_-)$. One can easily verify this using trilinear coordinates. This is also true for any permutation of A, B, and C, so we have

(I): the 6 lines $\mathcal{L}(A^*, B_+, C_-)$, $\mathcal{L}(A^*, B_-, C_+)$, $\mathcal{L}(B^*, C_+, A_-)$, $\mathcal{L}(B^*, C_-, A_+)$, $\mathcal{L}(C^*, A_+, B_-)$, $\mathcal{L}(C^*, A_-, B_+)$.

They all occur in Figures 1, 2, and 3. In fact the nine points A_+, A_-, A^*, ... themselves form the vertices of a $(9_2, 6_3)$ configuration.

It is easy to see other collinearities, namely 3 from each pair of triangles in perspective. For example, triangles $\mathbf{T}_+$ and $\mathbf{T}_-$ are in perspective from O, so we have

(II): the 3 lines $\mathcal{L}(A_+, O, A_-)$, $\mathcal{L}(B_+, O, B_-)$ and $\mathcal{L}(C_+, O, C_-)$.

See Figure 2.

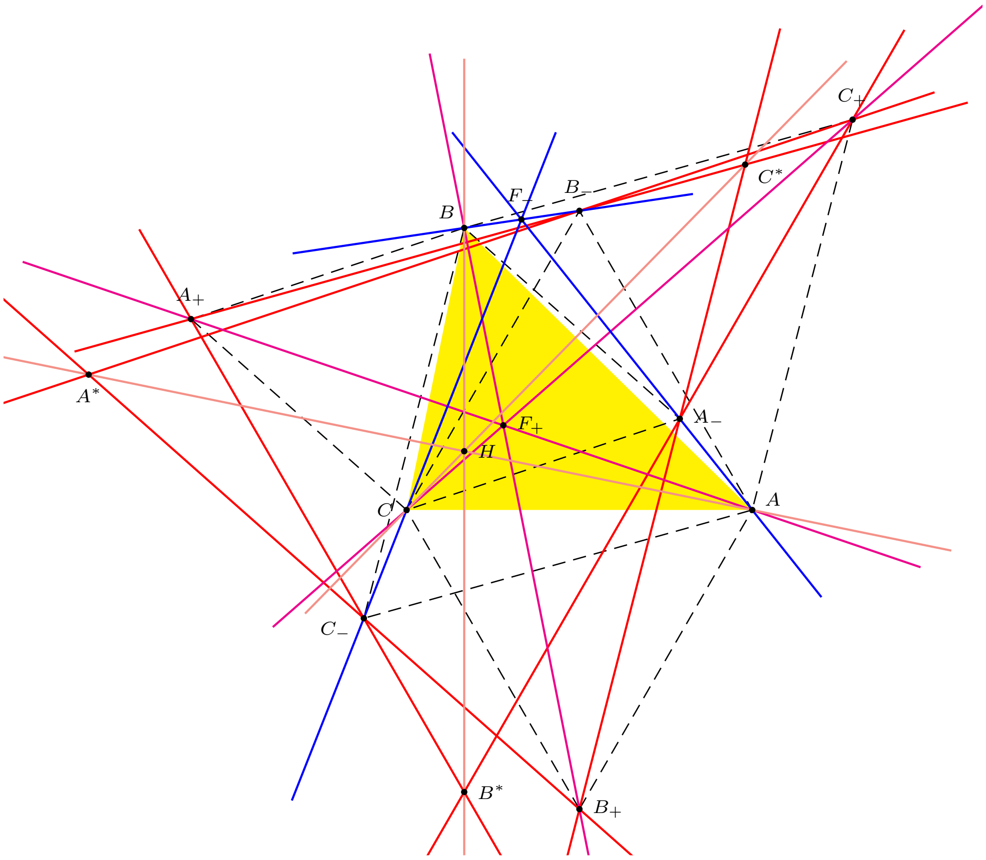

Figure 1. The Cremona-Richmond configuration

2.1. *The Cremona-Richmond configuration* (15_3). Consider the following sets of collinearities of three points:

(III): the 3 lines $\mathcal{L}(A, F_+, A_+)$, $\mathcal{L}(B, F_+, B_+)$ and $\mathcal{L}(C, F_+, C_+)$;
(IV): the 3 lines $\mathcal{L}(A, F_-, A_-)$, $\mathcal{L}(B, F_-, B_-)$ and $\mathcal{L}(C, F_-, C_-)$;
(V): the 3 lines $\mathcal{L}(A, H, A^*)$, $\mathcal{L}(B, H, B^*)$ and $\mathcal{L}(C, H, C^*)$.

The 15 points (A, B, C, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, H, $F_\pm$) and 15 lines in **(I), (III), (IV)**, and **(V)** form a figure which is called the Cremona-Richmond configuration [7]. See Figure 1. It has 3 lines meeting at each point with 3 points on each line, so it is self-dual with symbol (15_3). Inspection reveals that this configuration itself contains no triangles.

The reader may have noticed that the fifteen points in the configuration all lie on Neuberg's cubic curve, which is known to contain many triangle centers [7]. Recently a few papers, such as Pinkernell's [10] discussing Neuberg's cubic have

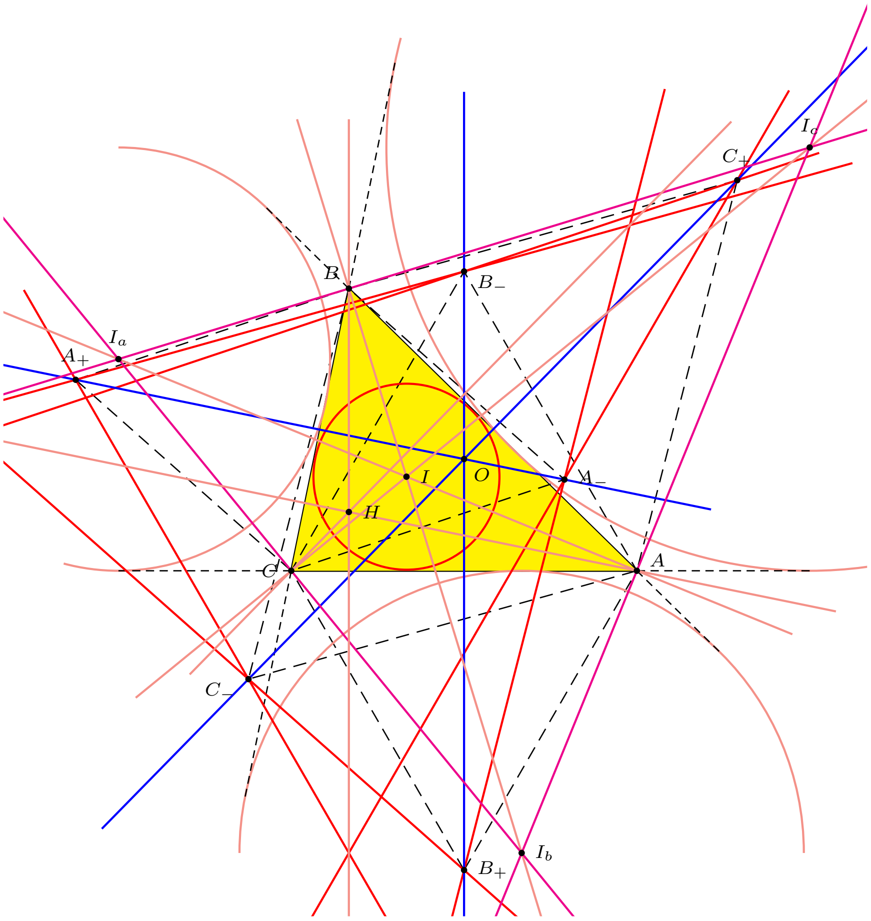

Figure 2

appeared, so we shall not elaborate on the curve itself. It has been known for a long time that many configurations are inscriptable in cubic curves, possibly first noticed by Schoenflies circa 1888 according to Feld [3]. However, it does not appear to be well-known that Neuberg's cubic in particular supports such configurations of familiar points. We shall exhibit two more configurations inscriptable in Neuberg's cubic.

2.2. *A* (18_3) *associated with the excentral triangle.* For another configuration, this one of the type (18_3), we employ the excentral triangle, that is, the triangle whose vertices are the excenters of $\mathbf{T}$. Denote the excenter opposite vertex A by I_a, etc., and denote the extriangle as $\mathbf{T}_{\mathrm{x}}$. Triangles $\mathbf{T}$ and $\mathbf{T}_{\mathrm{x}}$ are in perspective from the incenter, I. This introduces two more sets of collinearities involving the excenters:

(VI): the 3 lines $\mathcal{L}(A, I, I_a)$, $\mathcal{L}(B, I, I_b)$ and $\mathcal{L}(C, I, I_c)$;
(VII): the 3 lines $\mathcal{L}(I_b, A, I_c)$, $\mathcal{L}(I_c, B, I_a)$ and $\mathcal{L}(I_a, C, I_b)$.

The 18 lines of **(I), (II), (V), (VI), (VII)** and the 18 points A, B, C, I_a, I_b, I_c, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, O, H, and I form an (18_3) configuration. See Figure 2. There are enough points to suggest the outline of Neuberg's cubic, which is bipartite. The 10 points in the lower right portion of the figure lie on the ovoid portion of the curve. The 8 other points lie on the serpentine portion, which has an asymptote parallel to Euler's line (dashed). For other shapes of the basic triangle T, these points will not necessarily lie on the same components of the curve.

2.3. *A configuration* $(12_4, 16_3)$. Now we define two more sets of collinearities involving the isodynamic points:

(VIII): the 3 lines $\mathcal{L}(A^*, J_-, A_+)$, $\mathcal{L}(B^*, J_-, B_+)$ and $\mathcal{L}(C^*, J_-, C_+)$;
(IX): the 3 lines $\mathcal{L}(A^*, J_+, A_-)$, $\mathcal{L}(B^*, J_+, B_-)$ and $\mathcal{L}(C^*, J_+, C_-)$.

Among the centers of perspective we have defined so far, there is an additional collinearity, $\mathcal{L}(J_+, O, J_-)$, which is the Brocard axis. See Figure 3.

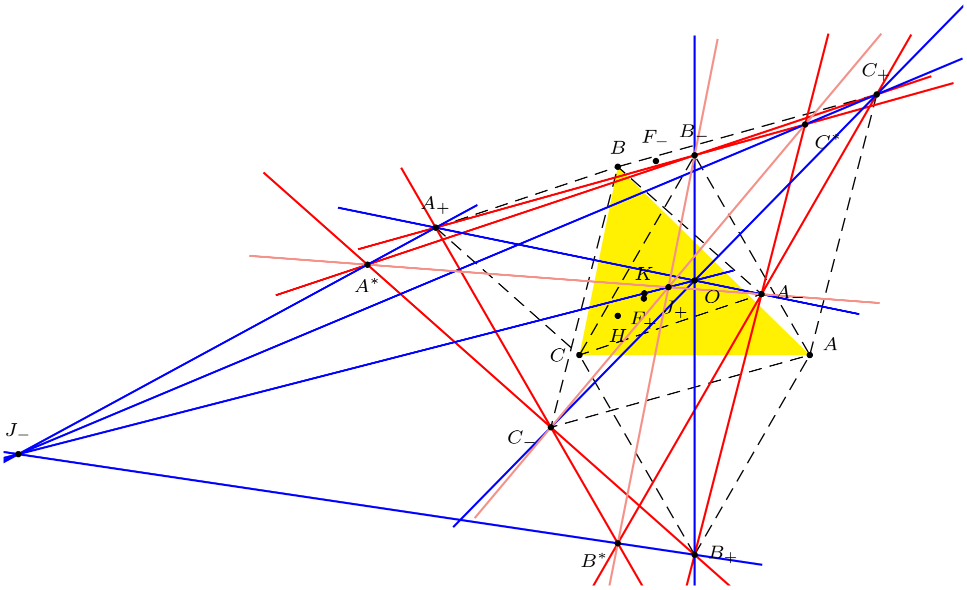

Figure 3

Using Weierstrass elliptic functions, Feld proved that within any bipartite cubic, a real configuration can be inscribed which has 12 points and 16 lines, with 4 lines meeting at each point and 3 points on each line [11], so that is, its symbol is $(12_4, 16_3)$. Now the Neuberg cubic of a non-equilateral triangle is bipartite, consisting of an ovoid portion and a serpentine portion whose asymptote is parallel to the Euler line of the triangle. Here one such inscriptable configuration consists of the following sets of lines: **(I), (II), (VIII), (IX)**, and the line, $\mathcal{L}(J_+, O, J_-)$. See Figure 3. The three triangles $\mathbf{T}_+$, $\mathbf{T}_-$, and $\mathbf{T}^*$ are pair-wise in perspective with collinear perspectors J_+, J_-, and O. The vertices of the basic triangle $\mathbf{T}$ are not in this configuration.

3. A Desargues configuration with triangle centers as vertices

There are so many collinearities involving triangle centers that we can also exhibit a Desargues (10_3) configuration with vertices consisting entirely of basic centers. Let K denote the symmedian (Lemoine's) point, N_p the center of the nine-point circle, G the centroid, N_+ the first Napoleon point, and N_- the second Napoleon point. Then the ten points F_+, F_-, J_+, J_-, N_+, N_-, K, G, H and N_p form the vertices of such a configuration. This is seen on noting that the triangles $F_-J_+N_+$ and $F_+J_-N_-$ are in perspective from K with the line of perspective $\mathcal{L}(G, N_p, H)$, which is Euler's line. See Figure 4. In a Desargues configuration any vertex may be chosen as the center of perspective of two suitable triangles. For simplicity we have chosen $\mathcal{K}$ in this example. Unlike the previous examples, Desargues configurations are not inscriptable in cubic curves [9].

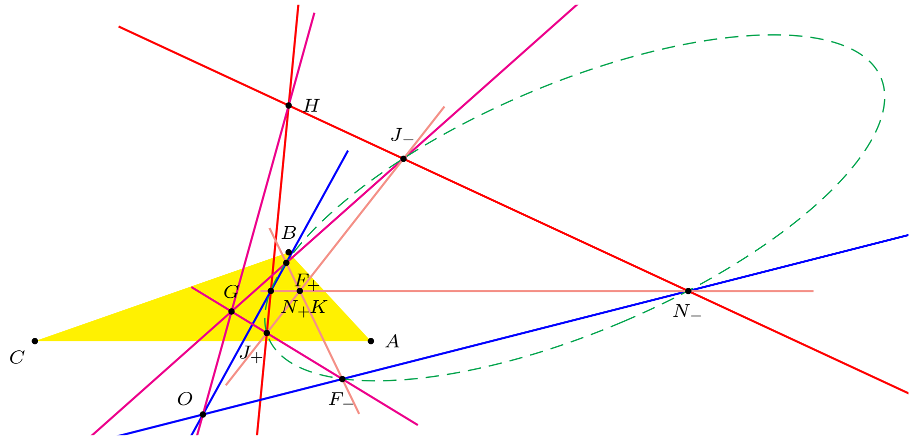

Figure 4

4. Configurations from Monge's theorem

Another way triangle centers form vertices of configurations arises from Monge's theorem [4, 11]. This theorem states that if we have three circles, then the 3 external centers of simitude (ecs) are collinear and that each external center of simitude is collinear with two of the internal centers of simitude (ics). These 4 collinearities form a $(4_3, 6_2)$ configuration, *i.e.*, a complete quadrilateral with the centers of similitude as vertices. This is best illustrated by an example. Suppose we have the circumcircle, the nine-point circle, and the incircle of a triangle. The ics of the circumcircle and the nine-point circle is the centroid, G, and their ecs is the orthocenter, H. The ics of the nine-point circle and the incircle is X_{12} in Kimberling's list and the ecs is Feuerbach's point, X_{11}. The ics of the circumcircle and the incircle is X_{55}, and the ecs is X_{56}. The lines of the configuration are then $\mathcal{L}(H, X_{56}, X_{11})$, $\mathcal{L}(G, X_{55}, X_{11})$, $\mathcal{L}(G, X_{56}, X_{12})$, and $\mathcal{L}(H, X_{55}, X_{12})$. This construction, of course, applies to any group of three circles related to the triangle. In the example given, the circles can be nested, so it may not be easy to see the centers of similitude. In such a case, the radii of the circles can be reduced in the same proportion to make the circles small enough that they do not overlap. The ecs's and ics's remain the same. The ecs of two such circles is the point where the two common external tangents meet, and the ics is the point where the two common internal tangents meet. When two of the circles have the same radii, their ics is the midpoint of the line joining their centers and their ecs is the point at infinity in the direction of the line joining their centers.

One may ask what happens when a fourth circle whose center is not collinear with any other two is also considered. Monge's theorem applies to each group of three circles. First it happens that the four lines containing only ecs's themselves form a $(6_2, 4_3)$ configuration. Second, when the twelve lines containing an ecs and two ics's are annexed, the result is a $(12_4, 16_3)$ configuration. This is a projection onto the plane of Reye's three-dimensional configuration, which arises from a three-dimensional analog of Monge's theorem for four spheres [4]. This is illustrated in Figure 5 with the vertices labelled with the points of Figure 3, which shows that these two $(12_4, 16_3)$ configurations are actually the same even though the representation in Figure 5 may not be inscriptable in a bipartite cubic. Evidently larger configurations arise by the same process when yet more circles are considered.

5. Final remarks

We have see that certain collections of collinear triangle points can be knitted together into highly symmetrical structures called configurations. Furthermore some relatively large configuratons such as the (18_3) shown above are inscriptable in low degree algebraic curves, in this case a cubic.

General information about configurations can be found in Hilbert and Cohn-Vossen [4]. Also we recommend Coxeter [1], which contains an extensive bibliography of related material pre-dating 1950.

The centers here appear in Kimberling [5, 6, 7, 8] as X_n for n below.

center	I	G	O	H	N_p	K	F_+	F_-	J_+	J_-	N_+	N_-
n	1	2	3	4	5	6	13	14	15	16	17	18

While not known by eponyms, X_{12}, X_{55}, and X_{56} are also geometrically significant in elementary ways [7, 8].

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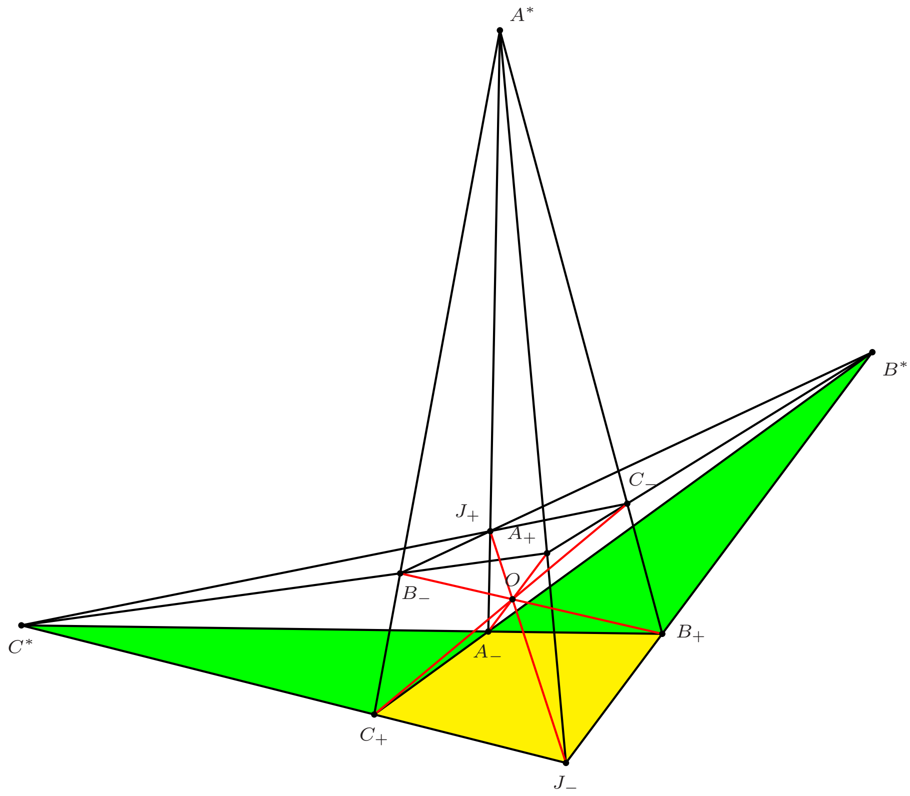

Figure 5

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