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On the Circumcenters of Cevasix Configurations

Alexei Myakishev and Peter Y. Woo

Abstract. We strengthen Floor van Lamoen's theorem that the 6 circumcenters of the cevasix configuration of the centroid of a triangle are concyclic by giving a proof which at the same time shows that the converse is also true with a minor qualification, *i.e.*, the circumcenters of the cevasix configuration of a point P are concyclic if and only if P is the centroid or the orthocenter of the triangle.

1. Introduction

Let P be a point in the plane of triangle ABC, with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. Triangle ABC is then divided by its cevians AA', BB', CC' into six triangles, giving rise to what Clark Kimberling [2, pp.257–260] called the *cevasix configuration* of P. See Figure 1. Floor van Lamoen has discovered that when P is the centroid of triangle ABC, the 6 circumcenters of the cevasix configuration are concyclic. See Figure 2. This was posed as a problem in the *American Mathematical Monthly* [3]. Solutions can be found in [3, 4]. In this note we study the converse.

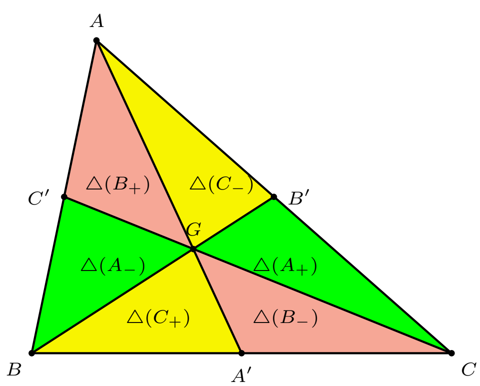

Figure 1

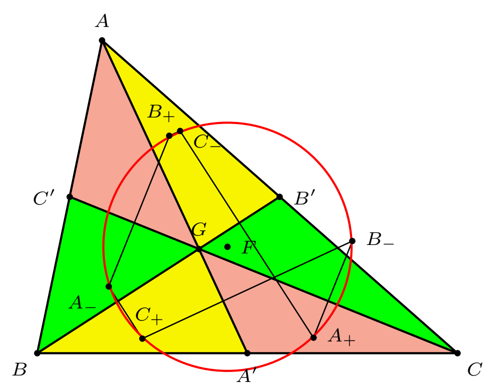

Figure 2

Theorem 1. *The circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of triangle ABC.*

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2. Preliminary results

We adopt the following notations.

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangle may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Lemma 2. *The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle in the line BC.*

Proof. Triangles APB' and APC' have the same circumcenter if and only if the four points A, B', P, C' are concyclic. In terms of directed angles, $\angle BPC = \angle B'PC' = \angle B'AC' = \angle CAB = -\angle BAC$. See, for example, [1, §§16–20]. It follows that the reflection of A in the line BC lies on the circumcircle of triangle PBC, and P lies on the reflection of the circumcircle in BC. The converse is clear. □

Thus, if $B_+ = C_-$ and $C_+ = A_-$, then necessarily P is the orthocenter H, and also $A_+ = B_-$. In this case, there are only three distinct circumcenters. They clearly lie on the nine-point circle of triangle ABC. We shall therefore assume $P \neq H$, so that there are at least five distinct points in the set $\{A_\pm, B_\pm, C_\pm\}$.

The next proposition appears in [2, p.259].

Proposition 3. *The 6 circumcenters of the cevasix configuration of P lie on a conic.*

Proof. We need only consider the case when these 6 circumcenters are all distinct. The circumcenters B_+ and C_- lie on the perpendicular bisector of the segment AP; similarly, B_- and C_+ lie on the perpendicular bisector of PA'. These two perpendicular bisectors are clearly parallel. This means that B_+C_- and B_-C_+ are parallel. Similarly, $C_+A_-//C_-A_+$ and $A_+B_-//A_-B_+$. The hexagon $A_+C_-B_+A_-C_+B_-$ has three pairs of parallel opposite sides. By the converse of Pascal's theorem, there is a conic passing through the six vertices of the hexagon. □

Proposition 4. *The vertices of a hexagon $A_+C_-B_+A_-C_+B_-$ with parallel opposite sides $B_+C_-//C_+B_-$, $C_+A_-//A_+C_-$, $A_+B_-//B_+A_-$ lie on a circle if and only if the main diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths.*

Proof. If the vertices are concyclic, then $A_+C_-A_-C_+$ is an isosceles trapezoid, and $A_+A_- = C_+C_-$. Similarly, $C_+B_-C_-B_+$ is also an isosceles trapezoid, and $C_+C_- = B_+B_-$.

Conversely, consider the triangle XYZ bounded by the three diagonals A_+A_-, B_+B_- and C_+C_-. If these diagonals are equal in length, then the trapezoids $A_+C_-A_-C_+$, $C_+B_-C_-B_+$ and $B_+A_-B_-A_+$ are isosceles. From these we immediately conclude that the common perpendicular bisector of A_+C_- and A_-C_+

is the bisector of angle XYZ. Similarly, the common perpendicular bisector of B_+C_- and B_-C_+ is the bisector of angle X, and that of A_+B_- and A_-B_+ the bisector of angle Z. These three perpendicular bisectors clearly intersect at a point, the incenter of triangle XYZ, which is equidistant from the six vertices of the hexagon. □

Proposition 5. *The vector sum* $\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = \mathbf{0}$ *if and only if* P *is the centroid.*

Proof. Suppose with reference to triangle ABC, the point P has *absolute* barycentric coordinates $uA + vB + wC$, where $u + v + w = 1$. Then,

$$A' = \frac{1}{v+w}(vB + wC), \quad B' = \frac{1}{w+u}(wC + uA), \quad C' = \frac{1}{u+v}(uA + vB).$$

From these,

$$\begin{aligned}&\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'}\\ =&(A' + B' + C') - (A + B + C)\\ =&\frac{u^2 - vw}{(w+u)(u+v)} \cdot A + \frac{v^2 - wu}{(u+v)(v+w)} \cdot B + \frac{w^2 - uv}{(v+w)(w+u)} \cdot C.\end{aligned}$$

This is zero if and only if

$$u^2 - vw = v^2 - wu = w^2 - uv = 0,$$

and $u = v = w = \frac{1}{3}$ since they are all real, and $u + v + w = 1$. □

We denote by π_a, π_b, π_c the orthogonal projections on the lines AA', BB', CC' respectively.

Proposition 6.

$$\begin{aligned}\pi_b(\mathbf{A_+A_-}) &= -\tfrac{1}{2}\mathbf{BB'}, & \pi_c(\mathbf{A_+A_-}) &= \tfrac{1}{2}\mathbf{CC'},\\ \pi_c(\mathbf{B_+B_-}) &= -\tfrac{1}{2}\mathbf{CC'}, & \pi_a(\mathbf{B_+B_-}) &= \tfrac{1}{2}\mathbf{AA'},\\ \pi_a(\mathbf{C_+C_-}) &= -\tfrac{1}{2}\mathbf{AA'}, & \pi_b(\mathbf{C_+C_-}) &= \tfrac{1}{2}\mathbf{BB'}.\end{aligned} \tag{1}$$

Proof. The orthogonal projections of A_+ and A_- on the cevian BB' are respectively the midpoints of the segments PB' and BP. Therefore,

$$\pi_b(\mathbf{A_+A_-}) = \frac{B+P}{2} - \frac{P+B'}{2} = -\frac{B'-B}{2} = -\frac{1}{2}\mathbf{BB'}.$$

The others follow similarly. □

3. Proof of Theorem 1

Sufficiency part. Let P be the centroid G of triangle ABC. By Proposition 4, it is enough to prove that the diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths. By Proposition 5, we can construct a triangle $A^*B^*C^*$ whose sides as vectors $\mathbf{B^*C^*}$, $\mathbf{C^*A^*}$ and $\mathbf{A^*B^*}$ are equal to the medians $\mathbf{AA'}$, $\mathbf{BB'}$, $\mathbf{CC'}$ respectively.

Consider the vector $\mathbf{A}^*\mathbf{Q}$ equal to $\mathbf{A}_+\mathbf{A}_-$. By Proposition 6, the orthogonal projections of $\mathbf{A}_+\mathbf{A}_-$ on the two sides C^*A^* and A^*B^* are the midpoints of the sides. This means that Q is the circumcenter of triangle $A^*B^*C^*$, and the length of $\mathbf{A}_+\mathbf{A}_-$ is equal to the circumradius of triangle $A^*B^*C^*$. The same is true for the lengths of $\mathbf{B}_+\mathbf{B}_-$ and $\mathbf{C}_+\mathbf{C}_-$. The case $P = H$ is trivial.

Necessity part. Suppose the 6 circumcenters $A_\pm$, $B_\pm$, $C_\pm$ lie on a circle. By Proposition 3, the diagonals A_+A_-, B_+B_-, and C_+C_- have equal lengths. We show that $\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = 0$, so that P is the centroid of triangle ABC by Proposition 5. In terms of scalar products, we rewrite equation (1) as

$$\begin{aligned}
\mathbf{A}_+\mathbf{A}_- \cdot \mathbf{BB}' &= -\tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}', & \mathbf{A}_+\mathbf{A}_- \cdot \mathbf{CC}' &= \tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}', \\
\mathbf{B}_+\mathbf{B}_- \cdot \mathbf{CC}' &= -\tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}', & \mathbf{B}_+\mathbf{B}_- \cdot \mathbf{AA}' &= \tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}', \qquad (2) \\
\mathbf{C}_+\mathbf{C}_- \cdot \mathbf{AA}' &= -\tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}', & \mathbf{C}_+\mathbf{C}_- \cdot \mathbf{BB}' &= \tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}'.
\end{aligned}$$

From these, $(\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}' = 0$, and $\mathbf{AA}'$ is orthogonal to $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$. Since $\mathbf{B}_+\mathbf{B}_-$, and $\mathbf{C}_+\mathbf{C}_-$ have equal lengths, $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$ and $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-$ are orthogonal. We may therefore write $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_- = k\mathbf{AA}'$ for a scalar k. From (2) above,

$$\begin{aligned}
k\mathbf{AA}' \cdot \mathbf{AA}' =& (\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}' \\
=& \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}' + \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}' \\
=& \mathbf{AA}' \cdot \mathbf{AA}'.
\end{aligned}$$

From this, $k = 1$ and we have

$$\mathbf{AA}' = \mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-.$$

The same reasoning shows that

$$\begin{aligned}
\mathbf{BB}' =& \mathbf{C}_+\mathbf{C}_- - \mathbf{A}_+\mathbf{A}_-, \\
\mathbf{CC}' =& \mathbf{A}_+\mathbf{A}_- - \mathbf{B}_+\mathbf{B}_-.
\end{aligned}$$

Combining the three equations, we have

$$\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = \mathbf{0}.$$

It follows from Proposition 5 that P must be the centroid of triangle ABC.

4. An alternative proof of Theorem 1

We present another proof of Theorem 1 by considering an auxiliary hexagon. Let $\mathcal{L}_a$ and $\mathcal{L}'_a$ be the lines perpendicular to AA' at A and A' respectively; similarly, $\mathcal{L}_b$, $\mathcal{L}'_b$, and $\mathcal{L}_c$ and $\mathcal{L}'_c$. Consider the points

$$\begin{aligned}
X_+ &= \mathcal{L}_c \cap \mathcal{L}'_b, & X_- &= \mathcal{L}_b \cap \mathcal{L}'_c, \\
Y_+ &= \mathcal{L}_a \cap \mathcal{L}'_c, & Y_- &= \mathcal{L}_c \cap \mathcal{L}'_a, \\
Z_+ &= \mathcal{L}_b \cap \mathcal{L}'_a, & Z_- &= \mathcal{L}_a \cap \mathcal{L}'_b.
\end{aligned}$$

Note that the circumcenters $A_\pm$, $B_\pm$, $C_\pm$ are respectively the midpoints of $PX_\pm$, $PY_\pm$, $PZ_\pm$. Hence, the six circumcenters are concyclic if and only if $X_\pm$, $Y_\pm$, $Z_\pm$ are concyclic.

In Figure 3, let $\angle CPA' = \angle APC' = \alpha$. Since angles $PA'Y_-$ and PCY_- are both right angles, the four points P, A', C, Y_- are concyclic and $\angle Z_+Y_-X_+ = \angle A'Y_-X_+ = \angle A'PC = \alpha$. Similarly, $\angle CPB' = \angle BPC' = \angle Y_-X_+Z_-$, and we denote the common measure by β.

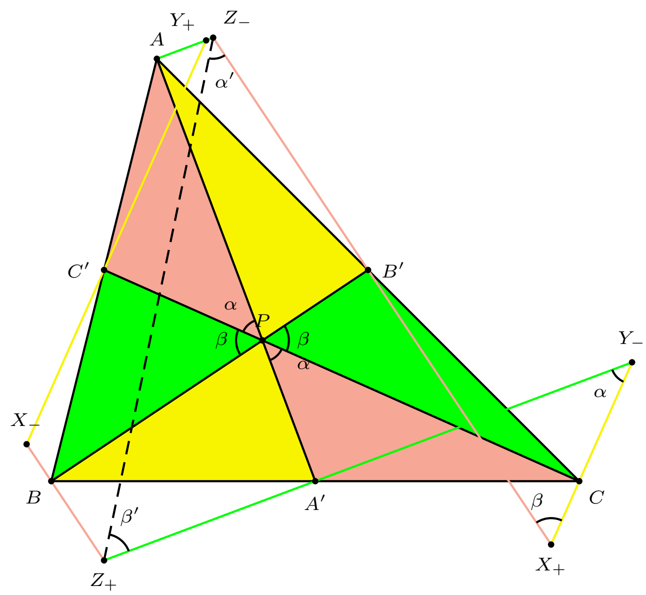

Figure 3

Lemma 7. *If the four points X_+, Y_-, Z_+, Z_- are concyclic, then P lies on the median through C.*

Proof. Let $x = \frac{AP}{AA'}$ and $y = \frac{BP}{BB'}$. If the four points X_+, Y_-, Z_+, Z_- are concyclic, then $\angle Z_+Z_-X_+ = \alpha$ and $\angle Y_-Z_+Z_- = \beta$. Now,

$$\frac{|BB'|}{|AA'|} = \frac{|Z_+Z_-| \cdot \sin\alpha'}{|Z_+Z_-| \cdot \sin\beta'} = \frac{\sin\alpha}{\sin\beta} = \frac{\frac{|AC'|}{|AP|}}{\frac{|BC'|}{|BP|}}.$$

It follows that

$$\frac{|BP|}{|BB'| \cdot |BC'|} = \frac{|AP|}{|AA'| \cdot |AC'|},$$

and, as a ratio of signed lengths,

$$\frac{BC'}{AC'} = -\frac{y}{x}. \tag{3}$$

Now applying Menelaus' theorem to triangle APC' with transversal $A'CB$, and triangle BGA' with transversal $B'CA$, we have

$$\frac{AA'}{A'P}\cdot\frac{PC}{CC'}\cdot\frac{C'B}{BA}=-1=\frac{BB'}{B'P}\cdot\frac{PC}{CC'}\cdot\frac{C'A}{AB}.$$

From this, $\frac{AA'}{A'P}\cdot BC'=\frac{BB'}{B'P}\cdot AC'$, or

$$\frac{BC'}{1-x}=-\frac{AC'}{1-y}. \qquad (4)$$

Comparing (3) and (4), we have $\frac{1-x}{1-y}=\frac{y}{x}$, $(x-y)(x+y-1)=0$. Either $x=y$ or $x+y=1$. It is easy to eliminate the possibility $x+y=1$. If P has homogeneous barycentric coordinates $(u:v:w)$ with reference to triangle ABC, then $x=\frac{v+w}{u+v+w}$ and $y=\frac{w+u}{u+v+w}$. Thus, $x+y=1$ requires $w=0$ and P lies on the sideline AB, contrary to the assumption. It follows that $x=y$, and from (3), C' is the midpoint of AB, and P lies on the median through C. □

The necessity part of Theorem 1 is now an immediate corollary of Lemma 7.

5. Concluding remark

We conclude with a remark on triangles for which two of the circumcenters of the cevasix configuration of the centroid coincide. Clearly, $B_+=C_-$ if and only if A, B', G, C' are concyclic. Equivalently, the image of G under the homothety $\mathsf{h}(A,2)$ lies on the circumcircle of triangle ABC. This point has homogeneous barycentric coordinates $(-1:2:2)$. Since the circumcircle has equation

$$a^2yz+b^2zx+c^2xy=0,$$

we have $2a^2=b^2+c^2$. There are many interesting properties of such triangles. We simply mention that it is similar to its own triangle of medians. Specifically,

$$m_a=\frac{\sqrt{3}}{2}a,\quad m_b=\frac{\sqrt{3}}{2}c,\quad m_c=\frac{\sqrt{3}}{2}b.$$

Editor's endnote. John H. Conway [5] has located the center of the Van Lamoen circle (of the circumcenters of the cevasix configuration of the centroid) as

$$F=\mathsf{m}N+\frac{\cot^2\omega}{12}\cdot(G-K),$$

where $\mathsf{m}N$ is the medial Ninecenter, [1] G the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. In particular, the parallel through F to the symmedian line GK hits the Euler line in $\mathsf{m}N$. See Figure 4. The point F has homogeneous barycentric coordinates

$$(10a^4-13a^2(b^2+c^2)+(4b^4-10b^2c^2+4c^4):\cdots:\cdots).$$

This appears as X_{1153} of [6].

[1] This is the point which divides OH in the ratio $1:3$.

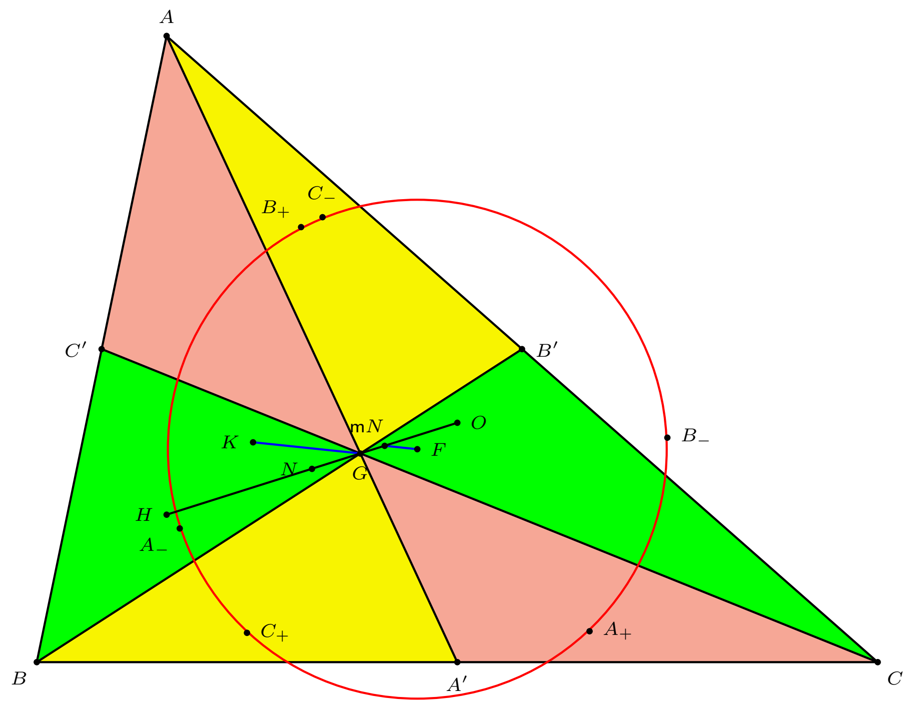

Figure 4

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Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: `alex_geom@mtu-net.ru`

Peter Y. Woo: Department of Mathematics, Biola University, 13800 Biola Avenue, La Mirada, California 90639, USA
E-mail address: `woobiola@yahoo.com`