

On the Circumcenters of Cevasix Configurations

Alexei Myakishev and Peter Y. Woo

Abstract. We strengthen Floor van Lamoen’s theorem that the 6 circumcenters of the cevasix configuration of the centroid of a triangle are concyclic by giving a proof which at the same time shows that the converse is also true with a minor qualification, *i.e.*, the circumcenters of the cevasix configuration of a point P are concyclic if and only if P is the centroid or the orthocenter of the triangle.

1. Introduction

Let P be a point in the plane of triangle ABC , with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. Triangle ABC is then divided by its cevians AA', BB', CC' into six triangles, giving rise to what Clark Kimberling [2, pp.257–260] called the *cevasix configuration* of P . See Figure 1. Floor van Lamoen has discovered that when P is the centroid of triangle ABC , the 6 circumcenters of the cevasix configuration are concyclic. See Figure 2. This was posed as a problem in the *American Mathematical Monthly* [3]. Solutions can be found in [3, 4]. In this note we study the converse.

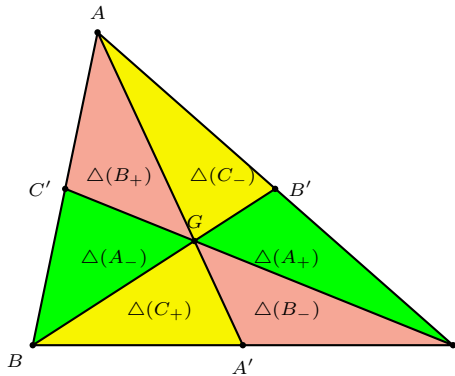


Figure 1

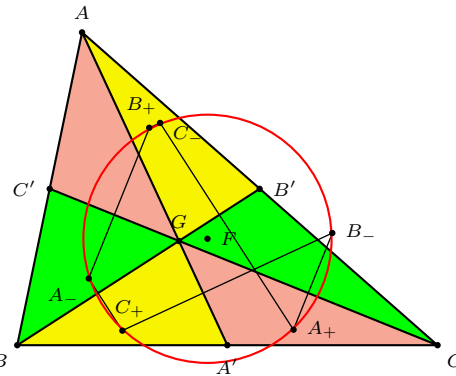


Figure 2

Theorem 1. *The circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of triangle ABC .*

2. Preliminary results

We adopt the following notations.

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\Delta(A_+)$	$\Delta(A_-)$	$\Delta(B_+)$	$\Delta(B_-)$	$\Delta(C_+)$	$\Delta(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangle may possibly share a common circumcenter only when they share a common vertex of triangle ABC .

Lemma 2. *The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle in the line BC .*

Proof. Triangles APB' and APC' have the same circumcenter if and only if the four points A, B', P, C' are concyclic. In terms of directed angles, $\angle BPC = \angle B'PC' = \angle B'AC' = \angle CAB = -\angle BAC$. See, for example, [1, §§16–20]. It follows that the reflection of A in the line BC lies on the circumcircle of triangle PBC , and P lies on the reflection of the circumcircle in BC . The converse is clear. \square

Thus, if $B_+ = C_-$ and $C_+ = A_-$, then necessarily P is the orthocenter H , and also $A_+ = B_-$. In this case, there are only three distinct circumcenters. They clearly lie on the nine-point circle of triangle ABC . We shall therefore assume $P \neq H$, so that there are at least five distinct points in the set $\{A_{\pm}, B_{\pm}, C_{\pm}\}$.

The next proposition appears in [2, p.259].

Proposition 3. *The 6 circumcenters of the cevaxix configuration of P lie on a conic.*

Proof. We need only consider the case when these 6 circumcenters are all distinct. The circumcenters B_+ and C_- lie on the perpendicular bisector of the segment AP ; similarly, B_- and C_+ lie on the perpendicular bisector of PA' . These two perpendicular bisectors are clearly parallel. This means that B_+C_- and B_-C_+ are parallel. Similarly, $C_+A_- // C_-A_+$ and $A_+B_- // A_-B_+$. The hexagon $A_+C_-B_+A_-C_+B_-$ has three pairs of parallel opposite sides. By the converse of Pascal's theorem, there is a conic passing through the six vertices of the hexagon. \square

Proposition 4. *The vertices of a hexagon $A_+C_-B_+A_-C_+B_-$ with parallel opposite sides $B_+C_- // C_+B_-$, $C_+A_- // A_+C_-$, $A_+B_- // B_+A_-$ lie on a circle if and only if the main diagonals A_+A_- , B_+B_- and C_+C_- have equal lengths.*

Proof. If the vertices are concyclic, then $A_+C_-A_-C_+$ is an isosceles trapezoid, and $A_+A_- = C_+C_-$. Similarly, $C_+B_-C_-B_+$ is also an isosceles trapezoid, and $C_+C_- = B_+B_-$.

Conversely, consider the triangle XYZ bounded by the three diagonals A_+A_- , B_+B_- and C_+C_- . If these diagonals are equal in length, then the trapezoids $A_+C_-A_-C_+$, $C_+B_-C_-B_+$ and $B_+A_-B_-A_+$ are isosceles. From these we immediately conclude that the common perpendicular bisector of A_+C_- and A_-C_+

is the bisector of angle XYZ . Similarly, the common perpendicular bisector of B_+C_- and B_-C_+ is the bisector of angle X , and that of A_+B_- and A_-B_+ the bisector of angle Z . These three perpendicular bisectors clearly intersect at a point, the incenter of triangle XYZ , which is equidistant from the six vertices of the hexagon. \square

Proposition 5. *The vector sum $\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = \mathbf{0}$ if and only if P is the centroid.*

Proof. Suppose with reference to triangle ABC , the point P has absolute barycentric coordinates $uA + vB + wC$, where $u + v + w = 1$. Then,

$$A' = \frac{1}{v+w}(vB + wC), \quad B' = \frac{1}{w+u}(wC + uA), \quad C' = \frac{1}{u+v}(uA + vB).$$

From these,

$$\begin{aligned} & \mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' \\ &= (A' + B' + C') - (A + B + C) \\ &= \frac{u^2 - vw}{(w+u)(u+v)} \cdot A + \frac{v^2 - wu}{(u+v)(v+w)} \cdot B + \frac{w^2 - uv}{(v+w)(w+u)} \cdot C. \end{aligned}$$

This is zero if and only if

$$u^2 - vw = v^2 - wu = w^2 - uv = 0,$$

and $u = v = w = \frac{1}{3}$ since they are all real, and $u + v + w = 1$. \square

We denote by π_a, π_b, π_c the orthogonal projections on the lines AA', BB', CC' respectively.

Proposition 6.

$$\begin{aligned} \pi_b(\mathbf{A}_+ \mathbf{A}_-) &= -\frac{1}{2} \mathbf{BB}', & \pi_c(\mathbf{A}_+ \mathbf{A}_-) &= \frac{1}{2} \mathbf{CC}', \\ \pi_c(\mathbf{B}_+ \mathbf{B}_-) &= -\frac{1}{2} \mathbf{CC}', & \pi_a(\mathbf{B}_+ \mathbf{B}_-) &= \frac{1}{2} \mathbf{AA}', \\ \pi_a(\mathbf{C}_+ \mathbf{C}_-) &= -\frac{1}{2} \mathbf{AA}', & \pi_b(\mathbf{C}_+ \mathbf{C}_-) &= \frac{1}{2} \mathbf{BB}'. \end{aligned} \quad (1)$$

Proof. The orthogonal projections of A_+ and A_- on the cevian BB' are respectively the midpoints of the segments PB' and BP . Therefore,

$$\pi_b(\mathbf{A}_+ \mathbf{A}_-) = \frac{B+P}{2} - \frac{P+B'}{2} = -\frac{B'-B}{2} = -\frac{1}{2} \mathbf{BB}'.$$

The others follow similarly. \square

3. Proof of Theorem 1

Sufficiency part. Let P be the centroid G of triangle ABC . By Proposition 4, it is enough to prove that the diagonals A_+A_- , B_+B_- and C_+C_- have equal lengths. By Proposition 5, we can construct a triangle $A^*B^*C^*$ whose sides as vectors $\mathbf{B}^*\mathbf{C}^*$, $\mathbf{C}^*\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{B}^*$ are equal to the medians \mathbf{AA}' , \mathbf{BB}' , \mathbf{CC}' respectively.

Consider the vector $\mathbf{A}^* \mathbf{Q}$ equal to $\mathbf{A}_+ \mathbf{A}_-$. By Proposition 6, the orthogonal projections of $\mathbf{A}_+ \mathbf{A}_-$ on the two sides $C^* A^*$ and $A^* B^*$ are the midpoints of the sides. This means that Q is the circumcenter of triangle $A^* B^* C^*$, and the length of $\mathbf{A}_+ \mathbf{A}_-$ is equal to the circumradius of triangle $A^* B^* C^*$. The same is true for the lengths of $\mathbf{B}_+ \mathbf{B}_-$ and $\mathbf{C}_+ \mathbf{C}_-$. The case $P = H$ is trivial.

Necessity part. Suppose the 6 circumcenters $A_{\pm}, B_{\pm}, C_{\pm}$ lie on a circle. By Proposition 3, the diagonals $A_+ A_-, B_+ B_-,$ and $C_+ C_-$ have equal lengths. We show that $\mathbf{A} \mathbf{A}' + \mathbf{B} \mathbf{B}' + \mathbf{C} \mathbf{C}' = 0$, so that P is the centroid of triangle ABC by Proposition 5. In terms of scalar products, we rewrite equation (1) as

$$\begin{aligned} \mathbf{A}_+ \mathbf{A}_- \cdot \mathbf{B} \mathbf{B}' &= -\frac{1}{2} \mathbf{B} \mathbf{B}' \cdot \mathbf{B} \mathbf{B}', & \mathbf{A}_+ \mathbf{A}_- \cdot \mathbf{C} \mathbf{C}' &= \frac{1}{2} \mathbf{C} \mathbf{C}' \cdot \mathbf{C} \mathbf{C}', \\ \mathbf{B}_+ \mathbf{B}_- \cdot \mathbf{C} \mathbf{C}' &= -\frac{1}{2} \mathbf{C} \mathbf{C}' \cdot \mathbf{C} \mathbf{C}', & \mathbf{B}_+ \mathbf{B}_- \cdot \mathbf{A} \mathbf{A}' &= \frac{1}{2} \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}', \\ \mathbf{C}_+ \mathbf{C}_- \cdot \mathbf{A} \mathbf{A}' &= -\frac{1}{2} \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}', & \mathbf{C}_+ \mathbf{C}_- \cdot \mathbf{B} \mathbf{B}' &= \frac{1}{2} \mathbf{B} \mathbf{B}' \cdot \mathbf{B} \mathbf{B}'. \end{aligned} \quad (2)$$

From these, $(\mathbf{B}_+ \mathbf{B}_- + \mathbf{C}_+ \mathbf{C}_-) \cdot \mathbf{A} \mathbf{A}' = 0$, and $\mathbf{A} \mathbf{A}'$ is orthogonal to $\mathbf{B}_+ \mathbf{B}_- + \mathbf{C}_+ \mathbf{C}_-$. Since $\mathbf{B}_+ \mathbf{B}_-$, and $\mathbf{C}_+ \mathbf{C}_-$ have equal lengths, $\mathbf{B}_+ \mathbf{B}_- + \mathbf{C}_+ \mathbf{C}_-$ and $\mathbf{B}_+ \mathbf{B}_- - \mathbf{C}_+ \mathbf{C}_-$ are orthogonal. We may therefore write $\mathbf{B}_+ \mathbf{B}_- - \mathbf{C}_+ \mathbf{C}_- = k \mathbf{A} \mathbf{A}'$ for a scalar k . From (2) above,

$$\begin{aligned} k \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}' &= (\mathbf{B}_+ \mathbf{B}_- - \mathbf{C}_+ \mathbf{C}_-) \cdot \mathbf{A} \mathbf{A}' \\ &= \frac{1}{2} \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}' + \frac{1}{2} \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}' \\ &= \mathbf{A} \mathbf{A}' \cdot \mathbf{A} \mathbf{A}'. \end{aligned}$$

From this, $k = 1$ and we have

$$\mathbf{A} \mathbf{A}' = \mathbf{B}_+ \mathbf{B}_- - \mathbf{C}_+ \mathbf{C}_-.$$

The same reasoning shows that

$$\mathbf{B} \mathbf{B}' = \mathbf{C}_+ \mathbf{C}_- - \mathbf{A}_+ \mathbf{A}_-,$$

$$\mathbf{C} \mathbf{C}' = \mathbf{A}_+ \mathbf{A}_- - \mathbf{B}_+ \mathbf{B}_-.$$

Combining the three equations, we have

$$\mathbf{A} \mathbf{A}' + \mathbf{B} \mathbf{B}' + \mathbf{C} \mathbf{C}' = \mathbf{0}.$$

It follows from Proposition 5 that P must be the centroid of triangle ABC .

4. An alternative proof of Theorem 1

We present another proof of Theorem 1 by considering an auxiliary hexagon. Let \mathcal{L}_a and \mathcal{L}'_a be the lines perpendicular to AA' at A and A' respectively; similarly, $\mathcal{L}_b, \mathcal{L}'_b,$ and \mathcal{L}_c and \mathcal{L}'_c . Consider the points

$$\begin{aligned} X_+ &= \mathcal{L}_c \cap \mathcal{L}'_b, & X_- &= \mathcal{L}_b \cap \mathcal{L}'_c, \\ Y_+ &= \mathcal{L}_a \cap \mathcal{L}'_c, & Y_- &= \mathcal{L}_c \cap \mathcal{L}'_a, \\ Z_+ &= \mathcal{L}_b \cap \mathcal{L}'_a, & Z_- &= \mathcal{L}_a \cap \mathcal{L}'_b. \end{aligned}$$

Note that the circumcenters $A_{\pm}, B_{\pm}, C_{\pm}$ are respectively the midpoints of $PX_{\pm}, PY_{\pm}, PZ_{\pm}$. Hence, the six circumcenters are concyclic if and only if $X_{\pm}, Y_{\pm}, Z_{\pm}$ are concyclic.

In Figure 3, let $\angle CPA' = \angle APC' = \alpha$. Since angles $PA'Y_-$ and PCY_- are both right angles, the four points P, A', C, Y_- are concyclic and $\angle Z_+Y_-X_+ = \angle A'Y_-X_+ = \angle A'PC = \alpha$. Similarly, $\angle CPB' = \angle BPC' = \angle Y_-X_+Z_-$, and we denote the common measure by β .

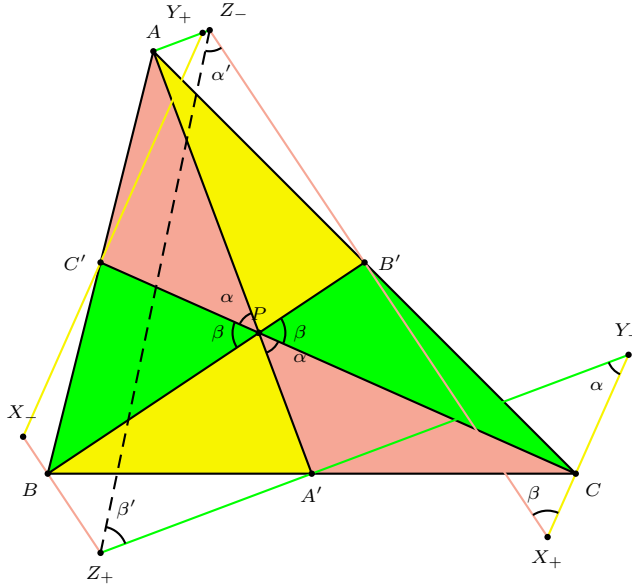


Figure 3

Lemma 7. *If the four points X_+, Y_-, Z_+, Z_- are concyclic, then P lies on the median through C .*

Proof. Let $x = \frac{AP}{AA'}$ and $y = \frac{BP}{BB'}$. If the four points X_+, Y_-, Z_+, Z_- are concyclic, then $\angle Z_+Z_-X_+ = \alpha$ and $\angle Y_-Z_+Z_- = \beta$. Now,

$$\frac{|BB'|}{|AA'|} = \frac{|Z_+Z_-| \cdot \sin \alpha'}{\sin \beta} = \frac{|AC'|}{|AP|} \cdot \frac{|AP|}{|BP|}.$$

It follows that

$$\frac{|BP|}{|BB'| \cdot |BC'|} = \frac{|AP|}{|AA'| \cdot |AC'|},$$

and, as a ratio of signed lengths,

$$\frac{BC'}{AC'} = -\frac{y}{x}. \tag{3}$$

Now applying Menelaus' theorem to triangle APC' with transversal $A'CB$, and triangle BGA' with transversal $B'CA$, we have

$$\frac{AA'}{A'P} \cdot \frac{PC}{CC'} \cdot \frac{C'B}{BA} = -1 = \frac{BB'}{B'P} \cdot \frac{PC}{CC'} \cdot \frac{C'A}{AB}.$$

From this, $\frac{AA'}{A'P} \cdot BC' = \frac{BB'}{B'P} \cdot AC'$, or

$$\frac{BC'}{1-x} = -\frac{AC'}{1-y}. \quad (4)$$

Comparing (3) and (4), we have $\frac{1-x}{1-y} = \frac{y}{x}$, $(x-y)(x+y-1) = 0$. Either $x = y$ or $x + y = 1$. It is easy to eliminate the possibility $x + y = 1$. If P has homogeneous barycentric coordinates $(u : v : w)$ with reference to triangle ABC , then $x = \frac{v+w}{u+v+w}$ and $y = \frac{w+u}{u+v+w}$. Thus, $x + y = 1$ requires $w = 0$ and P lies on the sideline AB , contrary to the assumption. It follows that $x = y$, and from (3), C' is the midpoint of AB , and P lies on the median through C . \square

The necessity part of Theorem 1 is now an immediate corollary of Lemma 7.

5. Concluding remark

We conclude with a remark on triangles for which two of the circumcenters of the cevasix configuration of the centroid coincide. Clearly, $B_+ = C_-$ if and only if A, B', G, C' are concyclic. Equivalently, the image of G under the homothety $h(A, 2)$ lies on the circumcircle of triangle ABC . This point has homogeneous barycentric coordinates $(-1 : 2 : 2)$. Since the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0,$$

we have $2a^2 = b^2 + c^2$. There are many interesting properties of such triangles. We simply mention that it is similar to its own triangle of medians. Specifically,

$$m_a = \frac{\sqrt{3}}{2}a, \quad m_b = \frac{\sqrt{3}}{2}c, \quad m_c = \frac{\sqrt{3}}{2}b.$$

Editor's endnote. John H. Conway [5] has located the center of the Van Lamoen circle (of the circumcenters of the cevasix configuration of the centroid) as

$$F = mN + \frac{\cot^2 \omega}{12} \cdot (G - K),$$

where mN is the medial Ninecenter,¹ G the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC . In particular, the parallel through F to the symmedian line GK hits the Euler line in mN . See Figure 4. The point F has homogeneous barycentric coordinates

$$(10a^4 - 13a^2(b^2 + c^2) + (4b^4 - 10b^2c^2 + 4c^4)) : \dots : \dots).$$

This appears as X_{1153} of [6].

¹This is the point which divides OH in the ratio $1 : 3$.

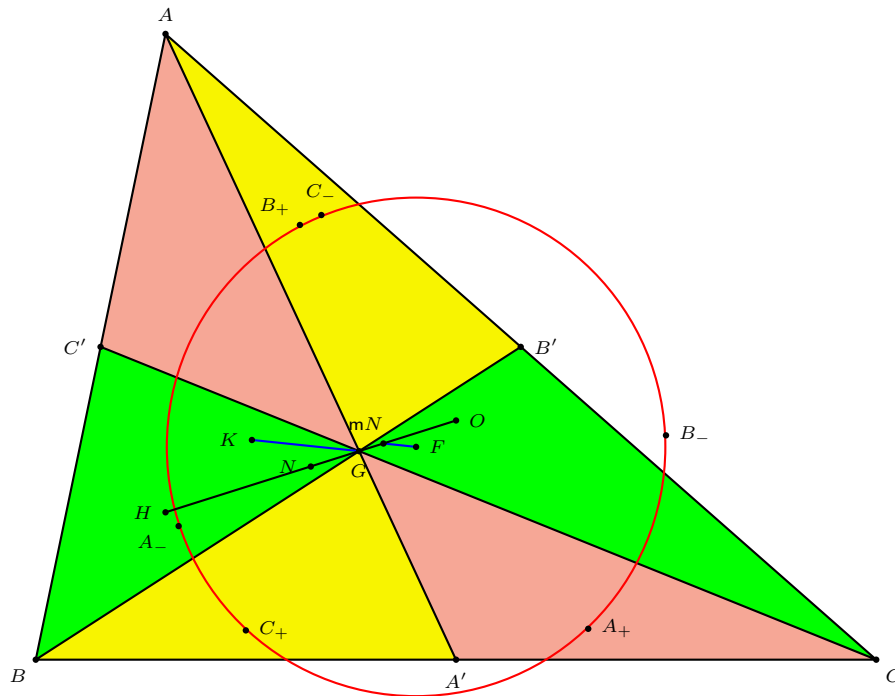


Figure 4

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Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex_geom@mtu-net.ru

Peter Y. Woo: Department of Mathematics, Biola University, 13800 Biola Avenue, La Mirada, California 90639, USA
E-mail address: woobiola@yahoo.com