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Napoleon Triangles and Kiepert Perspectors

Two examples of the use of complex number coordinates

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Abstract. In this paper we prove generalizations of the well known Napoleon Theorem and Kiepert Perspectors. We use complex numbers as coordinates to prove the generalizations, because this makes representation of isosceles triangles built on given segments very easy.

1. Introduction

In [1, XXVII] O. Bottema describes the famous (first) Fermat-Torricelli point of a triangle ABC. This point is found by attaching outwardly equilateral triangles to the sides of ABC. The new vertices form a triangle $A'B'C'$ that is perspective to ABC, that is, AA', BB' and CC' have a common point of concurrency, the perspector of ABC and $A'B'C'$. A lot can be said about this point, but for this paper we only need to know that the lines AA', BB' and CC' make angles of 60 degrees (see Figure 1), and that this is also the case when the equilateral triangles are pointed inwardly, which gives the second Fermat-Torricelli point.

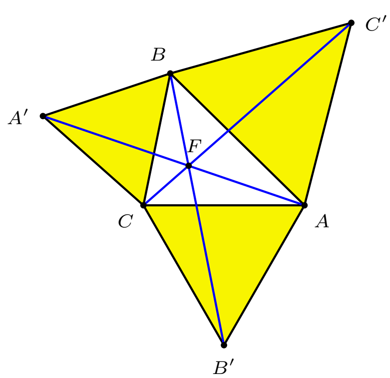

Figure 1

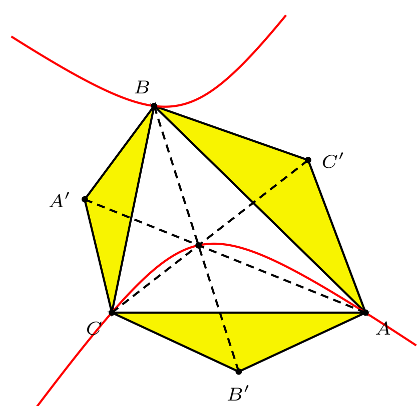

Figure 2

It is well known that to yield a perspector, the triangles attached to the sides of ABC do not need to be equilateral. For example they may be isosceles triangles with base angle ϕ, like Bottema tells us in [1, XI]. It was Ludwig Kiepert who studied these triangles - the perspectors with varying ϕ lie on a rectangular hyperbola

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named after Kiepert. See [4] for some further study on this hyperbola, and some references. See Figure 2. However, it is already sufficient for the lines AA', BB', CC' to concur when the attached triangles have oriented angles satisfying

$$\angle BAC' = \angle CAB', \qquad \angle ABC' = \angle CBA', \qquad \angle ACB = \angle BCA'.$$

When the attached triangles are equilateral, there is another nice geometric property: *the centroids of the triangles $A'BC$, $AB'C$ and ABC' form a triangle that is equilateral itself*, a fact that is known as Napoleon's Theorem. The triangles are referred to as the *first* and *second Napoleon triangles* (for the cases of outwardly and inwardly pointed attached triangles). See Figures 3a and 3b. The perspectors of these two triangles with ABC are called *first* and *second Napoleon points*. General informations on Napoleon triangles and Kiepert perspectors can be found in [2, 3, 5, 6].

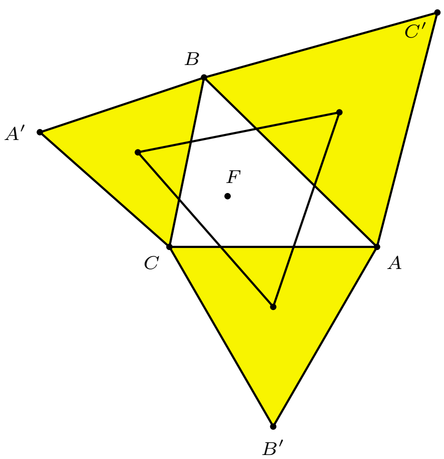

Figure 3a

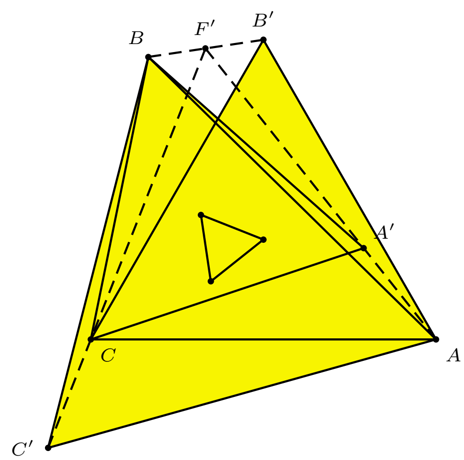

Figure 3b

2. The equation of a line in the complex plane

Complex coordinates are not that much different from the rectangular (x, y) - the two directions of the axes are now hidden in one complex number, that we call the *affix* of a point. Of course such an affix just exists of a real (x) and imaginary (y) part - the complex number $z = p + qi$ in fact resembles the point (p, q).

If $z = p + qi$, then the number $\overline{z} = p - qi$ is called complex conjugate of z. The combination of z and $\overline{z}$ is used to make formulas, since we do not have x and y anymore! A parametric formula for the line through the points a_1 and a_2 is $z = a_1 + t(a_2 - a_1)$, where t runs through the *real* numbers. The complex conjugate of this formula is $\overline{z} = \overline{a_1} + t(\overline{a_2} - \overline{a_1})$. Elimination of t from these two formulas gives the formula for the line through the points with affixes a_1 and a_2:

$$z(\overline{a_1 - a_2}) - \overline{z}(a_1 - a_2) + (a_1\overline{a_2} - \overline{a_1}a_2) = 0.$$

3. Isosceles triangle on a segment

Let the points A and B have affixes a and b. We shall find the affix of the point C for which ABC is an isosceles triangle with base angle ϕ and apex C. The midpoint of AB has affix $\frac{1}{2}(a+b)$. The distance from this midpoint to C is equal to $\frac{1}{2}|AB|\tan\phi$. With this we find the affix for C as

$$c = \frac{a+b}{2} + i\tan\phi \cdot \frac{b-a}{2} = \frac{1-i\tan\phi}{2}a + \frac{1+i\tan\phi}{2}b = \overline{\chi}a + \chi b$$

where $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that $\chi + \overline{\chi} = 1$.

The special case that ABC is equilateral, yields for χ the sixth root of unity $\zeta = \frac{1}{2} + \frac{i}{2}\sqrt{3} = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. This number ζ is a sixth root of unity, because it satisfies

$$\zeta^6 = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1.$$

It also satisfies the identities $\zeta^3 = -1$ and $\zeta \cdot \overline{\zeta} = \zeta + \overline{\zeta} = 1$. Depending on orientation one can find two vertices C that together with AB form an equilateral triangle, for which we have respectively $c = \zeta a + \overline{\zeta} b$ (negative orientation) and $c = \overline{\zeta} a + \zeta b$ (positive orientation). From this one easily derives

Proposition 1. *The complex numbers a, b and c are affixes of an equilateral triangle if and only if*

$$a + \zeta^2 b + \zeta^4 c = 0$$

for positive orientation or

$$a + \zeta^4 b + \zeta^2 c = 0$$

for negative orientation.

4. Napoleon triangles

We shall generalize Napoleon's Theorem, by extending the idea of the use of centroids. Napoleon triangles were indeed built in a triangle ABC by attaching to the sides of a triangle equilateral triangles, and taking the centroids of these. We now start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$, and attach equilateral triangles to the connecting segments between the A's, the B's and the C's. This seems to be entirely different, but Napoleon's Theorem will be a special case by starting with triangles BCA and CAB.

So we start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$ with affixes a_k, b_k and c_k for the vertices. The centroids Z_k have affixes $z_k = \frac{1}{3}(a_k + b_k + c_k)$. Now we attach positively orientated equilateral triangles to segments A_1A_2, B_1B_2 and C_1C_2 having A_{3+}, B_{3+}, C_{3+} as third vertices. In the same way we find A_{3-}, B_{3-}, C_{3-} from equilateral triangles with negative orientation. We find as affixes

$$a_{3+} = \zeta a_2 + \overline{\zeta} a_1$$

and

$$a_{3-} = \zeta a_1 + \overline{\zeta} a_2,$$

and similar expressions for b_{3+}, b_{3-}, c_{3+} and c_{3-}. The centroids Z_{3+} and Z_{3-} now have affixes $z_{3+} = \zeta z_2 + \overline{\zeta} z_1$ and $z_{3-} = \zeta z_1 + \overline{\zeta} z_2$ respectively, from which we

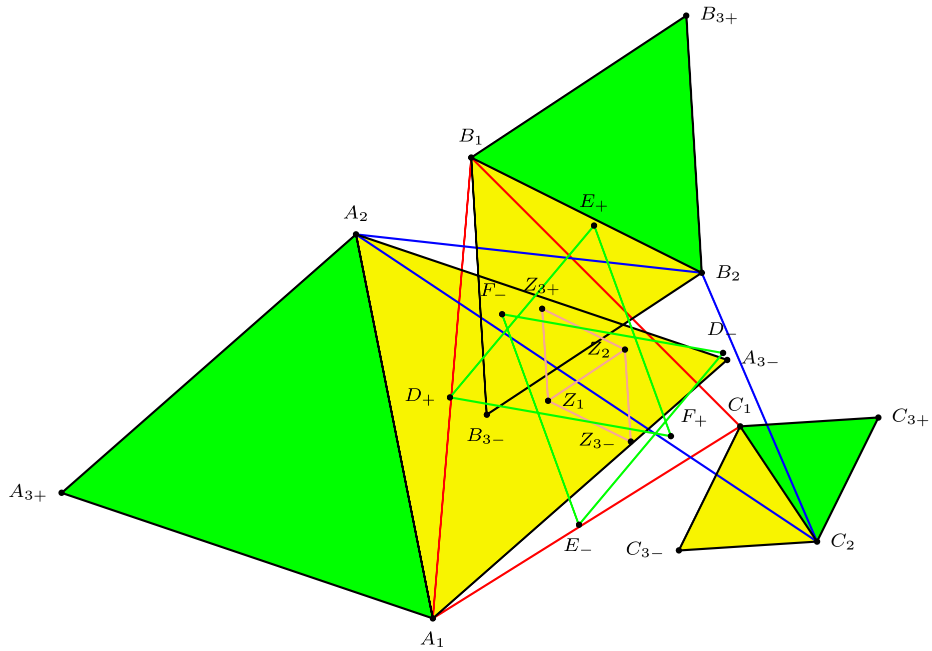

Figure 4

see that $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ are equilateral triangles of positive and negative orientation respectively.

We now work with the following centroids:
D_+, E_+ and F_+ of triangles $B_1C_2A_{3+}$, $C_1A_2B_{3+}$ and $A_1B_2C_{3+}$ respectively;
D_-, E_- and F_- of triangles $C_1B_2A_{3-}$, $A_1C_2B_{3-}$ and $B_1A_2C_{3-}$ respectively.

For these we claim

Theorem 2. *Given triangles $A_kB_kC_k$ and points Z_k for $k = 1,\ 2,\ 3+,\ 3-$ and $D_\pm E_\pm F_\pm$ as described above, triangles $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation, congruent and parallel, and their centroids coincide with the centroids of $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ respectively. (See Figure 4).*

Proof. To prove this we find the following affixes

$$d_+ = \tfrac{1}{3}(b_1 + c_2 + \zeta a_2 + \overline{\zeta} a_1), \qquad d_- = \tfrac{1}{3}(b_2 + c_1 + \zeta a_1 + \overline{\zeta} a_2),$$

$$e_+ = \tfrac{1}{3}(c_1 + a_2 + \zeta b_2 + \overline{\zeta} b_1), \qquad e_- = \tfrac{1}{3}(c_2 + a_1 + \zeta b_1 + \overline{\zeta} b_2),$$

$$f_+ = \tfrac{1}{3}(a_1 + b_2 + \zeta c_2 + \overline{\zeta} c_1), \qquad f_- = \tfrac{1}{3}(a_2 + b_1 + \zeta c_1 + \overline{\zeta} c_2).$$

Using Proposition 1 it is easy to show that $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation. For instance, the expression $d_+ + \zeta^4 e_+ + \zeta^2 f_+$ has as 'coefficient' of b_1 the number $\frac{1}{3}(1 + \zeta^4\overline{\zeta}) = 0$. We also find that

$$d_+ - e_+ = e_- - d_- = \overline{\zeta}(a_1 - a_2) + \zeta(b_1 - b_2) + (c_2 - c_1),$$

from which we see that D_+E_+ and D_-E_- are equal in length and directed oppositely. Finally it is easy to check that $\frac{1}{3}(d_+ + e_+ + f_+) = \frac{1}{3}(z_1 + z_2 + z_{3+})$ and $\frac{1}{3}(d_- + e_- + f_-) = \frac{1}{3}(z_1 + z_2 + z_{3-})$, and the theorem is proved. □

We can make a variation of Theorem 2 if in the creation of $D_\pm E_\pm F_\pm$ we interchange the roles of $A_{3+}B_{3+}C_{3+}$ and $A_{3-}B_{3-}C_{3-}$. The roles of Z_{3+} and Z_{3-} change as well, and the equilateral triangles found have positive orientation.

We note that if the centroids Z_1 and Z_2 coincide, then they coincide with Z_{3+} and Z_{3-}, so that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon, of which the center coincides with Z_1 and Z_2.

Napoleon's Theorem is a special case. If we take $A_1B_1C_1 = BCA$ and $A_2B_2C_2 = CAB$, then $D_+E_+F_+$ is the second Napoleon Triangle, and indeed appears equilateral. We get as a bonus that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon. Now D_- is the centroid of AAA_{3-}, that is, D_- is the point on AA_{3-} such that $AD_- : D_-A_{3-} = 1 : 2$. In similar ways we find E_- and F_-. Triangles ABC and $A_{3-}B_{3-}C_{3-}$ have the first point of Fermat-Torricelli F_1 as perspector, and the lines AA_{3-}, BB_{3-} and CC_{3-} make angles of 60 degrees. From this it is easy to see (congruent inscribed angles) that F_1 must be on the circumcircle of $D_-E_-F_-$ and thus also on the circumcircle of $D_+E_+F_+$. See Figure 5. In the same way, now using the variation of Theorem 2, we see that the second Fermat-Torricelli point lies on the circumcircle of the first Napoleon Triangle.

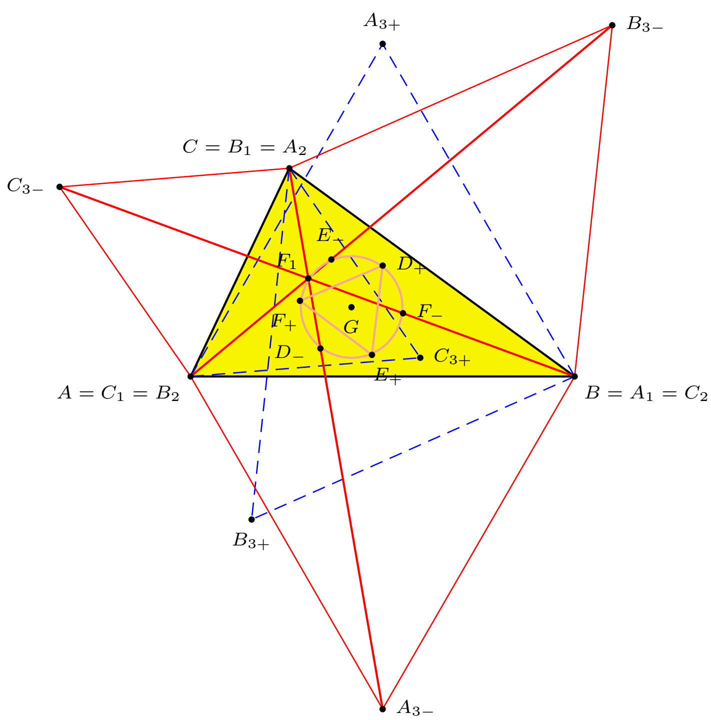

Figure 5

5. Kiepert perspectors

To generalize the Kiepert perspectors we start with two triangles as well. We label these ABC and $A'B'C'$ to distinguish from Theorem 2. These two triangles we take to be directly congruent (hence A corresponds to A', etc.) and of the same orientation. This means that the two triangles can be mapped to each other by a combination of a rotation and a translation (in fact one of both is sufficient). We now attach isosceles triangles to segments connecting ABC and $A'B'C'$. While we usually find Kiepert perspectors on a line, for example, from A to the apex of an isosceles triangle built on BC, now we start from the apex of an isosceles triangle on AA' and go to the apex of an isosceles triangle on on BC'. This gives the following theorem:

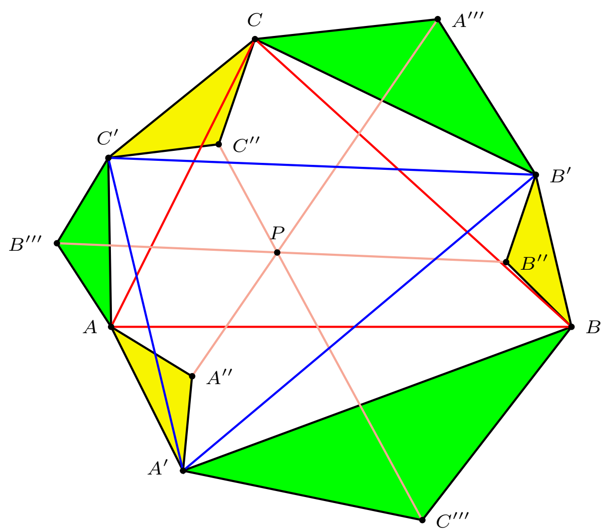

Figure 6

Theorem 3. *Given two directly congruent triangles ABC and $A'B'C'$ with the same orientation, attach to the segments AA', BB', CC', CB', AC' and BA' similar isosceles triangles with the same orientation and apexes A'', B'', C'', A''', B''' and C'''. The lines $A''A'''$, $B''B'''$ and $C''C'''$ are concurrent, so triangles $A''B''C''$ and $A'''B'''C'''$ are perspective. (See Figure 6).*

Proof. For the vertices A, B and C we take the affixes a, b and c. Because triangles ABC and $A'B'C'$ are directly congruent and of equal orientation, we can get $A'B'C'$ by applying on ABC a rotation about the origin, followed by a translation. This rotation about the origin can be represented by multiplication by a number τ on the unit circle, so that $\tau\overline{\tau} = 1$. The translation is represented by addition with a number σ. So the affixes of A', B' and C' are the numbers $\tau a + \sigma$, $\tau b + \sigma$ and $\tau c + \sigma$.

We take for the base angles of the isosceles triangle ϕ again, and we let $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that the affix for A'' is $(\overline{\chi} + \chi\tau)a + \chi\sigma$. For A''' we find $\overline{\chi}c + \chi\tau b + \chi\sigma$. The equation of the line $A''A'''$ we can find after some calculations as

$$\begin{aligned}&(\chi\overline{a} + \overline{\chi\tau a} - \chi\overline{c} - \overline{\chi\tau b})z - (\overline{\chi}a + \chi\tau a - \overline{\chi}c - \chi\tau b)\overline{z}\\ &+(\overline{\chi} + \chi\tau)a(\chi\overline{c} + \overline{\chi\tau b}) + \chi\sigma(\chi\overline{c} + \overline{\chi\tau b}) + \overline{\chi\sigma}(\overline{\chi} + \chi\tau)a\\ &-(\chi + \overline{\chi\tau})\overline{a}(\overline{\chi}c + \chi\tau b) - \overline{\chi\sigma}(\overline{\chi}c + \chi\tau b) - \chi\sigma(\chi + \overline{\chi\tau})\overline{a}\\ &=0.\end{aligned}$$

In a similar fashion we find for $B''B'''$,

$$\begin{aligned}&(\chi\overline{b} + \overline{\chi\tau b} - \chi\overline{a} - \overline{\chi\tau c})z - (\overline{\chi}b + \chi\tau b - \overline{\chi}a - \chi\tau c)\overline{z}\\ &+(\overline{\chi} + \chi\tau)b(\chi\overline{a} + \overline{\chi\tau c}) + \chi\sigma(\chi\overline{a} + \overline{\chi\tau c}) + \overline{\chi\sigma}(\overline{\chi} + \chi\tau)b\\ &-(\chi + \overline{\chi\tau})\overline{b}(\overline{\chi}a + \chi\tau c) - \overline{\chi\sigma}(\overline{\chi}a + \chi\tau c) - \chi\sigma(\chi + \overline{\chi\tau})\overline{b}\\ &=0,\end{aligned}$$

and for $C''C'''$,

$$\begin{aligned}&(\chi\overline{c} + \overline{\chi\tau c} - \chi\overline{b} - \overline{\chi\tau a})z - (\overline{\chi}c + \chi\tau c - \overline{\chi}b - \chi\tau a)\overline{z}\\ &+(\overline{\chi} + \chi\tau)c(\chi\overline{b} + \overline{\chi\tau a}) + \chi\sigma(\chi\overline{b} + \overline{\chi\tau a}) + \overline{\chi\sigma}(\overline{\chi} + \chi\tau)c\\ &-(\chi + \overline{\chi\tau})\overline{c}(\overline{\chi}b + \chi\tau a) - \overline{\chi\sigma}(\overline{\chi}b + \chi\tau a) - \chi\sigma(\chi + \overline{\chi\tau})\overline{c}\\ &=0.\end{aligned}$$

We must do some more effort to see what happens if we add the three equations. Our effort is rewarded by noticing that the sum gives $0 = 0$. The three equations are dependent, so the lines are concurrent. This proves the theorem. □

We end with a question on the locus of the perspector for varying ϕ. It would have been nice if the perspector would, like in Kiepert's hyperbola, lie on an equilateral hyperbola. This, however, does not seem to be generally the case.

References

[1] O. Bottema, *Hoofdstukken uit de elementaire meetkunde*, 2e druk, Epsilon Utrecht, 1987.
[2] H. S. M. Coxeter and S. L. Greitzer, *Geometery Revisted*, Math. Assoc. America, 1967.
[3] D. Klingens, Homepage, http://www.pandd.demon.nl/meetkunde.htm
[4] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.
[5] E. Weisstein, *MathWorld*, available at http://mathworld.wolfram.com/.
[6] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin, London, 1991.

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