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The Lucas Circles and the Descartes Formula

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Abstract. We determine the radii of the three circles each tangent to the circumcircle of a given triangle at a vertex, and mutually tangent to each other externally. The calculations are then reversed to give the radii of the two Soddy circles associated with three circles tangent to each other externally.

1. The Lucas circles

Consider a triangle ABC with circumcircle $\mathcal{C}$. We set up a coordinate system with the circumcenter O at the origin and A, B, C represented by complex numbers of moduli R, the circumradius. If the lengths of the sides BC, CA, AB are a, b, c respectively, then

$$||A-B|| = c \quad \text{and} \quad \langle A, B\rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Analogous relations hold for the pairs B, C and C, A. Let $0 \le \alpha < R$, and consider the circle $\mathcal{C}_A(\alpha)$ with center $\frac{R-\alpha}{R}\cdot A$ and radius α. This is internally tangent to the circumcircle at A, and is the image of $\mathcal{C}$ under the homothety $\mathsf{h}(A, \frac{\alpha}{R})$. See Figure 1. For real numbers β, γ satisfying $0 \le \beta, \gamma < R$, we consider the circles $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ analogously defined. Now, the circles $\mathcal{C}_A(\alpha)$ and $\mathcal{C}_B(\beta)$ are tangent externally if and only if

$$\left|\left|\frac{R-\alpha}{R}A - \frac{R-\beta}{R}B\right|\right| = \alpha + \beta.$$

This is equivalent, by a simple application of (1), to

$$c^2 = \frac{4\alpha\beta}{(R-\alpha)(R-\beta)}.$$

Therefore, the three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ are tangent externally to each other if and only if

$$a^2 = \frac{4R^2\beta\gamma}{(R-\beta)(R-\gamma)}, \quad b^2 = \frac{4R^2\gamma\alpha}{(R-\gamma)(R-\alpha)}, \quad c^2 = \frac{4R^2\alpha\beta}{(R-\alpha)(R-\beta)}. \tag{2}$$

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These equations can be solved for the radii α, β, and γ in terms of a, b, c, and R. In fact, multiplying the equations in (2), we obtain

$$abc = \frac{8R^3\alpha\beta\gamma}{(R-\alpha)(R-\beta)(R-\gamma)}.$$

Consequently,

$$\frac{\alpha}{R-\alpha} = \frac{bc}{2Ra}, \quad \frac{\beta}{R-\beta} = \frac{ca}{2Rb}, \quad \frac{\gamma}{R-\gamma} = \frac{ab}{2Rc}.$$

From these, we obtain

$$\alpha = \frac{bc}{2Ra+bc}\cdot R, \quad \beta = \frac{ca}{2Rb+ca}\cdot R, \quad \gamma = \frac{ab}{2Rc+ab}\cdot R. \qquad (3)$$

Denote by $\triangle$ the area of triangle ABC, and h_a, h_b, h_c its three altitudes. We have $2\triangle = a\cdot h_a = b\cdot h_b = c\cdot h_c$. Since $abc = 4R\triangle$, the expression for α in (3) can be rewritten as

$$\frac{\alpha}{R} = \frac{abc}{2Ra^2+abc} = \frac{4R\triangle}{2Ra^2+4R\triangle} = \frac{2\triangle}{a^2+2\triangle} = \frac{a\cdot h_a}{a^2+a\cdot h_a} = \frac{h_a}{a+h_a}.$$

Therefore, the homothety $\mathsf{h}(A, \frac{\alpha}{R})$ is the one that contracts the square on the side BC (externally) into the inscribed square on this side. See Figure 1. The same is true for the other two circles. The three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$, $\mathcal{C}_C(\gamma)$ are therefore the Lucas circles considered in [3]. See Figure 2.

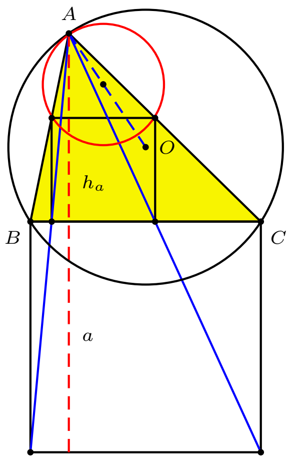

Figure 1

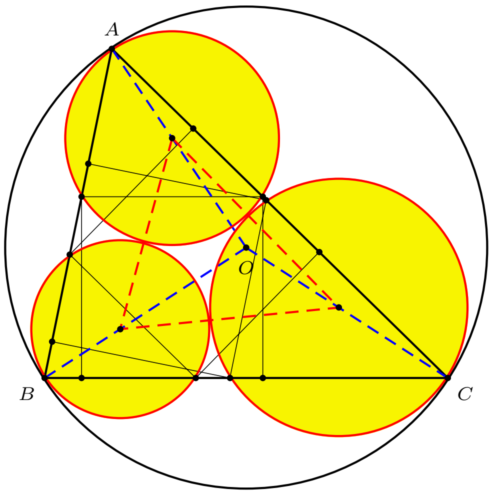

Figure 2

2. Another triad of circles

A simple modification of the above calculations shows that for positive numbers α', β', γ', the images of the circumcircle $\mathcal{C}$ under the homotheties $\mathsf{h}(A, -\frac{\alpha'}{R})$, $\mathsf{h}(B, -\frac{\beta'}{R})$ and $\mathsf{h}(C, -\frac{\gamma'}{R})$ (each tangent to $\mathcal{C}$ at a vertex) are tangent to each other if and only if

$$\alpha' = \frac{bc}{2Ra - bc} \cdot R, \quad \beta' = \frac{ca}{2Rb - ca} \cdot R, \quad \gamma' = \frac{ab}{2Rc - ab} \cdot R. \tag{4}$$

The tangencies are all external provided $2Ra - bc$, $2Rb - ca$ and $2Rc - ab$ are all positive. These quantities are essentially the excesses of the sides over the corresponding altitudes:

$$2Ra - bc = \frac{bc}{a}(a - h_a), \quad 2Rb - ca = \frac{ca}{b}(b - h_b), \quad 2Rc - ab = \frac{ab}{c}(c - h_c).$$

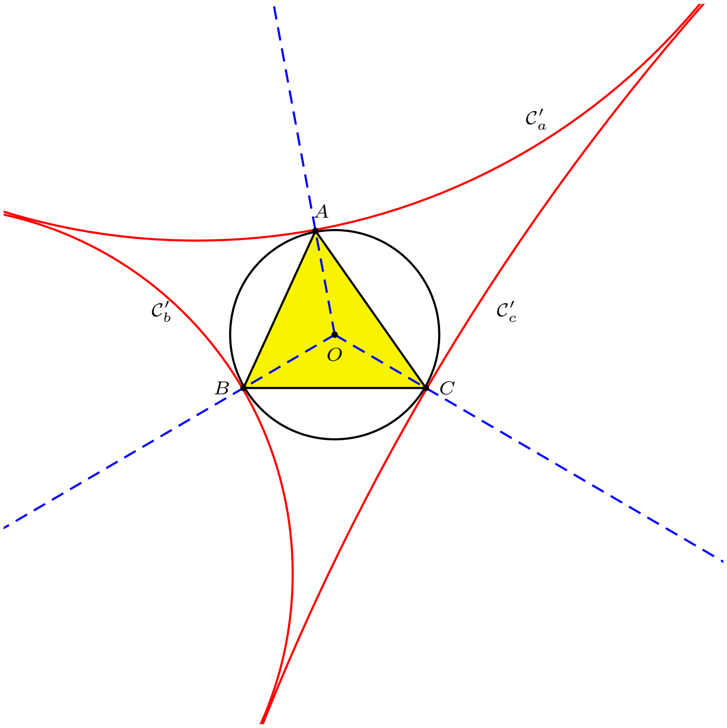

Figure 3

It may occur that one of them is negative. In that case, the tangencies with the corresponding circle are all internal. See Figure 4.

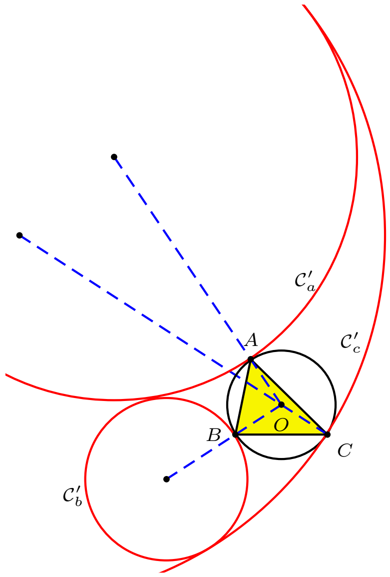

Figure 4

3. Inscribed squares

Consider the triad of circles in §2. The homothety $\mathsf{h}(A, -\frac{\alpha'}{R})$ transforms the square erected on BC on the same side of A into an inscribed square since $\frac{-\alpha'}{R} = \frac{-h_a}{a-h_a}$. See Figure 5.

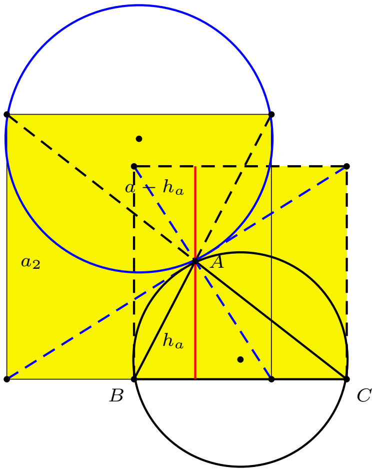

Figure 5

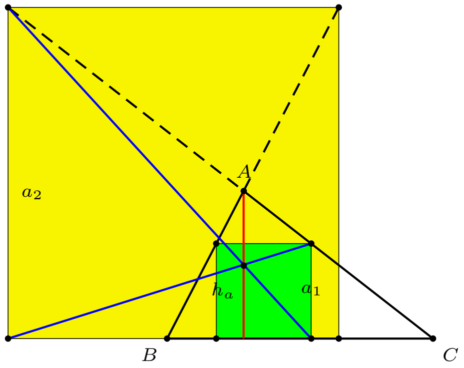

Figure 6

Denote by a_1 and a_2 the lengths of sides of the two inscribed squares on BC, under the homotheties $\mathsf{h}(A, \frac{\alpha}{R})$ and $\mathsf{h}(A, -\frac{\alpha'}{R})$ respectively, *i.e.*, $a_1 = \frac{\alpha}{R} \cdot a$ and

$a_2 = \frac{\alpha'}{R} \cdot a$. Making use of (3) and (4), we have

$$\frac{1}{a_1} + \frac{1}{a_2} = \left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)\frac{R}{a} = \frac{4a}{bc} \cdot \frac{R}{a} = \frac{a}{\triangle} = \frac{2}{h_a}.$$

This means that the altitude h_a is the harmonic mean of the lengths of the sides of the two inscribed squares on the side BC. See Figure 6.

4. The Descartes formula

We reverse the calculations in §§1,2 to give a proof of the Descartes formula. See, [2, pp.90–92]. Given three circles of radii α, β, γ tangent to each other externally, we determine the radii of the two Soddy circles tangent to each of them. See, for example, [1, pp.13–16]. We first seek the radius R of the circle tangent *internally* to each of them, the *outer* Soddy circle. Regard, in equation (3), R, a, b, c as unknowns, and write $\triangle$ for the area of the unknown triangle ABC whose vertices are the points of tangency. Thus, by Heron's formula,

$$16\triangle^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \tag{5}$$

In terms of $\triangle$, (3) can be rewritten as

$$\alpha = \frac{2\triangle}{a^2 + 2\triangle} \cdot R, \qquad \beta = \frac{2\triangle}{b^2 + 2\triangle} \cdot R, \qquad \gamma = \frac{2\triangle}{c^2 + 2\triangle} \cdot R,$$

or

$$a^2 = \frac{2(R-\alpha)\triangle}{\alpha}, \qquad b^2 = \frac{2(R-\beta)\triangle}{\beta}, \qquad c^2 = \frac{2(R-\gamma)\triangle}{\gamma}. \tag{6}$$

Substituting these into (5) and simplifying, we obtain

$$\begin{aligned}&\alpha^2\beta^2\gamma^2 + 2\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)R\\ &+(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2)R^2 = 0.\end{aligned}$$

Dividing throughout by $\alpha^2\beta^2\gamma^2 \cdot R^2$, we have

$$\frac{1}{R^2} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)\frac{1}{R} + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha}\right) = 0.$$

Since $R > \alpha, \beta, \gamma$, we have

$$\frac{1}{R} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

This is positive if and only if

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha} > 0. \tag{7}$$

This is the condition necessary and sufficient for the existence of a circle tangent *internally* to each of the given circles.

By reversing the calculations in §2, the radius of the circle tangent to the three given circles externally, the *inner* Soddy circle, is given by

$$\frac{1}{R'} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

If condition (7) is not satisfied, both Soddy circles are tangent to each of the given circles externally.

References

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