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On the Kosnita Point and the Reflection Triangle

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Abstract. The Kosnita point of a triangle is the isogonal conjugate of the nine-point center. We prove a few results relating the reflections of the vertices of a triangle in their opposite sides to triangle centers associated with the Kosnita point.

1. Introduction

By the Kosnita point of a triangle we mean the isogonal conjugate of its nine-point center. The name Kosnita point originated from J. Rigby [5].

Theorem 1 (Kosnita)**.** *Let ABC be a triangle with the circumcenter O, and X, Y, Z be the circumcenters of triangles BOC, COA, AOB. The lines AX, BY, CZ concur at the isogonal conjugate of the nine-point center.*

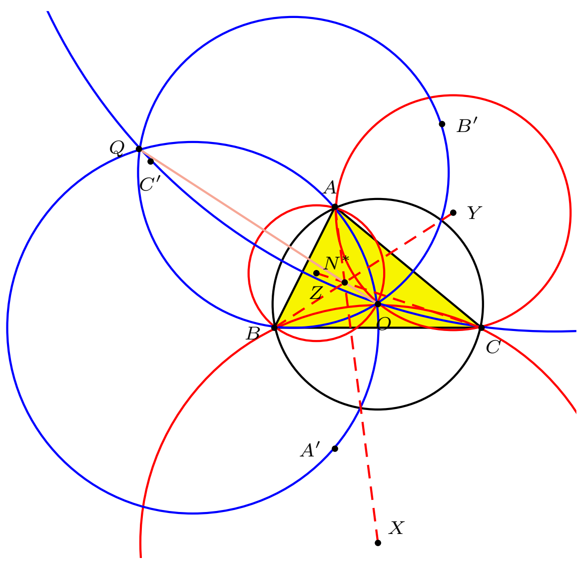

Figure 1

We denote the nine-point center by N and the Kosnita point by N^*. Note that N^* is an infinite point if and only if the nine-point center is on the circumcircle. We study this special case in §5 below. The points N and N^* appear in [3] as X_5 and X_{54} respectively. An old theorem of J. R. Musselman [4] relates the Kosnita

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point to the reflections A', B', C' of A, B, C in their opposite sides BC, CA, AB respectively.

Theorem 2 (Musselman)**.** *The circles AOA', BOB', COC' pass through the inversive image of the Kosnita point in the circumcircle of triangle ABC.*

This common point of the three circles is the triangle center X_{1157} in [3], which we denote by Q in Figure 1. The following theorem gives another triad of circles containing this point. It was obtained by Paul Yiu [7] by computations with barycentric coordinates. We give a synthetic proof in §2.

Theorem 3 (Yiu)**.** *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through the inversive image of the Kosnita point in the circumcircle.*

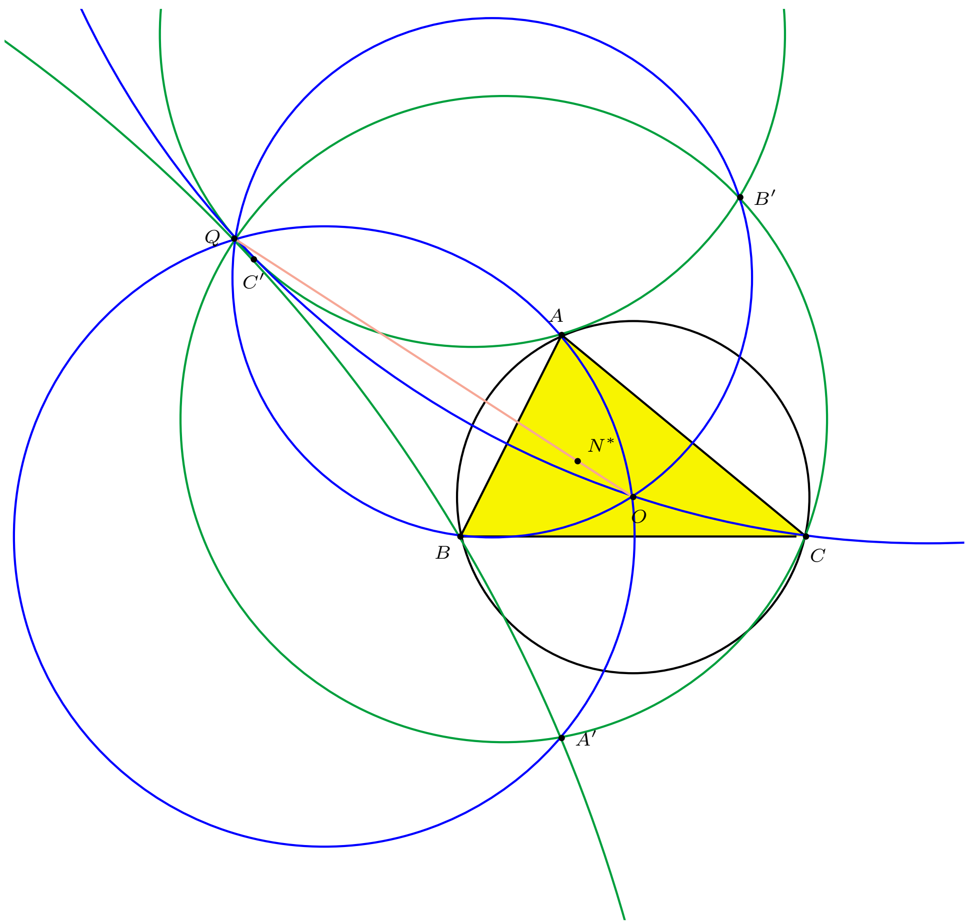

Figure 2

On the other hand, it is clear that the circles $A'BC$, $B'CA$, and $C'AB$ pass through the orthocenter of triangle ABC. It is natural to inquire about the circumcenter of the *reflection triangle* $A'B'C'$. A very simple answer is provided by the following characterization of $A'B'C'$ by G. Boutte [1].

Theorem 4 (Boutte)**.** *Let G be the centroid of ABC. The reflection triangle $A'B'C'$ is the image of the pedal triangle of the nine-point center N under the homothety* $\mathsf{h}(G, 4)$.

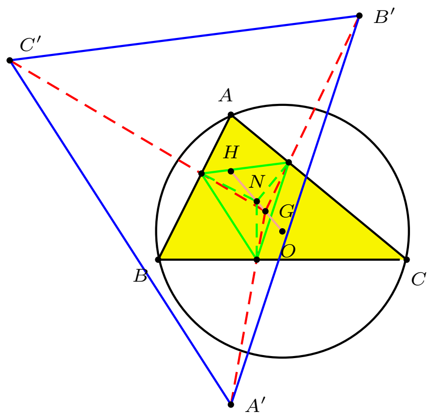

Figure 3

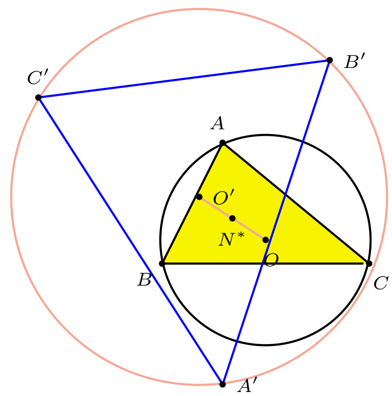

Figure 4

Corollary 5. *The circumcenter of the reflection triangle $A'B'C'$ is the reflection of the circumcenter in the Kosnita point.*

2. Proof of Theorem 3

Denote by Q the inverse of the Kosnita point N^* in the circumcircle. By Theorem 2, Q lies on the circles BOB' and COC'. So $\angle B'QO = \angle B'BO$ and $\angle C'QO = \angle C'CO$. Since $\angle B'QC' = \angle B'QO + \angle C'QO$, we get

$$\begin{aligned}\angle B'QC' =& \angle B'BO + \angle C'CO \\ =& \left(\angle CBB' - \angle CBO\right) + \left(\angle BCC' - \angle BCO\right) \\ =& \angle CBB' + \angle BCC' - (\angle CBO + \angle BCO) \\ =& \angle CBB' + \angle BCC' - (\pi - \angle BOC) \\ =& \angle CBB' + \angle BCC' - \pi + \angle BOC.\end{aligned}$$

But we have $\angle CBB' = \frac{\pi}{2} - C$ and $\angle BCC' = \frac{\pi}{2} - B$. Moreover, from the central angle theorem we get $\angle BOC = 2A$. Thus,

$$\begin{aligned}\angle B'QC' =& \left(\frac{\pi}{2} - C\right) + \left(\frac{\pi}{2} - B\right) - \pi + 2A \\ =& \pi - B - C - \pi + 2A = 2A - B - C \\ =& 3A - (A + B + C) = 3A - \pi,\end{aligned}$$

and consequently

$$\pi - \angle B'QC' = \pi - (3A - \pi) = 2\pi - 3A.$$

But on the other hand, $\angle BAC' = \angle BAC = A$ and $\angle CAB' = A$, so

$$\angle B'AC' = 2\pi - \left(\angle BAC' + \angle BAC + \angle CAB'\right) = 2\pi - (A + A + A) = 2\pi - 3A.$$

Consequently, $\angle B'AC' = \pi - \angle B'QC'$. Thus, Q lies on the circle $AB'C'$. Similar reasoning shows that Q also lies on the circles $BC'A'$ and $CA'B'$.

This completes the proof of Theorem 3.

Remark. In general, if a triangle ABC and three points A', B', C' are given, and the circles $A'BC$, $B'CA$, and $C'AB$ have a common point, then the circles $AB'C'$, $BC'A'$, and $CA'B'$ also have a common point. This can be proved with some elementary angle calculations. In our case, the common point of the circles $A'BC$, $B'CA$, and $C'AB$ is the orthocenter of ABC, and the common point of the circles $AB'C'$, $BC'A'$, and $CA'B'$ is Q.

3. Proof of Theorem 4

Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, and A_2, B_2, C_2 the midpoints of B_1C_1, C_1A_1, A_1B_1. It is clear that $A_2B_2C_2$ is the image of ABC under the homothety $\mathsf{h}(G, \frac{1}{4})$. Denote by X the image of A' under this homothety. We show that this is the pedal of the nine-point center N on BC.

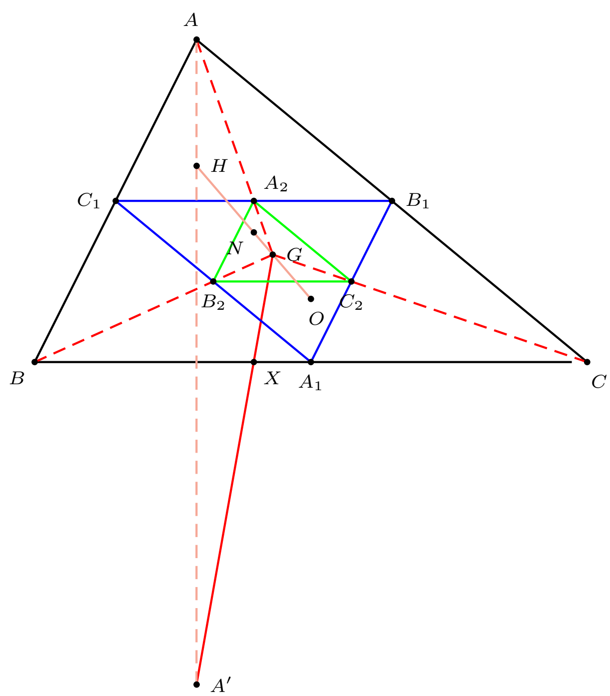

Figure 5

First, note that X, being the reflection of A_2 in B_2C_2, lies on BC. This is because A_2X is perpendicular to B_2C_2 and therefore to BC. The distance from

X to A_2 is twice of that from A_2 to B_2C_2. This is equal to the distance between the parallel lines B_2C_2 and BC.

The segment A_2X is clearly the perpendicular bisector of B_1C_1. It passes through the circumcenter of triangle $A_1B_1C_1$, which is the nine-point N of triangle ABC. It follows that X is the pedal of N on BC. For the same reasons, the images of B', C' under the same homothety $\mathsf{h}(G, \frac{1}{4})$ are the pedals of N on CA and AB respectively.

This completes the proof of Theorem 4.

4. Proof of Corollary 5

It is well known that the circumcenter of the pedal triangle of a point P is the midpoint of the segment PP^*, P^* being the isogonal conjugate of P. See, for example, [2, pp.155–156]. Applying this to the nine-point center N, we obtain the circumcenter of the reflection triangle $A'B'C'$ as the image of the midpoint of NN^* under the homothety $\mathsf{h}(G, 4)$. This is the point

$$\begin{aligned}
G + 4\left(\frac{N+N^*}{2} - G\right) =& 2(N+N^*) - 3G \\
=& 2N^* + 2N - 3G \\
=& 2N^* + (O+H) - (2\cdot O + H) \\
=& 2N^* - O,
\end{aligned}$$

the reflection of O in the Kosnita point N^*. Here, H is orthocenter, and we have made use of the well known facts that N is the midpoint of OH and G divides OH in the ratio $HG : GO = 2 : 1$.

This completes the proof of Corollary 5.

This point is the point X_{195} of [3]. Barry Wolk [6] has verified this theorem by computer calculations with barycentric coordinates.

5. Triangles with nine-point center on the circumcircle

Given a circle $O(R)$ and a point N on its circumference, let H be the reflection of O in N. For an arbitrary point P on the minor arc of the circle $N(\frac{R}{2})$ *inside* $O(R)$, let (i) A be the intersection of the *segment* HP with $O(R)$, (ii) the perpendicular to HP at P intersect $O(R)$ at B and C. Then triangle ABC has nine-point center N on its circumcircle $O(R)$. See Figure 6. This can be shown as follows. It is clear that $O(R)$ is the circumcircle of triangle ABC. Let M be the midpoint of BC so that OM is orthogonal to BC and parallel to PH. Thus, $OMPH$ is a (self-intersecting) trapezoid, and the line joining the midpoints of PM and OH is parallel to PH. Since the midpoint of OH is N and PH is orthogonal to BC, we conclude that N lies on the perpendicular bisector of PM. Consequently, $NM = NP = \frac{R}{2}$, and M lies on the circle $N(\frac{R}{2})$. This circle is the nine-point circle of triangle ABC, since it passes through the pedal P of A on BC and through the midpoint M of BC and has radius $\frac{R}{2}$.

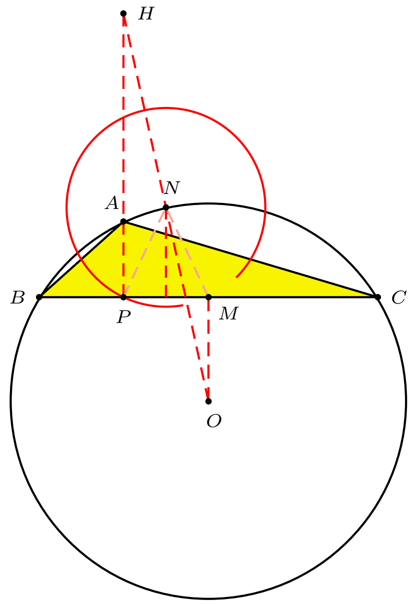

Figure 6

Remark. As P traverses the minor arc which the intersection of $N(\frac{R}{2})$ with the interior of $O(R)$, the line $\mathcal{L}$ passes through a fixed point, which is the reflection of O in H.

Theorem 6. *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*

(1) *The reflection triangle $A'B'C'$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles AOA', BOB', COC' are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through O.*

References

[1] G. Boutte, Hyacinthos message 3997, September 28, 2001.
[2] R. A. Johnson, *Modern Geometry*, 1929; reprinted as *Advanced Euclidean Geometry*, Dover Publications, 1960.
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[5] J. Rigby, Brief notes on some forgotten geometrical theorems, *Mathematics & Informatics Quarterly*, 7 (1997) 156–158.
[6] B. Wolk, Hyacinthos message 6432, January 26, 2003.
[7] P. Yiu, Hyacinthos message 4533, December 12, 2001.

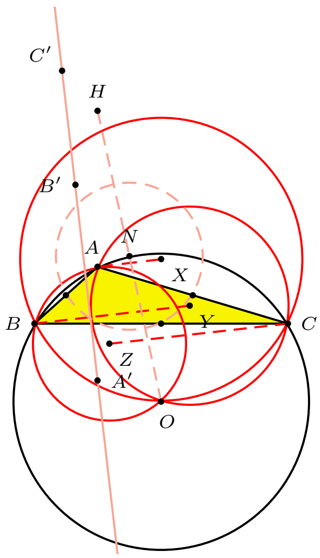

Figure 7

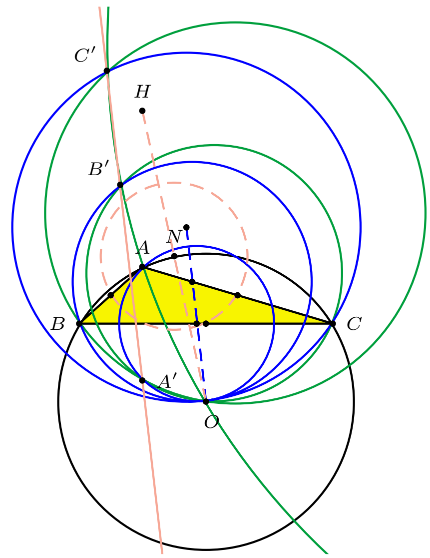

Figure 8

Added in proof: Bernard Gibert has kindly communincated the following results. Let A_1 be the intersection of the lines OA' and $B'C'$, and similarly define B_1 and C_1. Denote, as in §1, by Q be the inverse of the Kosnita point in the circumcircle.

Theorem 7 (Gibert)**.** *The lines AA_1, BB_1, CC_1 concur at the isogonal conjugate of Q.*

This is the point X_{1263} in [3]. The points A, B, C, A', B', C', O, Q, A_1, B_1, C_1 all lie on the Neuberg cubic of triangle ABC, which is the isogonal cubic with pivot the infinite point of the Euler line. This cubic is also the locus of all points whose reflections in the sides of triangle ABC form a triangle perspective to ABC. The point Q is the unique point whose triangle of reflections has perspector on the circumcircle. This perspector, called the Gibert point X_{1141} in [3], lies on the line joining the nine-point center to the Kosnita point.

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