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A Note on the Schiffler Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. We prove two interesting properties of the Schiffler point.

1. Main results

The Schiffler point is the intersection of four Euler lines. Let I be the incenter of triangle ABC. The Schiffler point S is the point common to the Euler lines of triangles IBC, ICA, IAB, and ABC. See [1, p.70]. Not much is known about S. In this note, we prove two interesting properties of this point.

Theorem 1. *Let A and I_1 be the circumcenter and A-excenter of triangle ABC, and A_1 the intersection of OI_1 and BC. Similarly define B_1 and C_1. The lines AA_1, BB_1 and CC_1 concur at the Schiffler point S.*

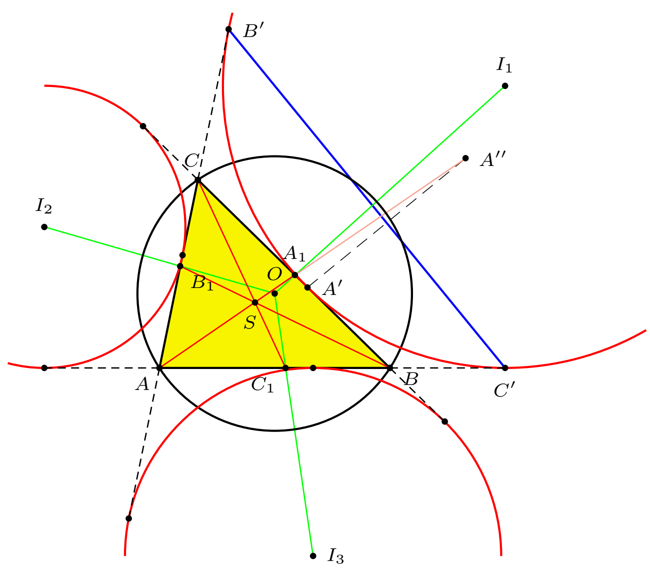

Figure 1

Theorem 2. *Let A', B', C' be the touch points of the A-excircle and BC, CA, AB respectively, and A'' the reflection of A' in $B'C'$. Similarly define B'' and C''. The lines AA'', BB'' and CC'' concur at the Schiffler point S.*

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We make use of trilinear coordinates with respect to triangle ABC. According to [1, p.70], the Schiffler point has coordinates

$$\left(\frac{1}{\cos B+\cos C}:\frac{1}{\cos C+\cos A}:\frac{1}{\cos A+\cos B}\right).$$

2. Proof of Theorem 1

We show that AA_1 passes through the Schiffler point S. Because

$$O=(\cos A:\cos B:\cos C)\quad\text{and}\quad I_1=(-1:1:1),$$

the line OI_1 is given by

$$(\cos B-\cos C)\alpha-(\cos C+\cos A)\beta+(\cos A+\cos B)\gamma=0.$$

The line BC is given by $\alpha=0$. Hence the intersection of OI_1 and BC is

$$A_1=(0:\cos A+\cos B:\cos A+\cos C).$$

The collinearity of A_1, S and A follows from

$$\begin{aligned}&\begin{vmatrix}1 & 0 & 0\\ 0 & \cos A+\cos B & \cos A+\cos C\\ \frac{1}{\cos B+\cos C} & \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B}\end{vmatrix}\\ =&\begin{vmatrix}\cos A+\cos B & \cos A+\cos C\\ \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B}\end{vmatrix}\\ =&0.\end{aligned}$$

This completes the proof of Theorem 1.

Remark. It is clear from the proof above that more generally, if P is a point with trilinear coordinates $(p:q:r)$, and A_1, B_1, C_1 the intersections of PI_a with BC, PI_2 with CA, PI_3 with AB, then the lines AA_1, BB_1, CC_1 intersect at a point with trilinear coordinates $\left(\frac{1}{q+r}:\frac{1}{r+p}:\frac{1}{p+q}\right)$. If P is the symmedian point, for example, this intersection is the point $X_{81}=\left(\frac{1}{b+c}:\frac{1}{c+a}:\frac{1}{a+b}\right)$.

3. Proof of Theorem 2

We deduce Theorem 2 as a consequence of the following two lemmas.

Lemma 3. *The line OI_1 is the Euler line of triangle $A'B'C'$.*

Proof. Triangle ABC is the tangential triangle of $A'B'C'$. It is known that the circumcenter of the tangential triangle lies on the Euler line. See, for example, [1, p.71]. It follows that OI_1 is the Euler line of triangle $A'B'C'$. □

Lemma 4. *Let A^* be the reflection of vertex A of triangle ABC with respect to BC, $A_1B_1C_1$ be the tangential triangle of ABC. Then the Euler line of ABC and line A_1A^* intersect line B_1C_1 in the same point.*

Proof. As is well known, the vertices of the tangential triangle are given by

$$A_1 = (-a : b : c), \quad B_1 = (a : -b : c), \quad C_1 = (a : b : -c).$$

The line B_1C_1 is given by $c\beta + b\gamma = 0$. According to [1, p.42], the Euler line of triangle ABC is given by

$$a(b^2-c^2)(b^2+c^2-a^2)\alpha+b(c^2-a^2)(c^2+a^2-b^2)\beta+c(a^2-b^2)(a^2+b^2-c^2)\gamma = 0.$$

Now, it is not difficult to see that

$$\begin{aligned} A^* =&(-1 : 2\cos C : 2\cos B) \\ =&(-abc : c(a^2+b^2-c^2) : b(c^2+a^2-b^2)). \end{aligned}$$

The equation of the line A^*A_1 is then

$$\begin{vmatrix} -abc & 2c(a^2+b^2-c^2) & 2b(c^2+a^2-b^2) \\ -a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = 0.$$

After simplification, this is

$$-(b^2-c^2)(b^2+c^2-a^2)\alpha + ab(a^2-b^2)\beta - ac(a^2-c^2)\gamma = 0.$$

Now, the lines B_1C_1, A^*A_1, and the Euler line are concurrent if the determinant

$$\begin{vmatrix} 0 & c & b \\ -(b^2-c^2)(b^2+c^2-a^2) & ab(a^2-b^2) & -ac(a^2-c^2) \\ a(b^2-c^2)(b^2+c^2-a^2) & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix}$$

is zero. Factoring out $(b^2-c^2)(b^2+c^2-a^2)$, we have

$$\begin{aligned} &\begin{vmatrix} 0 & c & b \\ -1 & ab(a^2-b^2) & -ac(a^2-c^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} \\ =&-c\begin{vmatrix} -1 & -ac(a^2-c^2) \\ a & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} + b\begin{vmatrix} -1 & ab(a^2-b^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) \end{vmatrix} \\ =&c^2((a^2-b^2)(a^2+b^2-c^2) - a^2(a^2-c^2)) \\ &-b^2((c^2-a^2)(c^2+a^2-b^2) + a^2(a^2-b^2)) \\ =&c^2 \cdot b^2(c^2-b^2) - b^2 \cdot c^2(c^2-b^2) \\ =&0. \end{aligned}$$

This confirms that the three lines are concurrent. □

To prove Theorem 2, it is enough to show that the line AA'' in Figure 1 contains S. Now, triangle $A'B'C'$ has tangential triangle ABC and Euler line OI_1 by Lemma 3. By Lemma 4, the lines OI_1, AA'' and BC are concurrent. This means that the line AA'' contains A_1. By Theorem 1, this line contains S.

Reference

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: emelyanov@kaluga.ru

Tatiana Emelyanova: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: emelyanov@kaluga.ru