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Harcourt's Theorem

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Abstract. We give a proof of Harcourt's theorem that if the signed distances from the vertices of a triangle of sides a, b, c to a tangent of the incircle are a_1, b_1, c_1, then $aa_1 + bb_1 + cc_1$ is twice of the area of the triangle. We also show that there is a point on the circumconic with center I whose distances to the sidelines of ABC are precisely a_1, b_1, c_1. An application is given to the extangents triangle formed by the external common tangents of the excircles.

1. Harcourt's Theorem

The following interesting theorem appears in F. G.-M.[1, p.750] as Harcourt's theorem.

Theorem 1 (Harcourt)**.** *If the distances from the vertices A, B, C to a tangent to the incircle of triangle ABC are a_1, b_1, c_1 respectively, then the algebraic sum $aa_1 + bb_1 + cc_1$ is twice of the area of triangle ABC.*

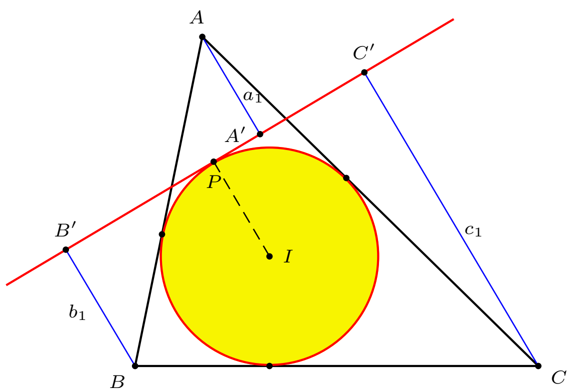

Figure 1

The distances are signed. Distances to a line from points on opposite sides are opposite in sign, while those from points on the same side have the same sign. For the tangent lines to the incircle, we stipulate that the distance from the incenter is positive. For example, in Figure 1, when the tangent line ℓ separates the vertex A from B and C, a_1 is negative while b_1 and c_1 are positive. With this sign convention, Harcourt's theorem states that

$$aa_1 + bb_1 + cc_1 = 2\triangle, \tag{1}$$

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where $\triangle$ is the area of triangle ABC.

We give a simple proof of Harcourt's theorem by making use of homogeneous barycentric coordinates with reference to triangle ABC. First, we establish a fundamental formula.

Proposition 2. *Let ℓ be a line passing through a point P with homogeneous barycentric coordinates $(x : y : z)$. If the signed distances from the vertices A, B, C to a line ℓ are d_1, d_2, d_3 respectively, then*

$$d_1 x + d_2 y + d_3 z = 0. \tag{2}$$

Proof. It is enough to consider the case when ℓ separates A from B and C. We take d_1 as negative, and d_2, d_3 positive. See Figure 2. If A' is the trace of P on the side line BC, it is well known that

$$\frac{AP}{PA'} = \frac{x}{y+z}.$$

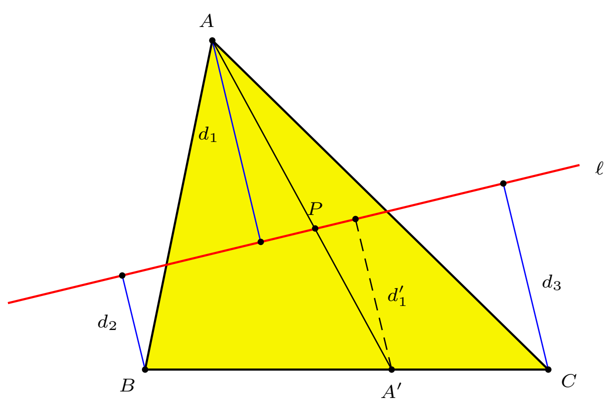

Figure 2

Since $\frac{BA'}{A'C} = \frac{z}{y}$, the distance from A' to ℓ is

$$d'_1 = \frac{yd_2 + zd_3}{y+z}.$$

Since $\frac{-d_1}{d'_1} = \frac{AP}{PA'} = \frac{y+z}{x}$, the equation (2) follows. □

Proof of Harcourt's theorem. We apply Proposition 2 to the line ℓ through the incenter $I = (a : b : c)$ parallel to the tangent. The signed distances from A, B, C to ℓ are $d_1 = a_1 - r$, $d_2 = a_2 - r$, and $d_3 = a_3 - r$. From these,

$$\begin{aligned} aa_1 + bb_1 + cc_1 =& a(d_1 + r) + b(d_2 + r) + c(d_3 + r) \\ =& (ad_1 + bd_2 + cd_3) + (a + b + c)r \\ =& 2\triangle, \end{aligned}$$

since $ad_1 + bd_2 + cd_3 = 0$ by Proposition 2.

2. Harcourt's theorem for the excircles

Harcourt's theorem for the incircle and its proof above can be easily adapted to the excircles.

Theorem 3. *If the distances from the vertices A, B, C to a tangent to the A-excircle of triangle ABC are a_1, b_1, c_1 respectively, then $-aa_1 + bb_1 + cc_1 = 2\triangle$. Analogous statements hold for the B- and C-excircles.*

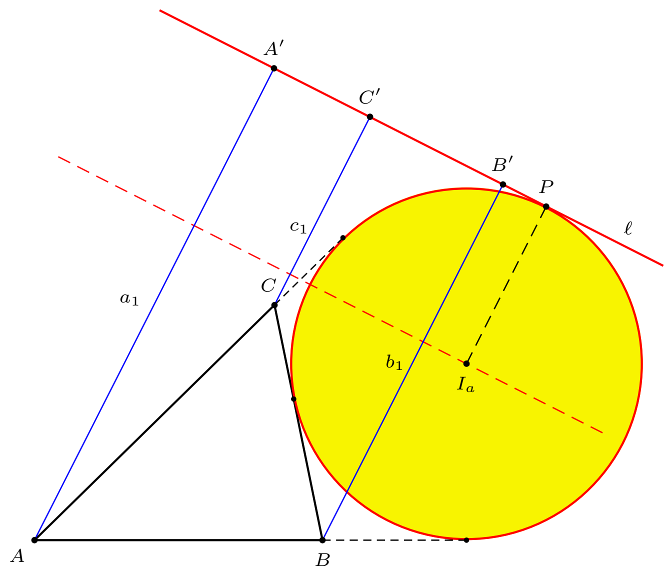

Figure 3

Proof. Apply Proposition 2 to the line ℓ through the excenter $I_a = (-a : b : c)$ parallel to the tangent. If the distances from A, B, C to ℓ are d_1, d_2, d_3 respectively, then

$$-ad_1 + bd_2 + cd_3 = 0.$$

Since $a_1 = d_1 + r_1$, $b_1 = d_2 + r_1$, $c_1 = d_3 + r_1$, where r_1 is the radius of the excircle, it easily follows that

$$\begin{aligned}
-aa_1 + bb_1 + cc_1 =& -a(d_1 + r_1) + b(d_2 + r_1) + c(d_3 + r_1)\\
=&(-ad_1 + bd_2 + cd_3) + r_1(-a + b + c)\\
=&r_1(-a + b + c)\\
=&2\triangle.
\end{aligned}$$

□

Consider the external common tangents of the excircles of triangle ABC. Let ℓ_a be the external common tangent of the B- and C-excircles. Denote by d_{a1}, d_{a2}, d_{a3} the distances from the A, B, C to this line. Clearly, $d_{a1} = h_a$, the altitude on BC. Similarly define ℓ_b, ℓ_c and the associated distances.

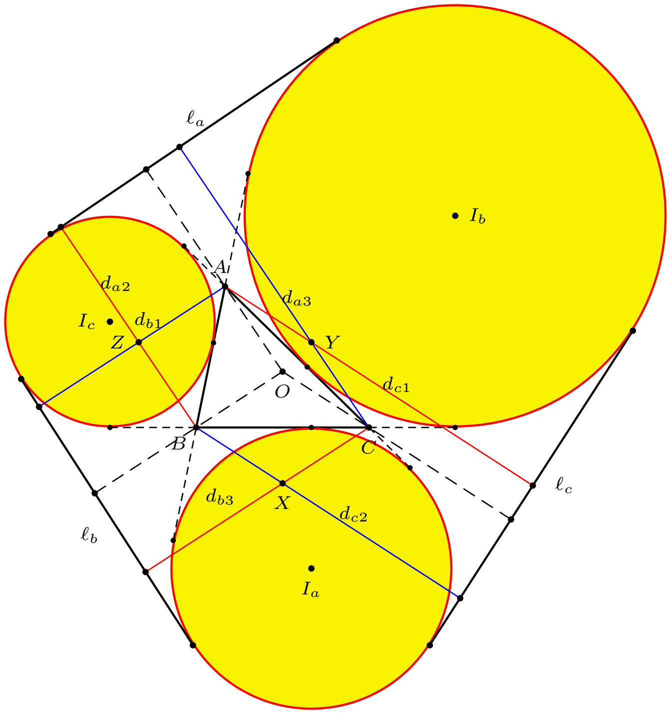

Figure 4

Theorem 4. $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$.

Proof. Applying Theorem 3 to the tangent ℓ_a of the B-excircle (respectively the C-excircle), we have

$$ad_{a1} - bd_{a2} + cd_{a3} = 2\triangle,$$
$$ad_{a1} + bd_{a2} - cd_{a3} = 2\triangle.$$

From these it is clear that $bd_{a2} = cd_{a3}$, and

$$\frac{d_{a2}}{d_{a3}} = \frac{c}{b}.$$

Similarly,

$$\frac{d_{b3}}{d_{b1}} = \frac{a}{c} \quad \text{and} \quad \frac{d_{c1}}{d_{c2}} = \frac{b}{a}.$$

Combining these three equations we have $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$. □

It is clear that the perpendiculars from A to ℓ_a, being the reflection of the A-altitude, passes through the circumcenter; similarly for the perpendiculars from B to ℓ_b and from C to ℓ_c.

Let X be the intersection of the perpendiculars from B to ℓ_c and from C to ℓ_b. Note that OB and CX are parallel, so are OC and BX. Since $OB = OC$, it follows that $OBXC$ is a rhombus, and $BX = CX = R$, the circumradius of triangle ABC. It also follows that X is the reflection of O in the side BC. Similarly, if Y is the intersection of the perpendiculars from C to ℓ_a and from A to ℓ_c, and Z that of the perpendiculars from A to ℓ_b and from B to ℓ_a, then XYZ is the triangle of reflections of the circumcenter O. As such, it is oppositely congruent to ABC, and the center of homothety is the nine-point center of triangle ABC.

3. The circum-ellipse with center I

Consider a tangent $\mathcal{L}$ to the incircle at a point P. If the signed distances from the vertices A, B, C to $\mathcal{L}$ are a_1, b_1, c_1, then by Harcourt's theorem, there is a point $P^\#$ whose signed distances to the sides BC, CA, AB are precisely a_1, b_1, c_1. What is the locus of the point $P^\#$ as P traverses the incircle? By Proposition 2, the barycentric equation of $\mathcal{L}$ is

$$a_1x + b_1y + c_1z = 0.$$

This means that the point with homogeneous barycentric coordinates $(a_1 : b_1 : c_1)$ is a point on the dual conic of the incircle, which is the circumconic with equation

$$(s-a)yz + (s-b)zx + (s-c)xy = 0. \tag{3}$$

The point $P^\#$ in question has barycentric coordinates $(aa_1 : bb_1 : cc_1)$. Since (a_1, b_1, c_1) satisfies (3), if we put $(x, y, z) = (aa_1, bb_1, cc_1)$, then

$$a(s-a)yz + b(s-b)zx + c(s-c)xy = 0.$$

Thus, the locus of $P^\#$ is the circumconic with perspector $(a(s-a) : b(s-b) : c(s-c))$. [1] It is an ellipse, and its center is, surprisingly, the incenter I.[2] We denote this circum-ellipse by $\mathcal{C}_I$. See Figure 5.

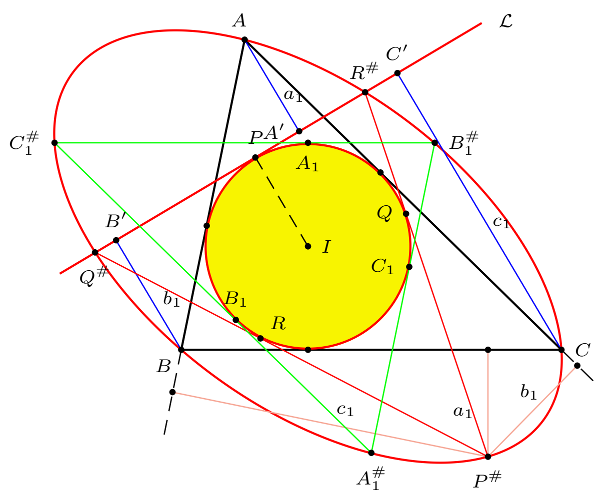

Figure 5

[1]This is the Mittenpunkt, the point X_9 in [4]. It can be constructed as the intersection of the lines joining the excenters to the midpoints of the corresponding sides of triangle ABC.

[2]In general, the center of the circumconic $pyz + qzx + rxy = 0$ is the point with homogeneous barycentric coordinates $(p(q + r - p) : q(r + p - q) : r(p + q - r))$.

Let A_1, B_1, C_1 be the antipodes of the points of tangency of the incircle with the sidelines. It is quite easy to see that $A_1^\#$, $B_1^\#$, $C_1^\#$ are the antipodes of A, B, C in the circum-ellipse $\mathcal{C}_I$. Note that $A_1^\# B_1^\# C_1^\#$ and ABC are oppositely congruent at I. It follows from Steiner's porism that if we denote the intersections of $\mathcal{L}$ and this ellipse by $Q^\#$ and $R^\#$, then the lines $P^\# Q^\#$ and $P^\# R^\#$ are tangent to the incircle at Q and R. This leads to the following construction of $P^\#$.

Construction. If the tangent to the incircle at P intersects the ellipse $\mathcal{C}_I$ at two points, the second tangents from these points to the incircle intersect at $P^\#$ on $\mathcal{C}_I$.

If the point of tangency P has coordinates $\left(\frac{u^2}{s-a} : \frac{v^2}{s-b} : \frac{w^2}{s-c}\right)$, with $u+v+w = 0$, then $P^\#$ is the point $\left(\frac{a(s-a)}{u} : \frac{b(s-b)}{v} : \frac{c(s-c)}{w}\right)$. In particular, if $\mathcal{L}$ is the common tangent of the incircle and the nine-point circle at the Feuerbach point, which has coordinates $((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$, then $P^\#$ is the point $\left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$. This is X_{100} of [3, 4]. It is a point on the circumcircle, lying on the half line joining the Feuerbach point to the centroid of triangle ABC. See [3, Figure 3.12, p.82].

4. The extangents triangle

Consider the external common tangent ℓ_a of the excircles (I_b) and (I_c). Let d_{a1}, d_{a2}, d_{a3} be the distances from A, B, C to this line. We have shown that $\frac{d_{a2}}{d_{a3}} = \frac{c}{b}$. On the other hand, it is clear that $\frac{d_{a1}}{d_{a2}} = \frac{b}{b+c}$. See Figure 6. It follows that

$$d_{a1} : d_{a2} : d_{a3} = bc : c(b+c) : b(b+c).$$

By Proposition 2, the barycentric equation of ℓ_a is

$$bcx + c(b+c)y + b(b+c)z = 0.$$

Similarly, the equations of ℓ_b and ℓ_c are

$$\begin{aligned} c(c+a)x + cay + a(c+a)z =& 0, \\ b(a+b)x + a(a+b)y + abz =& 0. \end{aligned}$$

These three external common tangents bound a triangle called the *extangents triangle* in [3]. The vertices are the points [3]

$$\begin{aligned} A' =& (-a^2 s : b(c+a)(s-c) : c(a+b)(s-b)), \\ B' =& (a(b+c)(s-c) : -b^2 s : c(a+b)(s-a)), \\ C' =& (a(b+c)(s-b) : b(c+a)(s-a) : -c^2 s). \end{aligned}$$

Let I_a' be the incenter of the reflection of triangle ABC in A. It is clear that the distances from A and I_a' to ℓ_a are respectively h_a and r. Since A is the midpoint of II_a', the distance from I to ℓ_a is $2h_a - r$.

[3]The *trilinear* coordinates of these vertices given in [3, p.162, §6.17] are not correct. The diagonal entries of the matrices should read $1 + \cos A$ etc. and $\frac{-a(a+b+c)}{(a-b+c)(a+b-c)}$ etc. respectively.

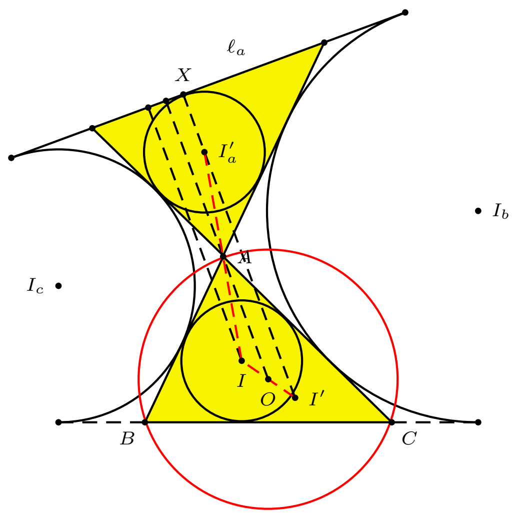

Figure 6

Now consider the reflection of I in O. We denote this point by I'. [4] Since the distances from I and O to ℓ_a are respectively $2h_a - r$ and $R + h_a$, it follows that the distance from I' to ℓ_a is $2(R + h_a) - (2h_a - r) = 2R + r$. For the same reason, the distances from I' to ℓ_b and ℓ_c are also $2R + r$. From this we deduce the following interesting facts about the extangents triangle.

Theorem 5. *The extangent triangle bounded by ℓ_a, ℓ_b, ℓ_c*

(1) *has incenter I' and inradius $2R + r$;*
(2) *is perspective with the excentral triangle at I';*
(3) *is homothetic to the tangential triangle at the internal center of similitude of the circumcircle and the incircle of triangle ABC, the ratio of the homothety being $\frac{2R+r}{R}$.*

Proof. It is enough to locate the homothetic center in (3). This is the point which divides $I'O$ in the ratio $2R + r : -R$, *i.e.*,

$$\frac{(2R + r)O - R(2O - I)}{R + r} = \frac{r \cdot O + R \cdot I}{R + r},$$

the internal center of similitude of the circumcircle and incircle of triangle ABC.[5] □

Remarks. (1) The statement that the extangents triangle has inradius $2R + r$ can also be found in [2, Problem 2.5.4].

(2) Since the excentral triangle has circumcenter I' and circumradius $2R$, it follows that the excenters and the incenters of the reflections of triangle ABC in A, B, C are concyclic. It is well known that since ABC is the orthic triangle of the

[4]This point appears as X_{40} in [4].
[5]This point appears as X_{55} in [4].

excentral triangle, the circumcircle of ABC is the nine-point circle of the excentral triangle.

(3) If the incircle of the extangents triangle touches its sides at X, Y, Z respectively, [6] then triangle XYZ is homothetic to ABC, again at the internal center of similitude of the circumcircle and the incircle.

(4) More generally, the reflections of the traces of a point P in the respective sides of the excentral triangle are points on the sidelines of the extangents triangle. They form a triangle perspective with ABC at the isogonal conjugate of P. For example, the reflections of the points of tangency of the excircles (traces of the Nagel point $(s-a : s-b : s-c)$) form a triangle with perspector $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, the external center of similitude of the circumcircle and the incircle.[7]

References

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[6]These are the reflections of the traces of the Gergonne point in the respective sides of the excentral triangle.

[7]This point appears as X_{56} in [4].