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Isotomic Inscribed Triangles and Their Residuals

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Abstract. We prove some interesting results on inscribed triangles which are isotomic. For examples, we show that the triangles formed by the centroids (respectively orthocenters) of their residuals have equal areas, and those formed by the circumcenters are congruent.

1. Isotomic inscribed triangles

The starting point of this investigation was the interesting observation that if we consider the points of tangency of the sides of a triangle with its incircle and excircles, we have two triangles of equal areas.

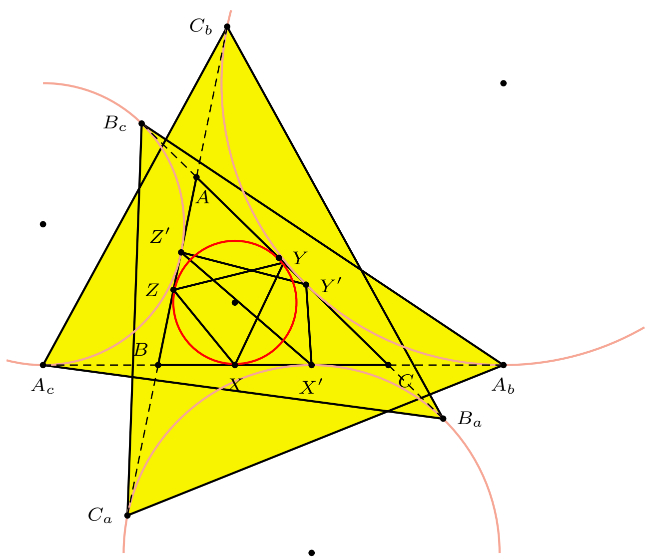

Figure 1

In Figure 1, X, Y, Z are the points of tangency of the incircle with the sides BC, CA, AB of triangle ABC, and X', Y', Z' those with the corresponding excircles. In [2], XYZ and $X'Y'Z'$ are called the intouch and extouch triangles of ABC respectively. That these two triangles have equal areas is best explained by the fact that each pair of points X, X'; Y, Y'; Z, Z' are isotomic on their respective sides, *i.e.*,

$$BX = X'C, \quad CY = Y'A, \quad AZ = Z'B. \tag{1}$$

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We shall say that XYZ and $X'Y'Z'$ are isotomic inscribed triangles. The following basic proposition follows from simple calculations with barycentric coordinates.

Proposition 1. *Isotomic inscribed triangles have equal areas.*

Proof. Let X, Y, Z be points on the sidelines BC, CA, AB dividing the sides in the ratios

$$BX : XC = x : 1-x, \quad CY : YA = y : 1-y, \quad AZ : ZB = z : 1-z.$$

In terms of barycentric coordinates with respect to ABC, we have

$$X = (1-x)B + xC, \quad Y = (1-y)C + yA, \quad Z = (1-z)A + zB. \tag{2}$$

The area of triangle XYZ, in terms of the area $\triangle$ of ABC, is

$$\begin{aligned}\triangle XYZ = &\begin{vmatrix} 0 & 1-x & x \\ y & 0 & 1-y \\ 1-z & z & 0 \end{vmatrix} \triangle \\ =&(1-(x+y+z)+(xy+yz+zx))\triangle \\ =&(xyz+(1-x)(1-y)(1-z))\triangle. \end{aligned} \tag{3}$$

See, for example, [4, Proposition 1]. If X', Y', Z' are points satisfying (1), then

$$BX' : X'C = 1-x : x, \quad CY' : Y'A = 1-y : y, \quad AZ' : Z'B = 1-z : z, \tag{4}$$

and

$$X' = xB + (1-x)C, \quad Y' = yC + (1-y)A, \quad Z' = zA + (1-z)B. \tag{5}$$

The area of triangle $X'Y'Z'$ can be obtained from (3) by replacing x, y, z by $1-x$, $1-y$, $1-z$ respectively. It is clear that this results in the same expression. This completes the proof of the proposition. □

Proposition 2. *The centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle.*

Proof. The expressions in (2) allow one to determine the centroid of triangle XYZ easily. This is the point

$$G_{XYZ} = \frac{1}{3}(X+Y+Z) = \frac{(1+y-z)A + (1+z-x)B + (1+x-y)C}{3}. \tag{6}$$

On the other hand, with the coordinates given in (5), the centroid of triangle $X'Y'Z'$ is

$$G_{X'Y'Z'} = \frac{1}{3}(X'+Y'+Z') = \frac{(1-y+z)A + (1-z+x)B + (1-x+y)C}{3}. \tag{7}$$

It follows easily that

$$\frac{1}{2}(G_{XYZ} + G_{X'Y'Z'}) = \frac{1}{3}(A+B+C) = G,$$

the centroid of triangle ABC. □

Corollary 3. *The intouch and extouch triangles have equal areas, and the midpoint of their centroids is the centroid of triangle ABC.*

Proof. These follow from the fact that the intouch triangle XYZ and the extouch triangle $X'Y'Z'$ are isotomic, as is clear from the following data, where a, b, c denote the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$.

$$\begin{aligned} BX &= X'C = s-b, & BX' &= XC = s-c,\\ CY &= Y'A = s-c, & CY' &= YA = s-a,\\ AZ &= Z'B = s-a, & AZ' &= ZB = s-b. \end{aligned}$$

□

In fact, we may take

$$x = \frac{s-b}{a}, \quad y = \frac{s-c}{b}, \quad z = \frac{s-a}{c},$$

and use (3) to obtain

$$\triangle XYZ = \triangle X'Y'Z' = \frac{2(s-a)(s-b)(s-c)}{abc}\triangle.$$

Let R and r denote respectively the circumradius and inradius of triangle ABC. Since $\triangle = rs$ and

$$R = \frac{abc}{4\triangle}, \qquad r^2 = \frac{(s-a)(s-b)(s-c)}{s},$$

we have

$$\triangle XYZ = \triangle X'Y'Z' = \frac{r}{2R} \cdot \triangle.$$

If we denote by A_b and A_c the points of tangency of the line BC with the B- and C-excircles, it is easy to see that A_b and A_c are isotomic points on BC. In fact,

$$BA_b = A_cC = s, \quad BA_c = A_bC = -(s-a).$$

Similarly, the other points of tangency B_c, B_a, C_a, C_b form pairs of isotomic points on the lines CA and AB respectively. See Figure 1.

Corollary 4. *The triangles $A_bB_cC_a$ and $A_cB_aC_b$ have equal areas. The centroids of these triangles are symmetric with respect to the centroid G of triangle ABC.*

These follow because $A_bB_cC_a$ and $A_cB_bC_a$ are isotomic inscribed triangles. Indeed,

$$\begin{aligned} BA_b : A_bC =& s : -(s-a) = 1 + \frac{s-a}{a} : -\frac{s-a}{a} = CA_c : A_cB,\\ CB_c : B_cA =& s : -(s-b) = 1 + \frac{s-b}{b} : -\frac{s-b}{b} = AB_a : B_aC,\\ AC_a : C_aB =& s : -(s-c) = 1 + \frac{s-c}{c} : -\frac{s-c}{c} = BC_b : C_bA. \end{aligned}$$

Furthermore, the centroids of the four triangles XYZ, $X'Y'Z'$, $A_bB_cC_a$ and $A_cB_aC_b$ form a parallelogram. See Figure 2.

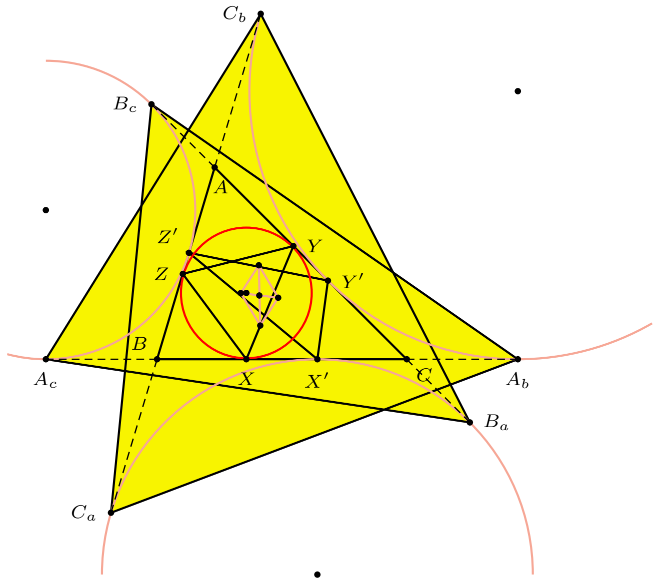

Figure 2

2. Triangles of residual centroids

For an inscribed triangle XYZ, we call the triangles AYZ, BZX, CXY its residuals. From (2, 5), we easily determine the centroids of these triangles.

$$G_{AYZ} = \frac{1}{3}((2+y-z)A + zB + (1-y)C),$$
$$G_{BZX} = \frac{1}{3}((1-z)A + (2+z-x)B + xC),$$
$$G_{CXY} = \frac{1}{3}(yA + (1-x)B + (2+x-y)C).$$

We call these the residual centroids of the inscribed triangle XYZ.

The following two propositions are very easily to established, by making the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$.

Proposition 5. *The triangles of residual centroids of isotomic inscribed triangles have equal areas.*

Proof. From the coordinates given above, we obtain the area of the triangle of residual centroids as

$$\frac{1}{27}\begin{vmatrix} 2+y-z & z & 1-y \\ 1-z & 2+z-x & x \\ y & 1-x & 2+x-y \end{vmatrix}\triangle$$
$$=\frac{1}{9}(3-x-y-z+xy+yz+zx)\triangle$$
$$=\frac{1}{9}(2+xyz+(1-x)(1-y)(1-z))\triangle$$

By effecting the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$, we obtain the area of the triangle of residual centroids of the isotomic inscribed triangle $X'Y'Z'$. This clearly remains unchanged. □

Proposition 6. *Let XYZ and $X'Y'Z'$ be isotomic inscribed triangles of ABC. The centroids of the following five triangles are collinear:*

- *G of triangle ABC,*
- *G_{XYZ} and $G_{X'Y'Z'}$ of the inscribed triangles,*
- *$\widetilde{G}$ and $\widetilde{G'}$ of the triangles of their residual centroids.*

Furthermore,

$$G_{XYZ}\widetilde{G} : \widetilde{G}G : G\widetilde{G'} : \widetilde{G'}G_{X'Y'Z'} = 1 : 2 : 2 : 1.$$

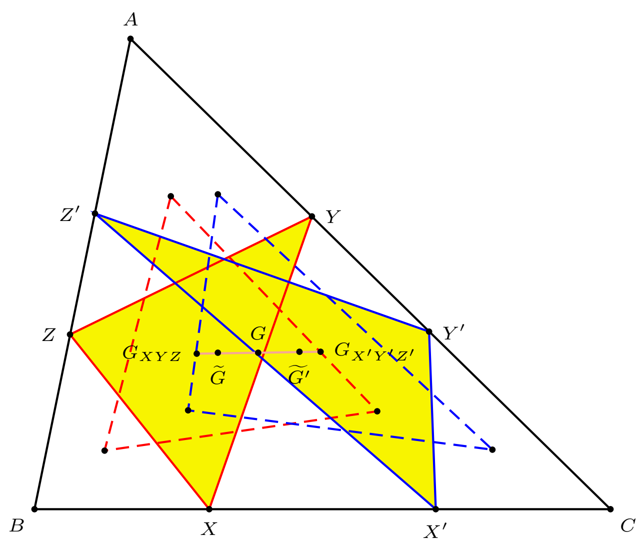

Figure 3

Proof. The centroid $\widetilde{G}$ is the point

$$\widetilde{G} = \frac{1}{9}((3+2y-2z)A + (3+2z-2x)B + (3+2x-2y)C).$$

We obtain the centroid $\widetilde{G'}$ by interchanging $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$. From these coordinates and those given in (6,7), the collinearity is clear, and it is easy to figure out the ratios of division. □

3. Triangles of residual orthocenters

Proposition 7. *The triangles of residual orthocenters of isotomic inscribed triangles have equal areas.*

See Figure 4. This is an immediate corollary of the following proposition (see Figure 5), which in turn is a special case of a more general situation considered in Proposition 8 below.

Proposition 8. *An inscribed triangle and its triangle of residual orthocenters have equal areas.*

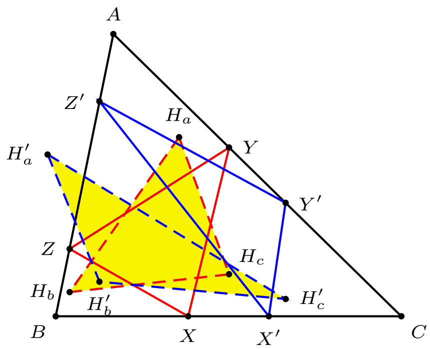

Figure 4

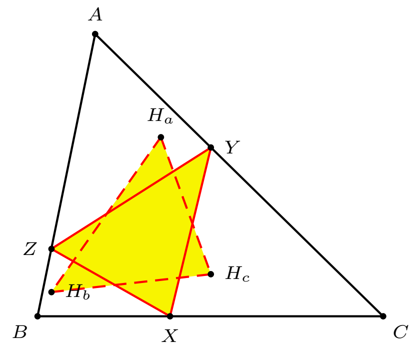

Figure 5

Proposition 9. *Given a triangle ABC, if pairs of parallel lines $\mathcal{L}_{1B}$, $\mathcal{L}_{1C}$ through B, C, $\mathcal{L}_{2C}$, $\mathcal{L}_{2A}$ through C, A, and $\mathcal{L}_{3A}$, $\mathcal{L}_{3B}$ through A, B are constructed, and if*

$$P_a = \mathcal{L}_{2C} \cap \mathcal{L}_{3B}, \qquad P_b = \mathcal{L}_{3A} \cap \mathcal{L}_{1C}, \qquad P_c = \mathcal{L}_{1B} \cap \mathcal{L}_{2A},$$

then the triangle $P_aP_bP_c$ has the same area as triangle ABC.

Proof. We write $Y = \mathcal{L}_{2C} \cap \mathcal{L}_{3A}$ and $Z = \mathcal{L}_{2A} \cap \mathcal{L}_{3B}$. Consider the parallelogram AZP_aY in Figure 6. If the points B and C divide the segments ZP_a and YP_a in the ratios

$$ZB : BP_a = v : 1 - v, \qquad YC : CP_a = w : 1 - w,$$

then it is easy to see that

$$\text{Area}(ABC) = \frac{1 + vw}{2} \cdot \text{Area}(AZP_aY). \tag{8}$$

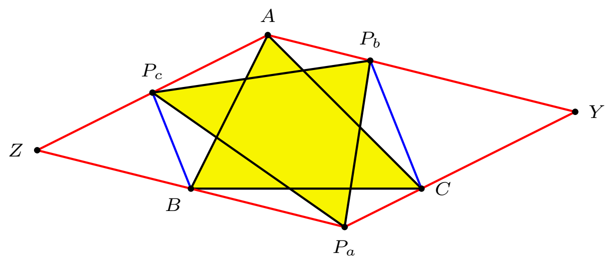

Figure 6

Now, P_b and P_c are points on AY and AZ such that BP_c and CP_b are parallel. If

$$YP_b : P_bA = v' : 1 - v', \qquad ZP_c : P_cA = w' : 1 - w',$$

then from the similarlity of triangles BZP_c and P_bYC, we have

$$ZB : ZP_c = YP_b : YC.$$

This means that $v : w' = v' : w$ and $v'w' = vw$. Now, in the same parallelogram AZP_aY, we have

$$\text{Area}(P_aP_bP_c) = \frac{1 + v'w'}{2} \cdot \text{Area}(AZP_aY).$$

From this we conclude that $P_aP_bP_c$ and ABC have equal areas.

□

4. Triangles of residual circumcenters

Consider the circumcircles of the residuals of an inscribed triangle XYZ. By Miquel's theorem, the circles AYZ, BZX, and CXY have a common point. Furthermore, the centers O_a, O_b, O_c of these circles form a triangle similar to ABC. See, for example, [1, p.134]. We prove the following interesting theorem.

Theorem 10. *The triangles of residual circumcenters of the isotomic inscribed triangles are congruent.*

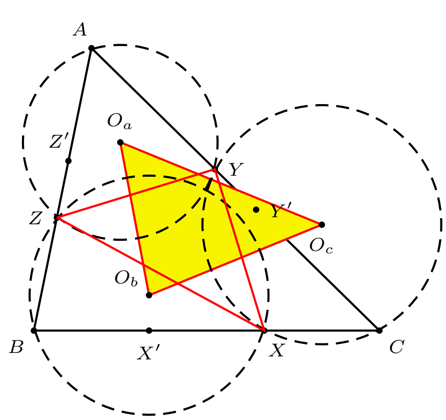

Figure 7A

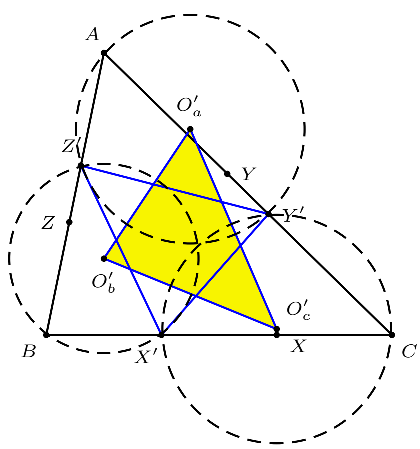

Figure 7B

We prove this theorem by calculations.

Lemma 11. *Let X, Y, Z be points on BC, CA, AB such that*

$$BX : XC = w : v, \quad CY : YA = u_c : w, \quad AZ : ZB = v : u_b.$$

The distance between the circumcenters O_b and O_c is the hypotenuse of a right triangle with one side $\frac{a}{2}$ and another side

$$\frac{(v-w)(u_b+v)(u_c+w)a^2 + (v+w)(w-u_c)(u_b+v)b^2 + (v+w)(w+u_c)(u_b-v)c^2}{8\triangle(u_b+v)(v+w)(w+u_c)} \cdot a. \tag{9}$$

Proof. The distance between O_b and O_c along the side BC is clearly $\frac{a}{2}$. We calculate their distance along the altitude on BC. The circumradius of BZX is clearly $R_b = \frac{ZX}{2\sin B}$. The distance of O_b above BC is

$$\begin{aligned}
R_b \cos BZX =& \frac{ZX\cos BZX}{2\sin B} = \frac{2BZ\cdot ZX\cos BZX}{4BZ\sin B} = \frac{BZ^2+ZX^2-BX^2}{4BZ\sin B}\\
=& \frac{BZ^2+BZ^2+BX^2-2BZ\cdot BX\cos B-BX^2}{4BZ\sin B}\\
=& \frac{BZ-BX\cos B}{2\sin B} = \frac{c(BZ-BX\cos B)}{4\triangle}\cdot a\\
=& \frac{c\left(\frac{u_b}{u_b+v}c - \frac{w}{v+w}a\cos B\right)}{4\triangle}\cdot a\\
=& \frac{u_b(v+w)2c^2 - w(u_b+v)(c^2+a^2-b^2)}{8\triangle(u_b+v)(v+w)}\cdot a\\
=& \frac{-(u_b+v)w(a^2-b^2)+(2u_bv+u_bw-vw)c^2}{8\triangle(u_b+v)(v+w)}\cdot a
\end{aligned}$$

By making the interchanges $b\leftrightarrow c$, $v\leftrightarrow w$, and $u_b\leftrightarrow u_c$, we obtain the distance of O_c above the same line as

$$\frac{-(u_c+w)v(a^2-c^2)+(2u_cw+u_cv-vw)b^2}{8\triangle(u_c+w)(v+w)}\cdot a.$$

The difference between these two is the expression given in (9) above. □

Consider now the isotomic inscribed triangle $X'Y'Z'$. We have

$$\begin{aligned}
BX' : X'C =& v : w,\\
CY' : Y'A =& w : u_c = \frac{vw}{u_c} : v,\\
AZ' : Z'B =& u_b : v = w : \frac{vw}{u_b}.
\end{aligned}$$

Let O_b' and O_c' be the circumcenters of $BZ'X'$ and $CX'Y'$. By making the following interchanges

$$v\leftrightarrow w, \qquad u_b\leftrightarrow \frac{vw}{u_b}, \qquad u_c\leftrightarrow \frac{vw}{u_c}$$

in (9), we obtain the distance between O_b' and O_c' along the altitude on BC as

$$\begin{aligned}
&\frac{(w-v)(\frac{vw}{u_b}+w)(\frac{vw}{u_c}+v)a^2+(v+w)(v-\frac{vw}{u_c})(\frac{vw}{u_b}+w)b^2+(v+w)(v+\frac{vw}{u_c})(\frac{vw}{u_b}-w)c^2}{8\triangle(\frac{vw}{u_b}+w)(v+w)(v+\frac{vw}{u_c})}\cdot a\\
=&\frac{(w-v)(v+u_b)(w+u_c)a^2+(v+w)(u_c-w)(v+u_b)b^2+(v+w)(w+u_c)(v-u_b)c^2}{8\triangle(v+u_b)(v+w)(u_c+w)}\cdot a.
\end{aligned}$$

Except for a reversal in sign, this is the same as (9).

From this we easily conclude that the segments O_bO_c and $O'_bO'_c$ are congruent. The same reasoning also yields the congruences of O_cO_a, $O'_cO'_a$, and of O_aO_b, $O'_aO'_b$. It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. This completes the proof of Theorem 9.

5. Isotomic conjugates

Let XYZ be the cevian triangle of a point P, *i.e.*, X, Y, Z are respectively the intersections of the line pairs AP, BC; BP, CA; CP, AB. By the residual centroids ((respectively orthocenters, circumcenters) of P, we mean those of its cevian triangle. If we construct points X', Y', Z' satisfying (1), then the lines AX', BY', CZ' intersect at a point P' called the isotomic conjugate of P. If the point P has homogeneous barycentric coordinates $(x : y : z)$, then P' has homogeneous barycentric coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. All results in the preceding sections apply to the case when XYZ and $X'Y'Z'$ are the cevian triangles of two isotomic conjugates. In particular, in the case of residual circumcenters in §4 above, if XYZ is the cevian triangle of P with homogeneous barycentric coordinates $(u : v : w)$, then

$$BX : XC = w : v, \qquad CY : YA = u : w, \qquad AZ : ZB = v : u.$$

By putting $u_b = u_c = u$ in (9) we obtain a necessary and sufficient condition for the line O_bO_c to be parallel to BC, namely,

$$(v-w)(u+v)(u+w)a^2+(v+w)(w-u)(u+v)b^2+(v+w)(w+u)(u-v)c^2 = 0.$$

This can be reorganized into the form

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

This is the equation of the Lucas cubic, consisting of points P for which the line joining P to its isotomic conjugate P' passes through the orthocenter H. The symmetry of this equation leads to the following interesting theorem.

Theorem 12. *The triangle of residual circumcenters of P is homothetic to ABC if and only if P lies on the Lucas cubic.*

It is well known that the Lucas cubic is the locus of point P whose cevian triangle is also the pedal triangle of a point Q. In this case, the circumcircles of AYZ, BZX and CXY intersect at Q, and the circumcenters O_a, O_b, O_c are the midpoints of the segments AQ, BQ, CQ. The triangle $O_aO_bO_c$ is homothetic to ABC at Q.

For example, if P is the Gergonne point, then $O_aO_bO_c$ is homothetic to ABC at the incenter I. The isotomic conjugate of P is the Nagel point, and $O'_aO'_bO'_c$ is homothetic to ABC at the reflection of I in the circumcenter O.

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