Isotomic Inscribed Triangles and Their Residuals

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Abstract. We prove some interesting results on inscribed triangles which are isotomic. For examples, we show that the triangles formed by the centroids (respectively orthocenters) of their residuals have equal areas, and those formed by the circumcenters are congruent.

1. Isotomic inscribed triangles

The starting point of this investigation was the interesting observation that if we consider the points of tangency of the sides of a triangle with its incircle and excircles, we have two triangles of equal areas.

In Figure 1, $X, Y, Z$ are the points of tangency of the incircle with the sides $BC, CA, AB$ of triangle $ABC$, and $X', Y', Z'$ those with the corresponding excircles. In [2], $XYZ$ and $X'Y'Z'$ are called the intouch and extouch triangles of $ABC$ respectively. That these two triangles have equal areas is best explained by the fact that each pair of points $X, X'; Y, Y'; Z, Z'$ are isotomic on their respective sides, i.e.,

\[ BX = X'C, \quad CY = Y'A, \quad AZ = Z'B. \] (1)

We shall say that $XYZ$ and $X'Y'Z'$ are isotomic inscribed triangles. The following basic proposition follows from simple calculations with barycentric coordinates.

**Proposition 1.** Isotomic inscribed triangles have equal areas.

**Proof.** Let $X, Y, Z$ be points on the sidelines $BC, CA, AB$ dividing the sides in the ratios


In terms of barycentric coordinates with respect to $ABC$, we have

$$X = (1 - x)B + xC, \quad Y = (1 - y)C + yA, \quad Z = (1 - z)A + zB.$$  \hspace{1cm} (2)

The area of triangle $XYZ$, in terms of the area $\triangle ABC$, is

$$\triangle XYZ = \begin{vmatrix} 0 & 1 - x & x \\ y & 0 & 1 - y \\ 1 - z & z & 0 \end{vmatrix} \triangle = (1 - (x + y + z) + (xy + yz + zx)) \triangle = (xyz + (1 - x)(1 - y)(1 - z)) \triangle.$$  \hspace{1cm} (3)

See, for example, [4, Proposition 1]. If $X', Y', Z'$ are points satisfying (1), then

$$BX' : X'C = 1 - x : x, \quad CY' : Y'A = 1 - y : y, \quad AZ' : Z'B = 1 - z : z,$$  \hspace{1cm} (4)

and

$$X' = xB + (1 - x)C, \quad Y' = yC + (1 - y)A, \quad Z' = zA + (1 - z)B.$$  \hspace{1cm} (5)

The area of triangle $X'Y'Z'$ can be obtained from (3) by replacing $x, y, z$ by $1 - x, 1 - y, 1 - z$ respectively. It is clear that this results in the same expression. This completes the proof of the proposition. \hfill $\square$

**Proposition 2.** The centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle.

**Proof.** The expressions in (2) allow one to determine the centroid of triangle $XYZ$ easily. This is the point

$$G_{XYZ} = \frac{1}{3}(X + Y + Z) = \frac{(1 + y - z)A + (1 + z - x)B + (1 + x - y)C}{3}.$$  \hspace{1cm} (6)

On the other hand, with the coordinates given in (5), the centroid of triangle $X'Y'Z'$ is

$$G_{X'Y'Z'} = \frac{1}{3}(X' + Y' + Z') = \frac{(1 - y + z)A + (1 - z + x)B + (1 - x + y)C}{3}.$$  \hspace{1cm} (7)

It follows easily that

$$\frac{1}{2}(G_{XYZ} + G_{X'Y'Z'}) = \frac{1}{3}(A + B + C) = G,$$

the centroid of triangle $ABC$. \hfill $\square$
Corollary 3. The intouch and extouch triangles have equal areas, and the midpoint of their centroids is the centroid of triangle $ABC$.

Proof. These follow from the fact that the intouch triangle $XYZ$ and the extouch triangle $X'Y'Z'$ are isotomic, as is clear from the following data, where $a$, $b$, $c$ denote the lengths of the sides $BC$, $CA$, $AB$ of triangle $ABC$, and $s = \frac{1}{2}(a+b+c)$.

\[
egin{align*}
BX &= X'C = s - b, & BX' &= XC = s - c, \\
CY &= Y'A = s - c, & CY' &= YA = s - a, \\
AZ &= Z'B = s - a, & AZ' &= ZB = s - b.
\end{align*}
\]

\[\square\]

In fact, we may take

\[
\begin{align*}
x &= \frac{s - b}{a}, & y &= \frac{s - c}{b}, & z &= \frac{s - a}{c},
\end{align*}
\]

and use (3) to obtain

\[
\triangle XYZ = \triangle X'Y'Z' = \frac{2(s - a)(s - b)(s - c)}{abc} \cdot \triangle.
\]

Let $R$ and $r$ denote respectively the circumradius and inradius of triangle $ABC$. Since $\triangle = rs$ and

\[
\begin{align*}
R &= \frac{abc}{4\triangle}, & r^2 &= \frac{(s - a)(s - b)(s - c)}{s},
\end{align*}
\]

we have

\[
\triangle XYZ = \triangle X'Y'Z' = \frac{r}{2R} \cdot \triangle.
\]

If we denote by $A_b$ and $A_c$ the points of tangency of the line $BC$ with the $B$- and $C$-excircles, it is easy to see that $A_b$ and $A_c$ are isotomic points on $BC$. In fact,

\[
BA_b = A_cC = s, \quad BA_c = A_bC = -(s - a).
\]

Similarly, the other points of tangency $B_c, B_a, C_a, C_b$ form pairs of isotomic points on the lines $CA$ and $AB$ respectively. See Figure 1.

Corollary 4. The triangles $A_bB_cC_a$ and $A_cB_aC_b$ have equal areas. The centroids of these triangles are symmetric with respect to the centroid $G$ of triangle $ABC$.

These follow because $A_bB_cC_a$ and $A_cB_aC_b$ are isotomic inscribed triangles. Indeed,

\[
\begin{align*}
BA_b : A_bC &= s : -(s - a) = 1 + \frac{s - a}{a} : -\frac{s - a}{a} = CA_c : A_cB, \\
CB_c : B_cA &= s : -(s - b) = 1 + \frac{s - b}{b} : -\frac{s - b}{b} = AB_a : B_aC, \\
AC_a : C_aB &= s : -(s - c) = 1 + \frac{s - c}{c} : -\frac{s - c}{c} = BC_b : C_bA.
\end{align*}
\]

Furthermore, the centroids of the four triangles $XYZ$, $X'Y'Z'$, $A_bB_cC_a$ and $A_cB_aC_b$ form a parallelogram. See Figure 2.
2. Triangles of residual centroids

For an inscribed triangle $XYZ$, we call the triangles $AYZ$, $BZX$, $CXY$ its residuals. From (2, 5), we easily determine the centroids of these triangles.

\[
G_{AYZ} = \frac{1}{3}((2 + y - z)A + zB + (1 - y)C),
\]

\[
G_{BZX} = \frac{1}{3}((1 - z)A + (2 + z - x)B + xC),
\]

\[
G_{CXY} = \frac{1}{3}(yA + (1 - x)B + (2 + x - y)C).
\]

We call these the residual centroids of the inscribed triangle $XYZ$.

The following two propositions are very easily to established, by making the interchanges $(x, y, z) \leftrightarrow (1 - x, 1 - y, 1 - z)$.

**Proposition 5.** The triangles of residual centroids of isotomic inscribed triangles have equal areas.

**Proof.** From the coordinates given above, we obtain the area of the triangle of residual centroids as

\[
\frac{1}{27} \begin{vmatrix} 2 + y - z & z & 1 - y \\ 1 - z & 2 + z - x & x \\ y & 1 - x & 2 + x - y \end{vmatrix} \triangle = \frac{1}{9}(3 - x - y - z + xy + yz + zx)\triangle
\]

\[
= \frac{1}{9}(2 + xyz + (1 - x)(1 - y)(1 - z))\triangle
\]
By effecting the interchanges \((x, y, z) \leftrightarrow (1-x, 1-y, 1-z)\), we obtain the area of the triangle of residual centroids of the isotomic inscribed triangle \(X'Y'Z'\). This clearly remains unchanged. \(\Box\)

**Proposition 6.** Let \(XYZ\) and \(X'Y'Z'\) be isotomic inscribed triangles of \(ABC\). The centroids of the following five triangles are collinear:

- \(G\) of triangle \(ABC\),
- \(G_{XYZ}\) and \(G_{X'Y'Z'}\) of the inscribed triangles,
- \(\widetilde{G}\) and \(\widetilde{G}'\) of the triangles of their residual centroids.

Furthermore,

\[G_{XYZ}\widetilde{G} : \widetilde{G}G : G\widetilde{G}' : \widetilde{G}'G_{X'Y'Z'} = 1 : 2 : 2 : 1.\]

**Proof.** The centroid \(\widetilde{G}\) is the point

\[\widetilde{G} = \frac{1}{9}(3 + 2y - 2z)A + (3 + 2z - 2x)B + (3 + 2x - 2y)C.\]

We obtain the centroid \(\widetilde{G}'\) by interchanging \((x, y, z) \leftrightarrow (1-x, 1-y, 1-z)\). From these coordinates and those given in (6,7), the collinearity is clear, and it is easy to figure out the ratios of division. \(\Box\)

3. Triangles of residual orthocenters

**Proposition 7.** The triangles of residual orthocenters of isotomic inscribed triangles have equal areas.

See Figure 4. This is an immediate corollary of the following proposition (see Figure 5), which in turn is a special case of a more general situation considered in Proposition 8 below.
Proposition 8. An inscribed triangle and its triangle of residual orthocenters have equal areas.

![Figure 4](image)

![Figure 5](image)

Proposition 9. Given a triangle $ABC$, if pairs of parallel lines $L_{2B}$, $L_{1C}$ through $B$, $C$, $L_{2C}$, $L_{2A}$ through $C$, $A$, and $L_{3A}$, $L_{3B}$ through $A$, $B$ are constructed, and if

$$ P_a = L_{2C} \cap L_{3B}, \quad P_b = L_{3A} \cap L_{1C}, \quad P_c = L_{1B} \cap L_{2A}, $$

then the triangle $P_aP_bP_c$ has the same area as triangle $ABC$.

**Proof.** We write $Y = L_{2C} \cap L_{3A}$ and $Z = L_{2A} \cap L_{3B}$. Consider the parallelogram $AZP_aY$ in Figure 6. If the points $B$ and $C$ divide the segments $ZP_a$ and $YP_a$ in the ratios

$$ ZB : BP_a = v : 1 - v, \quad YC : CP_a = w : 1 - w, $$

then it is easy to see that

$$ \text{Area}(ABC) = \frac{1 + vw}{2} \cdot \text{Area}(AZP_aY). \quad (8) $$

![Figure 6](image)

Now, $P_b$ and $P_c$ are points on $AY$ and $AZ$ such that $BP_a$ and $CP_b$ are parallel. If

$$ YP_b : P_bA = v' : 1 - v', \quad ZP_c : P_cA = w' : 1 - w', $$

then...
then from the similarity of triangles $BZP_c$ and $P_bYC$, we have

$ZB : ZP_c = YP_b : YC$.

This means that $v : w' = v' : w$ and $v'w' = vw$. Now, in the same parallelogram $AZP_aY$, we have

$$\text{Area}(P_aP_bP_c) = \frac{1 + v'w'}{2} \cdot \text{Area}(AZP_aY).$$

From this we conclude that $P_aP_bP_c$ and $ABC$ have equal areas. \hfill \Box

4. Triangles of residual circumcenters

Consider the circumcircles of the residuals of an inscribed triangle $XYZ$. By Miquel’s theorem, the circles $AYZ$, $BZX$, and $CXY$ have a common point. Furthermore, the centers $O_a$, $O_b$, $O_c$ of these circles form a triangle similar to $ABC$. See, for example, [1, p.134]. We prove the following interesting theorem.

**Theorem 10.** The triangles of residual circumcenters of the isotomic inscribed triangles are congruent.

We prove this theorem by calculations.

**Lemma 11.** Let $X, Y, Z$ be points on $BC, CA, AB$ such that

$$BX : XC = w : v, \quad CY : YA = u : w, \quad AZ : ZB = v : u_b.$$ 

The distance between the circumcenters $O_b$ and $O_c$ is the hypotenuse of a right triangle with one side $\frac{a}{2}$ and another side

$$\frac{(v - w)(u_b + v)(u_c + w)a^2 + (v + w)(w - u_c)(u_b + v)b^2 + (v + w)(w + u_c)(u_b - v)c^2}{8\Delta(u_b + v)(v + w)(w + u_c)}.$$ 

(9)
Proof. The distance between $O_b$ and $O_c$ along the side $BC$ is clearly $\frac{a}{2}$. We calculate their distance along the altitude on $BC$. The circumradius of $BZX$ is clearly $R_b = \frac{ZX}{2 \sin B}$. The distance of $O_b$ above $BC$ is

$$R_b \cos BZX = \frac{ZX \cos BZX}{2 \sin B} = \frac{2BZ \cdot BX \cos BZX}{4BZ \sin B} = \frac{BX^2 + ZX^2 - BX^2}{4BZ \sin B}$$

$$= \frac{BZ - BX \cos B}{2 \sin B} = \frac{c(BZ - BX \cos B)}{4\Delta} \cdot a$$

$$= \frac{c\left(\frac{u_b}{u_a + u_b} c - \frac{u_c}{u_c + u_a} a \cos B\right)}{4\Delta} \cdot a$$

$$= \frac{u_b(v + w)c^2 - w(u_b + v)(c^2 + a^2 - b^2)}{8\Delta(v + w)(u_a + v)} \cdot a$$

$$= \frac{(u_b + v)w(a^2 - b^2) + (2u_bw + u cw - vw)c^2}{8\Delta(v + w)(v + w)} \cdot a$$

By making the interchanges $b \leftrightarrow c$, $v \leftrightarrow w$, and $u_b \leftrightarrow u_c$, we obtain the distance of $O_c$ above the same line as

$$\frac{-(u_c + w)v(a^2 - c^2) + (2u_dw + u_cw - vw)b^2}{8\Delta(u_c + w)(v + w)} \cdot a.$$

The difference between these two is the expression given in (9) above. \hfill \Box

Consider now the isotomic inscribed triangle $X'Y'Z'$. We have

$$BX' : X'C = v : w,$$
$$CY' : Y'A = w : u_c = \frac{vw}{u_c} : v,$$
$$AZ' : Z'B = u_b : v = w : \frac{vw}{u_b}.$$

Let $O'_b$ and $O'_c$ be the circumcenters of $BZX'$ and $C'XY'$. By making the following interchanges

$$v \leftrightarrow w, \quad u_b \leftrightarrow \frac{vw}{u_b}, \quad u_c \leftrightarrow \frac{vw}{u_c}$$

in (9), we obtain the distance between $O'_b$ and $O'_c$ along the altitude on $BC$ as

$$\frac{(w - v)(\frac{vw}{u_b} + w)(\frac{vw}{u_c} + v)a^2 + (v + w)(v - \frac{vw}{u_b})(\frac{vw}{u_c} + w)b^2 + (v + w)(v + \frac{vw}{u_b})(\frac{vw}{u_c} - w)c^2}{8\Delta(v + w)(v + w)} \cdot a$$

$$= \frac{(w - v)(v + u_b)(w + u_c)a^2 + (v + w)(u_c - w)(v + u_b)b^2 + (v + w)(w + u_c)(v - u_c)c^2}{8\Delta(v + u_b)(v + w)(u_c + w)} \cdot a.$$
From this we easily conclude that the segments $O_bO_c$ and $O'_bO'_c$ are congruent. The same reasoning also yields the congruences of $O_bO_a$, $O'_bO'_a$, and of $O_aO_b$, $O'_aO'_b$. It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. This completes the proof of Theorem 9.

5. Isotomic conjugates

Let $XYZ$ be the cevian triangle of a point $P$, i.e., $X$, $Y$, $Z$ are respectively the intersections of the line pairs $AP$, $BC$; $BP$, $CA$; $CP$, $AB$. By the residual centroids (respectively orthocenters, circumcenters) of $P$, we mean those of its cevian triangle. If we construct points $X'$, $Y'$, $Z'$ satisfying (1), then the lines $AX'$, $BY'$, $CZ'$ intersect at a point $P'$ called the isotomic conjugate of $P$. If the point $P$ has homogeneous barycentric coordinates $(x : y : z)$, then $P'$ has homogeneous barycentric coordinates \( \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) \). All results in the preceding sections apply to the case when $XYZ$ and $X'Y'Z'$ are the cevian triangles of two isotomic conjugates. In particular, in the case of residual circumcenters in §4 above, if $XYZ$ is the cevian triangle of $P$ with homogeneous barycentric coordinates $(u : v : w)$, then

\[
BX : XC = w : v, \quad CY : YA = u : w, \quad AZ : ZB = v : u.
\]

By putting $u_b = u_c = u$ in (9) we obtain a necessary and sufficient condition for the line $O_bO_c$ to be parallel to $BC$, namely,

\[
(v-w)(u+v)(u+w)a^2+(v+w)(w-u)(u+v)b^2+(v+w)(w+u)(u-v)c^2 = 0.
\]

This can be reorganized into the form

\[
(b^2+c^2-a^2)w(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.
\]

This is the equation of the Lucas cubic, consisting of points $P$ for which the line joining $P$ to its isotomic conjugate $P'$ passes through the orthocenter $H$. The symmetry of this equation leads to the following interesting theorem.

**Theorem 12.** The triangle of residual circumcenters of $P$ is homothetic to $ABC$ if and only if $P$ lies on the Lucas cubic.

It is well known that the Lucas cubic is the locus of point $P$ whose cevian triangle is also the pedal triangle of a point $Q$. In this case, the circumcircles of $AYZ$, $BZX$ and $CXY$ intersect at $Q$, and the circumcenters $O_a$, $O_b$, $O_c$ are the midpoints of the segments $AQ$, $BQ$, $CQ$. The triangle $O_aO_bO_c$ is homothetic to $ABC$ at $Q$.

For example, if $P$ is the Gergonne point, then $O_aO_bO_c$ is homothetic to $ABC$ at the incenter $I$. The isotomic conjugate of $P$ is the Nagel point, and $O'_aO'_bO'_c$ is homothetic to $ABC$ at the reflection of $I$ in the circumcenter $O$.

References


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