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# The M-Configuration of a Triangle

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**Abstract**. We give an easy construction of points $A_a$, $B_a$, $C_a$ on the sides of a triangle $ABC$ such that the figure M path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. We study the configuration consisting of the three figures M of a triangle, and define an interesting mapping of triangle centers associated with such an M-configuration.

## 1. Introduction

Given a triangle $ABC$, we consider points $A_a$ on the line $BC$, $B_a$ on the half line $CA$, and $C_a$ on the half line $BA$ such that $BC_a = C_aA_a = A_aB_a = B_aC$. We shall refer to $BC_aA_aB_aC$ as M$_a$, because it looks like the letter M when triangle $ABC$ is acute-angled. See Figures 1a. Figure 1b illustrates the case when the triangle is obtuse-angled. Similarly, we also have M$_b$ and M$_c$. The three figures M$_a$, M$_b$, M$_c$ constitute the M-configuration of triangle $ABC$. See Figure 2.

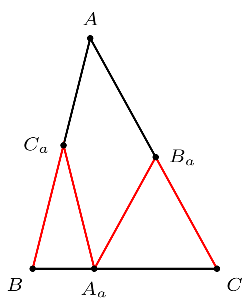

Figure 1a

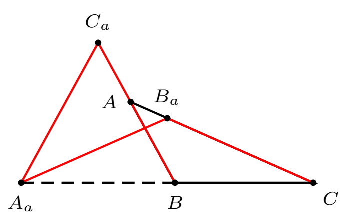

Figure 1b

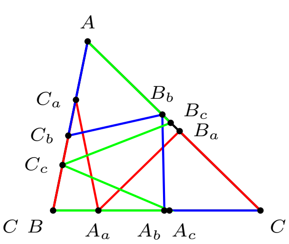

Figure 2

**Proposition 1.** *The lines $AA_a$, $BB_a$, $CC_a$ concur at the point with homogeneous barycentric coordinates*

$$\left(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}\right).$$

*Proof.* Let $l_a$ be the length of $BC_a = C_aA_a = A_aB_a = B_aC$. It is clear that the directed length $BA_a = 2l_a \cos B$ and $A_aC = 2l_a \cos C$, and $BA_a : A_aC = \cos B : \cos C$. For the same reason, $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. It follows by Ceva's theorem that the lines $AA_a$, $BB_a$, $CC_a$ concur at the point with homogeneous barycentric coordinates given above.[1] □

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[1]This point appears in [3] as $X_{92}$.

*Remark.* Since $2l_a \cos B + 2l_a \cos C = a = 2R \sin A$, where $R$ is the circumradius of triangle $ABC$,

$$l_a = \frac{a}{2(\cos B + \cos C)} = \frac{R \sin A}{\cos B + \cos C} = \frac{R \cos \frac{A}{2}}{\cos \frac{B-C}{2}}. \tag{1}$$

For later use, we record the absolute barycentric coordinates of $A_a$, $B_a$, $C_a$ in terms of $l_a$:

$$\begin{aligned} A_a =& \frac{2l_a}{a}(\cos C \cdot B + \cos B \cdot C), \\ B_a =& \frac{1}{b}(l_a \cdot A + (b - l_a)C), \\ C_a =& \frac{1}{c}(l_a \cdot A + (c - l_a)B). \end{aligned} \tag{2}$$

## 2. Construction of $\mathbf{M}_a$

**Proposition 2.** *Let $A'$ be the intersection of the bisector of angle $A$ with the circumcircle of triangle $ABC$.*

(a) *$A_a$ is the intersection of $BC$ with the parallel to $AA'$ through the orthocenter $H$.*

(b) *$B_a$ (respectively $C_a$) is the intersection of $CA$ (respectively $BA$) with the parallel to $CA'$ (respectively $BA'$) through the circumcenter $O$.*

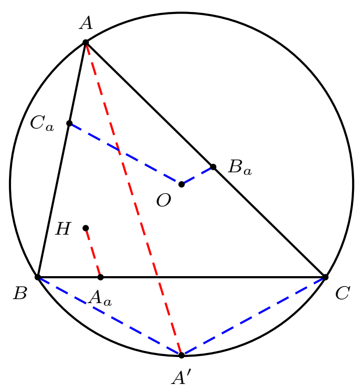

Figure 3a

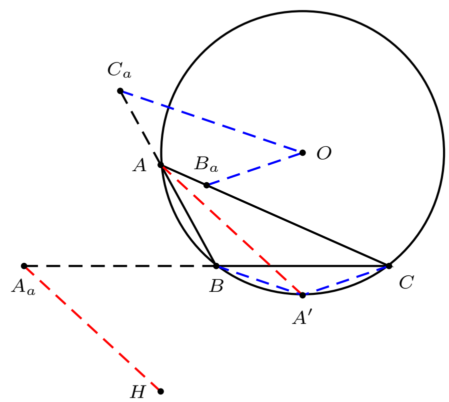

Figure 3b

*Proof.* (a) The line joining $A_a = (0 : \cos C : \cos B)$ to $H = \left(\frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C}\right)$ has equation

$$\begin{vmatrix} 0 & \cos C & \cos B \\ \frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \\ x & y & z \end{vmatrix} = 0.$$

This simplifies to

$$-(b-c)x\cos A + a(y\cos B - z\cos C) = 0.$$

It has infinite point

$$\begin{aligned}&(-a(\cos B+\cos C) : a\cos C-(b-c)\cos A : (b-c)\cos A + a\cos B)\\ =&(-a(\cos B+\cos C) : b(1-\cos A) : c(1-\cos A)).\end{aligned}$$

It is clear that this is the same as the infinite point $(-(b+c) : b : c)$, which is on the line joining $A$ to the incenter.

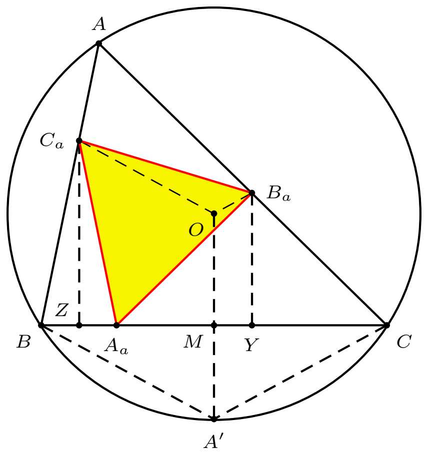


Figure 4

(b) Let $M$ be the midpoint of $BC$, and $Y$, $Z$ the pedals of $B_a$, $C_a$ on $BC$. See Figure 4. We have

$$\begin{aligned}OM =&\frac{a}{2}\cot A = l_a(\cos B+\cos C)\cot A,\\ C_aZ =&l_a\sin B,\\ MZ =&\frac{a}{2} - l_a\cos B = l_a(\cos B+\cos C) - l_a\cos B = l_a\cos C.\end{aligned}$$

From this the acute angle between the line $C_aO$ and $BC$ has tangent ratio

$$\begin{aligned}\frac{C_aZ-OM}{MZ} =&\frac{\sin B-(\cos B+\cos C)\cot A}{\cos C}\\ =&\frac{\sin B\sin A-(\cos B+\cos C)\cos A}{\cos C\sin A}\\ =&\frac{-\cos(A+B)-\cos C\cos A}{\cos C\sin A} = \frac{\cos C(1-\cos A)}{\cos C\sin A}\\ =&\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}.\end{aligned}$$

It follows that $C_aO$ makes an angle $\frac{A}{2}$ with the line $BC$, and is parallel to $BA'$. The same reasoning shows that $B_aO$ is parallel to $CA'$. □

## 3. Circumcenters in the M-configuration

Note that $\angle B_aA_aC_a = \angle A$. It is clear that the circumcircles of $B_aA_aC_a$ and $B_aAC_a$ are congruent. The circumradius is

$$R_a = \frac{l_a}{2\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)} = \frac{l_a}{2\cos\frac{A}{2}} = \frac{R}{2\cos\frac{B-C}{2}} \tag{3}$$

from (1).

**Proposition 3.** *The circumcircle of triangle $AB_aC_a$ contains* (i) *the circumcenter $O$ of triangle $ABC$,* (ii) *the orthocenter $H_a$ of triangle $A_aB_aC_a$, and* (iii) *the midpoint of the arc $BAC$.*

*Proof.* (i) is an immediate corollary of Proposition 2(b) above.

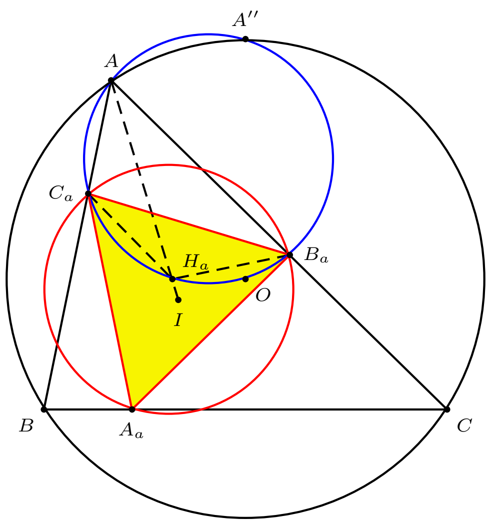


Figure 5

(ii) Let $H_a$ be the orthocenter of triangle $A_aB_aC_a$. It is clear that

$$\angle B_aH_aC_a = \pi - \angle B_aA_aC_a = \pi - \angle BAC = \pi - \angle C_aAB_a.$$

It follows that $H_a$ lies on the circumcircle of $AB_aC_a$. See Figure 5. Since the triangle $A_aB_aC_a$ is isosceles, $B_aH_a = C_aH_a$, and the point $H_a$ lies on the bisector of angle $A$.

(iii) Let $A''$ be the midpoint of the arc $BAC$. By a simple calculation, $\angle AA''O = \frac{\pi}{2} - \frac{1}{2}|B-C|$. Also, $\angle AC_aO = \frac{\pi}{2} + \frac{1}{2}|B-C|$.[2] This shows that $A''$ also lies on the circle $AB_aOC_a$. □

The points $B_a$ and $C_a$ are therefore the intersections of the circle $OAA''$ with the sidelines $AC$ and $AB$. This furnishes another simple construction of the figure M$_a$.

[2]This is $C + \frac{A}{2}$ if $C \geq B$ and $B + \frac{A}{2}$ otherwise.

*Remarks.* (1) If we take into consideration also the other figures $M_b$ and $M_c$, we have three triangles $AB_aC_a$, $BC_bA_b$, $CA_cB_c$ with their circumcircles intersecting at $O$.

(2) We also have three triangles $A_aB_aCa$, $A_bB_bC_b$, $A_cB_cC_c$ with their orthocenters forming a triangle perspective with $ABC$ at the incenter $I$.

**Proposition 4.** *The circumcenter $O_a$ of triangle $A_aB_aC_a$ is equidistant from $O$ and $H$.*

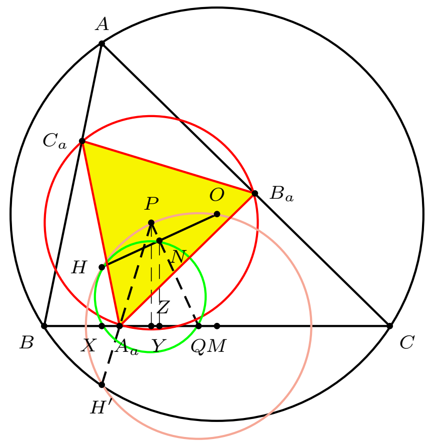


Figure 6

*Proof.* Construct the circle through $O$ and $H$ with center $Q$ on the line $BC$. We prove that the midpoint $P$ of the arc $OH$ on the opposite side of $Q$ is the circumcenter $O_a$ of triangle $A_aB_aC_a$. See Figure 6. It will follow that $O_a$ is equidistant from $O$ and $H$. Let $N$ be the midpoint of $OH$. Suppose the line $PQ$ makes an angle $\varphi$ with $BC$. Let $X$, $Y$, and $M$ be the pedals of $H$, $N$, $O$ on the line $BC$.

Since $H$, $X$, $Q$, $N$ are concyclic, and the diameter of the circle containing them is $QH = \frac{NX}{\sin\varphi} = \frac{R}{2\sin\varphi}$. This is the radius of the circle $OPH$.

By symmetry, the circle $OPH$ contains the reflection $H'$ of $H$ in the line $BC$.

$$\angle HH'P = \frac{1}{2}\angle HQP = \frac{1}{2}\angle HQN = \frac{1}{2}\angle HXN = \frac{1}{2}|B-C|.$$

Therefore, the angle between $H'P$ and $BC$ is $\frac{\pi}{2} - \frac{1}{2}|B-C|$. It is obvious that the angle between $A_aO_a$ and $BC$ is the same. But from Proposition 2(a), the angle between $HA_a$ and $BC$ is the same too, so is the angle between the reflection $H'A_a$ and $BC$. From these we conclude that $H'$, $A_a$, $O_a$ and $P$ are collinear. Now, let $Z$ be the pedal of $P$ on $BC$.

$$A_aP = \frac{PZ}{\cos\frac{1}{2}(B-C)} = \frac{QP\sin\varphi}{\cos\frac{1}{2}(B-C)} = \frac{R}{2\cos\frac{1}{2}(B-C)} = R_a.$$

Therefore, $P$ is the circumcenter $O_a$ of triangle $A_aB_aC_a$. □

Applying this to the other two figures $M_b$ and $M_c$, we obtain the following remarkable theorem about the M-configuration of triangle $ABC$.

**Theorem 5.** *The circumcenters of triangles $A_aB_aC_a$, $A_bB_bC_b$, and $A_cB_cC_c$ are collinear. The line containing them is the perpendicular bisector of the segment $OH$.*

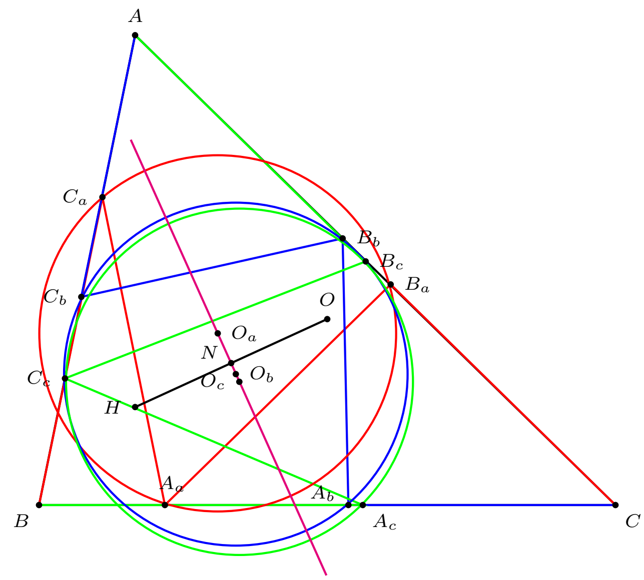


Figure 7

One can check without much effort that in homogeneous barycentric coordinates, the equation of this line is

$$\frac{\sin 3A}{\sin A}x + \frac{\sin 3B}{\sin B}y + \frac{\sin 3C}{\sin C}z = 0.$$

## 4. A central mapping

Let $P$ be a triangle center in the sense of Kimberling [2, 3], given in homogeneous barycentric coordinates $(f(a,b,c) : f(b,c,a) : f(c,a,b))$ where $f = f_P$ satisfies $f(a,b,c) = f(a,c,b)$. If the reference triangle $ABC$ is isosceles, say, with $AB = AC$, then $P$ lies on the perpendicular bisector of $BC$ and has coordinates of the form $(g_P : 1 : 1)$. The coordinate $g$ depends only on the shape of the isosceles triangle, and we express it as a function of the *base* angle. We shall call $g = g_P$ the *isoscelized form* of the triangle center function $f_P$. Let $P^*$ denote the isogonal conjugate of $P$.

**Lemma 6.** $g_{P^*}(B) = \frac{4\cos^2 B}{g_P(B)}$.

*Proof.* If $P = (g_P(B) : 1 : 1)$ for an isosceles triangle $ABC$ with $B = C$, then

$$P^* = \left(\frac{\sin^2 A}{g_P(B)} : \sin^2 B : \sin^2 B\right) = \left(\frac{4\cos^2 B}{g_P(B)} : 1 : 1\right)$$

since $\sin^2 A = \sin^2(\pi - 2B) = \sin^2 2B = 4\sin^2 B\cos^2 B$. □

Here are some examples.

| Center | $f_P$ | $g_P$ |
|---|---|---|
| centroid | $1$ | $1$ |
| incenter | $a$ | $2\cos B$ |
| circumcenter | $a^2(b^2+c^2-a^2)$ | $-2\cos 2B$ |
| orthocenter | $\frac{1}{b^2+c^2-a^2}$ | $\frac{-2\cos^2 B}{\cos 2B}$ |
| symmedian point | $a^2$ | $4\cos^2 B$ |
| Gergonne point | $\frac{1}{s-a}$ | $\frac{\cos B}{1-\cos B}$ |
| Nagel point | $s-a$ | $\frac{1-\cos B}{\cos B}$ |
| Mittenpunkt | $a(s-a)$ | $2(1-\cos B)$ |
| Spieker point | $b+c$ | $\frac{2}{1+2\cos B}$ |
| $X_{55}$ | $a^2(s-a)$ | $4\cos B(1-\cos B)$ |
| $X_{56}$ | $\frac{a^2}{s-a}$ | $\frac{4\cos^3 B}{1-\cos B}$ |
| $X_{57}$ | $\frac{a}{s-a}$ | $\frac{2\cos^2 B}{1-\cos B}$ |

Consider a triangle center given by a triangle center function with isoscelized form $g = g_P$. The triangle center of the isosceles triangle $C_aBA_a$ is the point $P_{a,b}$ with coordinates $(g(B) : 1 : 1)$ relative to $C_aBA_a$. Making use of the absolute barycentric coordinates of $A_a$, $B_a$, $C_a$ given in (2), it is easy to see that this is the point

$$P_{a,b} = \left(\frac{g(B)l_a}{c} : \frac{g(B)(c-l_a)}{c} + 1 + \frac{2l_a}{a}\cos C : \frac{2l_a}{a}\cos B\right).$$

The same triangle center of the isosceles triangle $B_aA_aC$ is the point

$$P_{a,c} = \left(\frac{g(C)l_a}{b} : \frac{2l_a}{a}\cos C : \frac{g(C)(b-l_a)}{b} + \frac{2l_a}{a}\cos B + 1\right).$$

It is clear that the lines $BP_{a,b}$ and $CP_{a,c}$ intersect at the point

$$\begin{aligned}P_a &= \left(\frac{g(B)g(C)l_a^2}{bc} : \frac{2g(B)l_a^2\cos C}{ca} : \frac{2g(C)l_a^2\cos B}{ab}\right)\\ &= (ag(B)g(C) : 2bg(B)\cos C : 2cg(C)\cos B)\\ &= \left(\frac{ag(B)g(C)}{2\cos B\cos C} : \frac{bg(B)}{\cos B} : \frac{cg(C)}{\cos C}\right).\end{aligned}$$

Figure 8 illustrates the case of the Gergonne point.

In the M-configuration, we may also consider the same triangle center (given in isoscelized form $g_P$ of the triangle center function) in the isosceles triangles . These are the point $P_{b,c}$, $P_{b,a}$, $P_{c,a}$, $P_{c,b}$. The pairs of lines $CP_{b,c}$, $AP_{b,a}$ intersecting at $P_b$ and $AP_{c,a}$, $BP_{c,b}$ intersecting at $P_c$. The coordinates of $P_b$ and $P_c$ can be

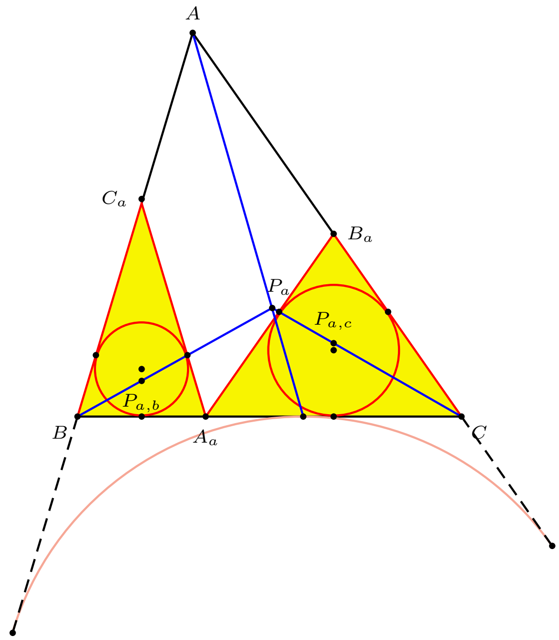


Figure 8

written down easily from those of $P_a$. From these coordinates, we easily conclude that that $P_aP_bP_c$ is perspective with triangle $ABC$ at the point

$$\Phi(P) = \left(\frac{ag_P(A)}{\cos A} : \frac{bg_P(B)}{\cos B} : \frac{cg_P(C)}{\cos C}\right)$$
$$= (g_P(A)\tan A : g_P(B)\tan B : g_P(C)\tan C)\,.$$

**Proposition 7.** $\Phi(P^*) = \Phi(P)^*$.

*Proof.* We make use of Lemma 6.

$$\begin{aligned}\Phi(P^*) =&(g_{P^*}(A)\tan A : g_{P^*}(B)\tan B : g_{P^*}(C)\tan C)\\ =&\left(\frac{4\cos^2 A}{g_P(A)}\tan A : \frac{4\cos^2 B}{g_P(B)}\tan B : \frac{4\cos^2 C}{g_P(C)}\tan C\right)\\ =&\left(\frac{\sin^2 A}{g_P(A)\tan A} : \frac{\sin^2 B}{g_P(B)\tan B} : \frac{\sin^2 C}{g_P(C)\tan C}\right)\\ =&\Phi(P)^*.\end{aligned}$$

□

We conclude with some examples.

| $P$ | $\Phi(P)$ | $P^*$ | $\Phi(P^*) = \Phi(P)^*$ |
|---|---|---|---|
| incenter | incenter | | |
| centroid | orthocenter | symmedian point | circumcenter |
| circumcenter | $X_{24}$ | orthocenter | $X_{68}$ |
| Gergonne point | Nagel point | $X_{55}$ | $X_{56}$ |
| Nagel point | $X_{1118}$ | $X_{56}$ | $X_{1259} = X^*_{1118}$ |
| Mittenpunkt | $X_{34}$ | $X_{57}$ | $X_{78} = X^*_{34}$ |

For the Spieker point, we have

$$\begin{aligned}\Phi(X_{10}) &= \left(\frac{\tan A}{1+2\cos A} : \frac{\tan B}{1+2\cos B} : \frac{\tan C}{1+2\cos C}\right) \\ &= \left(\frac{1}{a(b^2+c^2-a^2)(b^2+c^2-a^2+bc)} : \cdots : \cdots\right).\end{aligned}$$

This triangle center does not appear in the current edition of [3].

*Remark.* For $P = X_8$, the Nagel point, the point $P_a$ has an alternative description. Antreas P. Hatzipolakis [1] considered the incircle of triangle $ABC$ touching the sides $CA$ and $AB$ at $Y$ and $Z$ respectively, and constructed perpendiculars from $Y$, $Z$ to $BC$ intersecting the incircle again at $Y'$ and $Z'$. See Figure 9. It happens that $B$, $Z'$, $P_{a,b}$ are collinear; so are $C$, $Y'$, $P_{a,c}$. Therefore, $BZ'$ and $CY'$ intersect at $P_a$. The coordinates of $Y'$ and $Z'$ are

$$\begin{aligned}Y' &= (a^2(b+c-a)(c+a-b) : (a^2+b^2-c^2)^2 : (b+c)^2(a+b-c)(c+a-b)), \\ Z' &= (a^2(b+c-a)(a+b-c) : (b+c)^2(c+a-b)(a+b-c) : (a^2-b^2+c^2)^2).\end{aligned}$$

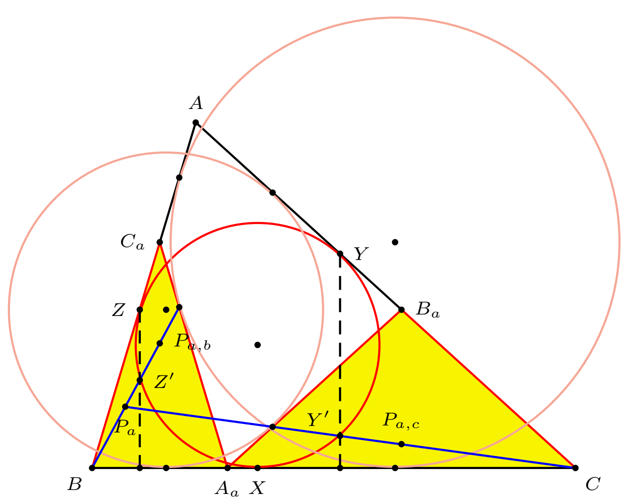


Figure 9

The lines $BZ'$ and $CY'$ intersect at

$$P_a = \left(a^2(b+c-a) : \frac{(a^2+b^2-c^2)^2}{c+a-b} : \frac{(a^2-b^2+c^2)^2}{a+b-c}\right)$$
$$= \left(\frac{a^2(b+c-a)}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} : \frac{1}{(c+a-b)(a^2-b^2+c^2)^2} : \frac{1}{(a+b-c)(a^2+b^2-c^2)^2}\right).$$

It was in this context that Hatzipolakis constructed the triangle center

$$X_{1118} = \left(\frac{1}{(b+c-a)(b^2+c^2-a^2)^2} : \cdots : \cdots\right).$$

## References

[1] A. P. Hatzipolakis, Hyacinthos message 5321, April 30, 2002.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, May 23, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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