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Rectangles Attached to Sides of a Triangle

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Abstract. We study the figure of a triangle with a rectangle attached to each side. In line with recent publications on special cases we find concurrencies and study homothetic triangles. Special attention is given to the cases in which the attached rectangles are similar, have equal areas and have equal perimeters, respectively.

1. Introduction

In recent publications [3, 4, 10, 11, 12] the configurations have been studied in which rectangles or squares are attached to the sides of a triangle. In these publications the rectangles are all similar. In this paper we study the more general case in which the attached rectangles are not necessarily similar. We consider a triangle ABC with attached rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a. Let u be the length of CA_c, positive if A_c and A are on opposite sides of BC, otherwise negative. Similarly let v and w be the lengths of AB_a and BC_b. We describe the shapes of these rectangles by the ratios

$$U = \frac{a}{u}, \qquad V = \frac{b}{v}, \qquad W = \frac{c}{w}. \tag{1}$$

The vertices of these rectangles are [1]

$$\begin{array}{ll} A_b = (-a^2 : S_C + SU : S_B), & A_c = (-a^2 : S_C : S_B + SU), \\ B_a = (S_C + SV : -b^2 : S_A), & B_c = (S_C : -b^2 : S_A + SV), \\ C_a = (S_B + SW : S_A : -c^2), & C_b = (S_B : S_A + SW : -c^2). \end{array}$$

Consider the flank triangles AB_aC_a, A_bBC_b and A_cB_cC. With the same reasoning as in [10], or by a simple application of Ceva's theorem, we can see that the triangle $H_aH_bH_c$ of orthocenters of the flank triangles is perspective to ABC with perspector

$$P_1 = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right) = (U : V : W). \tag{2}$$

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[1] All coordinates in this note are homogeneous barycentric coordinates. We adopt J. H. Conway's notation by letting $S = 2\Delta$ denote twice the area of ABC, while $S_A = \frac{-a^2+b^2+c^2}{2} = S\cot A$, $S_B = S\cot B$, $S_C = S\cot C$, and generally $S_{XY} = S_XS_Y$.

See Figure 1. On the other hand, the triangle $O_aO_bO_c$ of circumcenters of the flank triangles is clearly homothetic to ABC, the homothetic center being the point

$$P_2 = (au : bv : cw) = \left(\frac{a^2}{U} : \frac{b^2}{V} : \frac{c^2}{W}\right). \tag{3}$$

Clearly, P_1 and P_2 are isogonal conjugates.

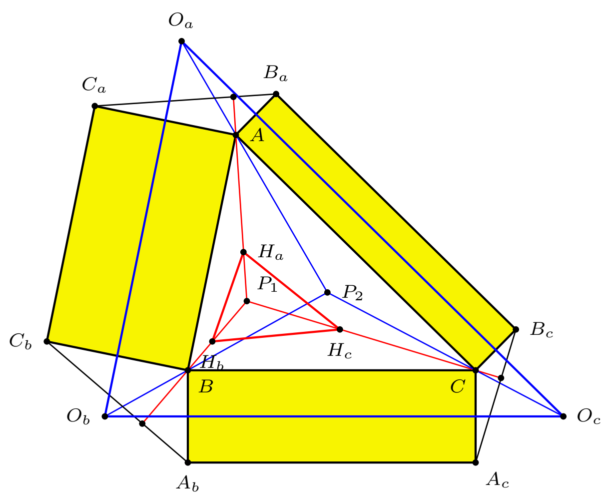

Figure 1

Now the perpendicular bisectors of B_aC_a, A_bC_b and A_cB_c pass through O_a, O_b and O_c respectively and are parallel to AP_1, BP_1 and CP_1 respectively. This shows that these perpendicular bisectors concur in a point P_3 on P_1P_2 satisfying

$$P_2P_1 : P_1P_3 = 2S : au + bv + cw,$$

where S is twice the area of ABC. See Figure 2. More explicitly,

$$\begin{aligned} P_3 =&(-a^2VW(V+W) + U^2(b^2W + c^2V) + 2SU^2VW \\ &: -b^2WU(W+U) + V^2(c^2U + a^2W) + 2SUV^2W) \\ &: -c^2UV(U+V) + W^2(a^2V + b^2U) + 2SUVW^2) \end{aligned} \tag{4}$$

This concurrency generalizes a similar result by Hoehn in [4], and was mentioned by L. Lagrangia [9]. It was also a question in the Bundeswettbewerb Mathematik Deutschland (German National Mathematics Competition) 1996, Second Round.

From the perspectivity of ABC and the orthocenters of the flank triangles, we see that ABC and the triangle $A'B'C'$ enclosed by the lines B_aC_a, A_bC_b and A_cB_c are orthologic. This means that the lines from the vertices of $A'B'C'$ to the corresponding sides of ABC are concurrent as well. The point of concurrency is the reflection of P_1 in O, *i.e.*,

$$P_4 = (-S_{BC}U + a^2 S_A(V + W) : \cdots : \cdots). \qquad (5)$$

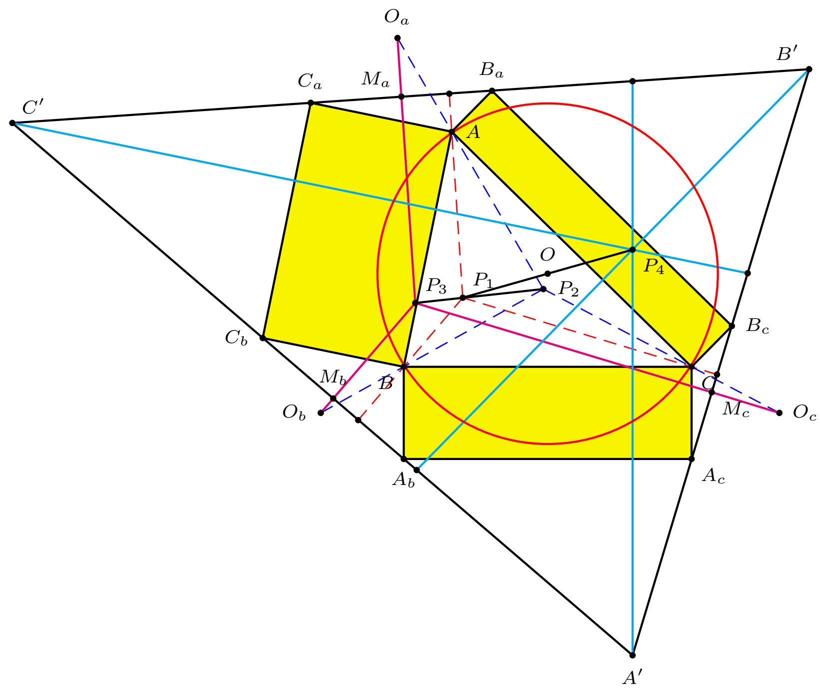

Figure 2

Remark. We record the coordinates of A'. Those of B' and C' can be written down accordingly.

$$\begin{aligned} A' =&(-(a^2S(U+V+W) + (a^2V + S_CU)(a^2W + S_BU)) \\ &:S_CS(U+V+W) + (b^2U + S_CV)(a^2W + S_BU) \\ &:S_BS(U+V+W) + (a^2V + S_CU)(c^2U + S_BW)). \end{aligned}$$

2. Special cases

We are mainly interested in three special cases.

2.1. *The similarity case.* This is the case when the rectangles are similar, *i.e.*, $U = V = W = t$ for some t. In this case, $P_1 = G$, the centroid, and $P_2 = K$, the symmedian point. As t varies,

$$P_3 = (b^2 + c^2 - 2a^2 + 2St : c^2 + a^2 - 2b^2 + 2St : a^2 + b^2 - 2c^2 + 2St)$$

traverses the line GK. The point P_4, being the reflection of G in O, is X_{376} in [7]. The triangle $M_aM_bM_c$ is clearly perspective with ABC at the orthocenter H. More interestingly, it is also perspective with the medial triangle at

$$((S_A + St)(a^2 + 2St) : (S_B + St)(b^2 + 2St) : (S_C + St)(c^2 + 2St)),$$

which is the complement of the Kiepert perspector

$$\left(\frac{1}{S_A+St}:\frac{1}{S_B+S_t}:\frac{1}{S_C+St}\right).$$

It follows that as t varies, this perspector traverses the Kiepert hyperbola of the medial triangle. See [8].

The case $t=1$ is the *Pythagorean* case, when the rectangles are squares erected externally. The perspector of $M_aM_bM_c$ and the medial triangle is the point

$$O_1=(2a^4-3a^2(b^2+c^2)+(b^2-c^2)^2-2(b^2+c^2)S:\cdots:\cdots),$$

which is the center of the circle through the centers of the squares. See Figure 3. This point appears as X_{641} in [7].

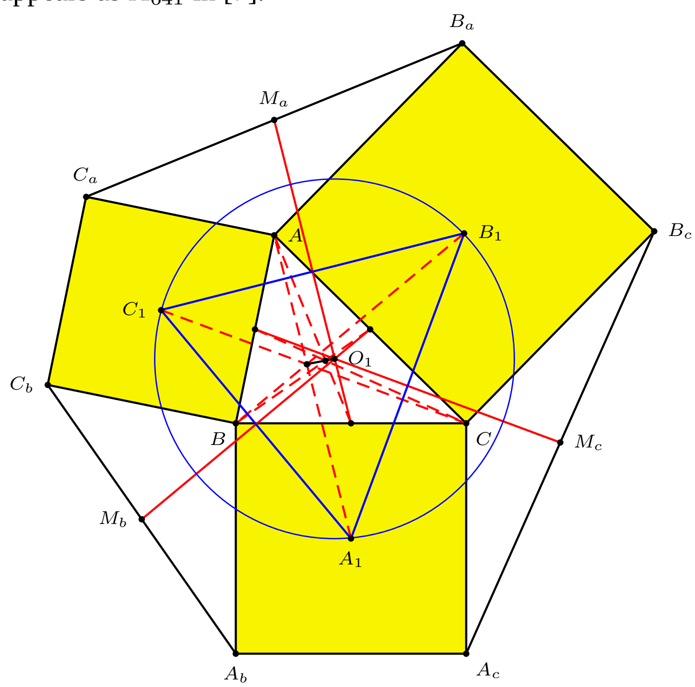

Figure 3

2.2. *The equiareal case.* When the rectangles have equal areas $\frac{T}{2}$, *i.e.*, $(U,V,W)=\left(\frac{2a^2}{T},\frac{2b^2}{T},\frac{2c^2}{T}\right)$, it is easy to see that $P_1=K$, $P_2=G$, and

$$\begin{aligned}P_4=&(a^2(-S_{BC}+S_A(b^2+c^2)):\cdots:\cdots)\\=&(a^2(a^4+2a^2(b^2+c^2)-(3b^4+2b^2c^2+3c^4)):\cdots:\cdots)\end{aligned}$$

is the reflection of K in O.[2] The *special equiareal case* is when $T=S$, the rectangles having the same area as triangle ABC. See Figure 4. In this case,

$$P_3=(6a^2-b^2-c^2:6b^2-c^2-a^2:6c^2-a^2-b^2).$$

[2]This point is not in the current edition of [7].

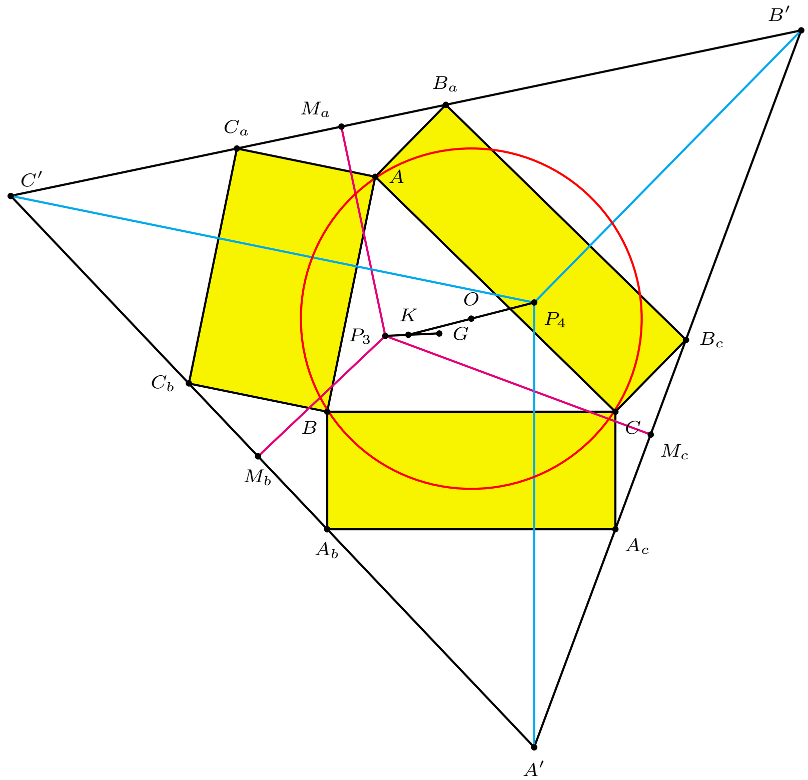

Figure 4

2.3. *The isoperimetric case.* This is the case when the rectangles have equal perimeters $2p$, *i.e.*, $(u, v, w) = (p - a, p - b, p - c)$. The *special isoperimetric* case is when $p = s$, the semiperimeter, the rectangles having the same perimeter as triangle ABC. In this case, $P_1 = X_{57}$, $P_2 = X_9$, the Mittenpunkt, and

$$\begin{aligned}
P_3 =&(a(bc(2a^2 - a(b + c) - (b - c)^2) + 4(s - b)(s - c)S) : \cdots : \cdots),\\
P_4 =&(a(a^6 - 2a^5(b + c) - a^4(b^2 - 10bc + c^2) + 4a^3(b + c)(b^2 - bc + c^2)\\
&- a^2(b^4 + 8b^3c - 2b^2c^2 + 8c^3b + c^4) - 2a(b + c)(b - c)^2(b^2 + c^2)\\
&+ (b + c)^2(b - c)^4) : \cdots : \cdots).
\end{aligned}$$

These points can be described in terms of division ratios as follows.[3]

$$\begin{aligned}
P_3X_{57} : X_{57}X_9 =& 4R + r : 2s,\\
P_4I : IX_{57} =& 4R : r.
\end{aligned}$$

3. A pair of homothetic triangles

Let A_1, B_1 and C_1 be the centers of the rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively, and $A_2B_2C_2$ the triangle bounded by the lines B_cC_b, C_aA_c and A_bB_a. Since, for instance, segments B_1C_1 and B_cC_b are homothetic through

[3]These points are not in the current edition of [7].

A, the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic. See Figure 5. Their homothetic center is the point

$$P_5 = \left(-a^2S_A(V+W) + U(S_B+SW)(S_C+SV) : \cdots : \cdots\right).$$

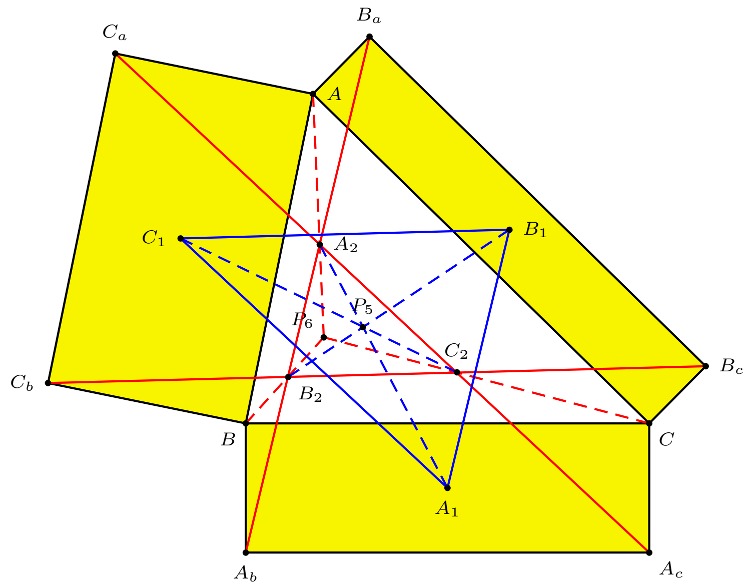

Figure 5

For the Pythagorean case with squares attached to triangles, *i.e.*, $U = V = W = 1$, Toshio Seimiya and Peter Woo [12] have proved the beautiful result that the areas Δ_1 and Δ_2 of $A_1B_1C_1$ and $A_2B_2C_2$ have geometric mean Δ. See Figure 5. We prove a more general result by computation using two fundamental area formulae.

Proposition 1. *For $i = 1, 2, 3$, let P_i be finite points with homogeneous barycentric coordinates $(x_i : y_i : z_i)$ with respect to triangle ABC. The oriented area of the triangle $P_1P_2P_3$ is*

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)} \cdot \Delta.$$

A proof of this proposition can be found in [1, 2].

Proposition 2. *For $i = 1, 2, 3$, let ℓ_i be a finite line with equation $p_ix + q_iy + r_iz = 0$. The oriented area of the triangle bounded by the three lines ℓ_1, ℓ_2, ℓ_3 is*

$$\frac{\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}^2}{D_1 \cdot D_2 \cdot D_3} \cdot \Delta,$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} p_1 & q_1 & r_1 \\ 1 & 1 & 1 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix}.$$

A proof of this proposition can be found in [5].

Theorem 3. $\frac{\Delta_1\Delta_2}{\Delta^2} = \frac{(U+V+W-UVW)^2}{4(UVW)^2}$.

Proof. The coordinates of A_1, B_1, C_1 are

$$\begin{aligned} A_1 =& (-a^2 : S_C + SU : S_B + SU), \\ B_1 =& (S_C + SV : -b^2 : S_A + SV), \\ C_1 =& (S_B + SW : S_A + SW : -c^2). \end{aligned}$$

By Proposition 1, the area of triangle $A_1B_1C_1$ is

$$\Delta_1 = \frac{S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV)}{4SUVW} \cdot \Delta. \qquad (6)$$

The lines B_cC_b, C_aA_c, A_bB_a have equations

$$\begin{aligned} (S(1-VW) - S_A(V+W))x + (S+S_BV)y + (S+S_CW)z =& 0, \\ (S+S_AU)x + (S(1-WU) - S_B(W+U))y + (S+S_CW)z =& 0, \\ (S+S_AU)x + (S+S_BV)y + (S(1-UV) - S_C(U+V))z =& 0. \end{aligned}$$

By Proposition 2, the area of the triangle bounded by these lines is

$$\Delta_2 = \frac{S(U+V+W-UVW)^2}{UVW(S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV))} \cdot \Delta. \quad (7)$$

From (6, 7), the result follows. □

Remarks. (1) The ratio of homothety is

$$\frac{-S(U+V+W-UVW)}{2(S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV))}.$$

(2) We record the coordinates of A_2 below. Those of B_2 and C_2 can be written down accordingly.

$$\begin{aligned} A_2 =& (-a^2((S+S_AU)(V+W) + SU(1-VW)) + (S_B+SW)(S_C+SV)U^2 \\ & : (S+S_AU)(SUV + S_C(U+V+W)) \\ & : (S+S_AU)(SUW + S_B(U+V+W))). \end{aligned}$$

From the coordinates of $A_2B_2C_2$ we see that this triangle is perspective to ABC at the point

$$P_6 = \left(\frac{1}{S_A(U+V+W) + SVW} : \cdots : \cdots\right).$$

4. Examples

4.1. *The similarity case.* If the rectangles are similar, $U = V = W = t$, then

$$P_6 = \left(\frac{1}{3S_A + St} : \frac{1}{3S_B + St} : \frac{1}{3S_C + St}\right)$$

traverses the Kiepert hyperbola. In the Pythagorean case, the homothetic center P_5 is the point

$$((S_B - S)(S_C - S) - 4S_{BC} : (S_C - S)(S_A - S) - 4S_{CA} : (S_A - S)(S_B - S) - 4S_{AB}).$$

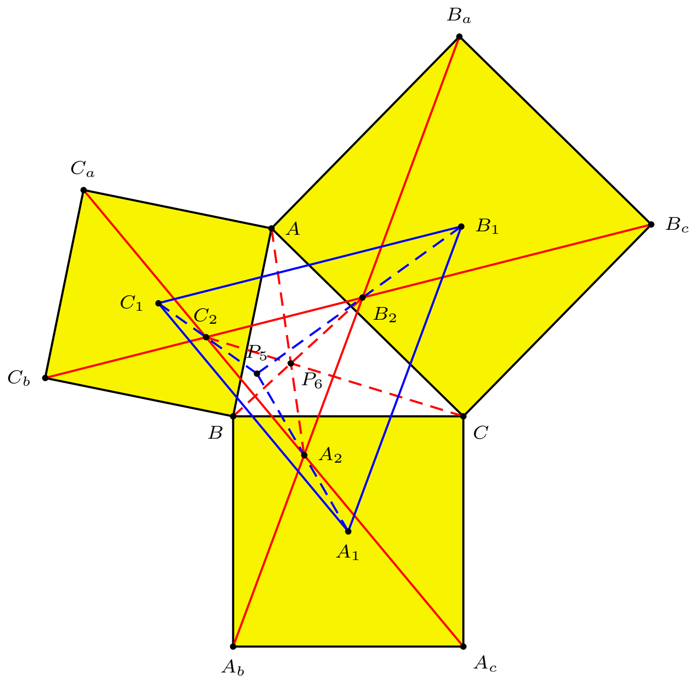

Figure 6

4.2. *The equiareal case.* For $(U, V, W) = (\frac{2a^2}{T}, \frac{2b^2}{T}, \frac{2c^2}{T})$, we have

$$P_6 = \left(\frac{1}{T(a^2 + b^2 + c^2)S_A + 2Sb^2c^2} : \cdots : \cdots\right).$$

This traverses the Jerabek hyperbola as T varies. When the rectangles have the same area as the triangle, the homothetic center P_5 is the point

$$(a^2((a^2 + 3b^2 + 3c^2)^2 - 4(4b^4 - b^2c^2 + 4c^4)) : \cdots : \cdots).$$

5. More homothetic triangles

Let $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$ be the circumcricles of rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively. See Figure 7. Since the circle $\mathcal{C}_A$ passes through B and C, its equation is of the form

$$a^2yz + b^2zx + c^2xy - px(x + y + z) = 0.$$

Since the same circle passes through A_b, we have $p = \frac{S_A U + S}{U} = S_A + \frac{S}{U}$. By the same method we derive the equations of the three circles:

$$a^2yz + b^2zx + c^2xy = (S_A + \frac{S}{U})x(x+y+z),$$
$$a^2yz + b^2zx + c^2xy = (S_B + \frac{S}{V})y(x+y+z),$$
$$a^2yz + b^2zx + c^2xy = (S_C + \frac{S}{W})z(x+y+z).$$

From these, the radical center of the three circles is the point

$$J = \left(\frac{1}{S_A + \frac{S}{U}} : \frac{1}{S_B + \frac{S}{V}} : \frac{1}{S_C + \frac{S}{W}}\right) = \left(\frac{U}{S_AU + S} : \frac{V}{S_BV + S} : \frac{W}{S_CW + S}\right).$$

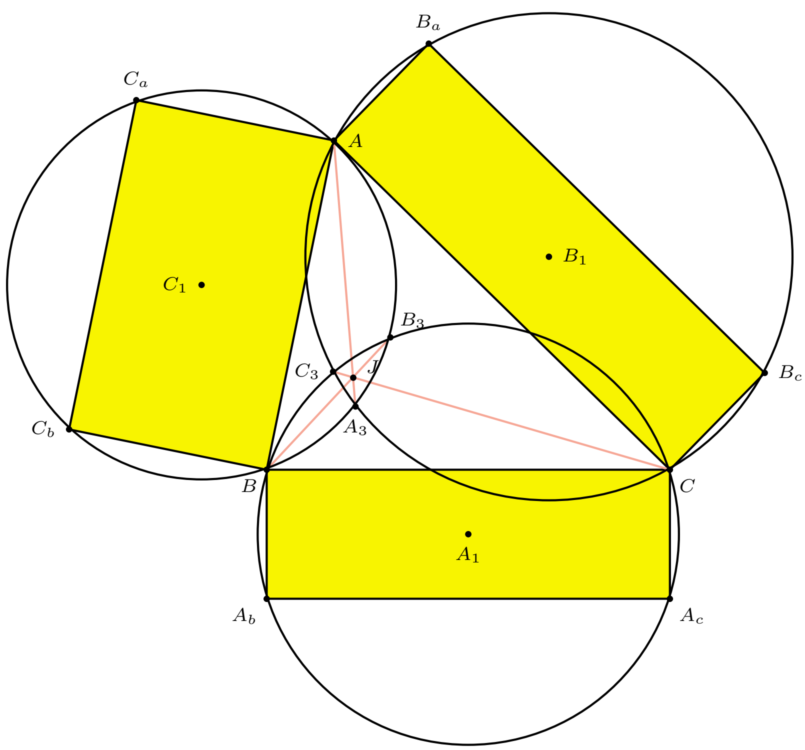

Figure 7

Note that the isogonal conjugate of J is the point

$$J^* = \left(a^2S_A + S\cdot\frac{a^2}{U} : b^2S_B + S\cdot\frac{b^2}{V} : c^2S_C + S\cdot\frac{c^2}{W}\right).$$

It lies on the line joining O to P_2. In fact,

$$P_2J^* : J^*O = 2S : au + bv + cw = P_2P_1 : P_1P_3.$$

The circles $\mathcal{C}_B$ and $\mathcal{C}_C$ meet at A and a second point A_3, which is the reflection of A in B_1C_1. See Figure 8. In homogeneous barycentric coordinates,

$$A_3 = \left(\frac{V+W}{S_A(V+W)-S(1-VW)} : \frac{V}{S_BV+S} : \frac{W}{S_CW+S}\right).$$

Similarly we have points B_3 and C_3. Clearly, the radical center J is the perspector of ABC and $A_3B_3C_3$.

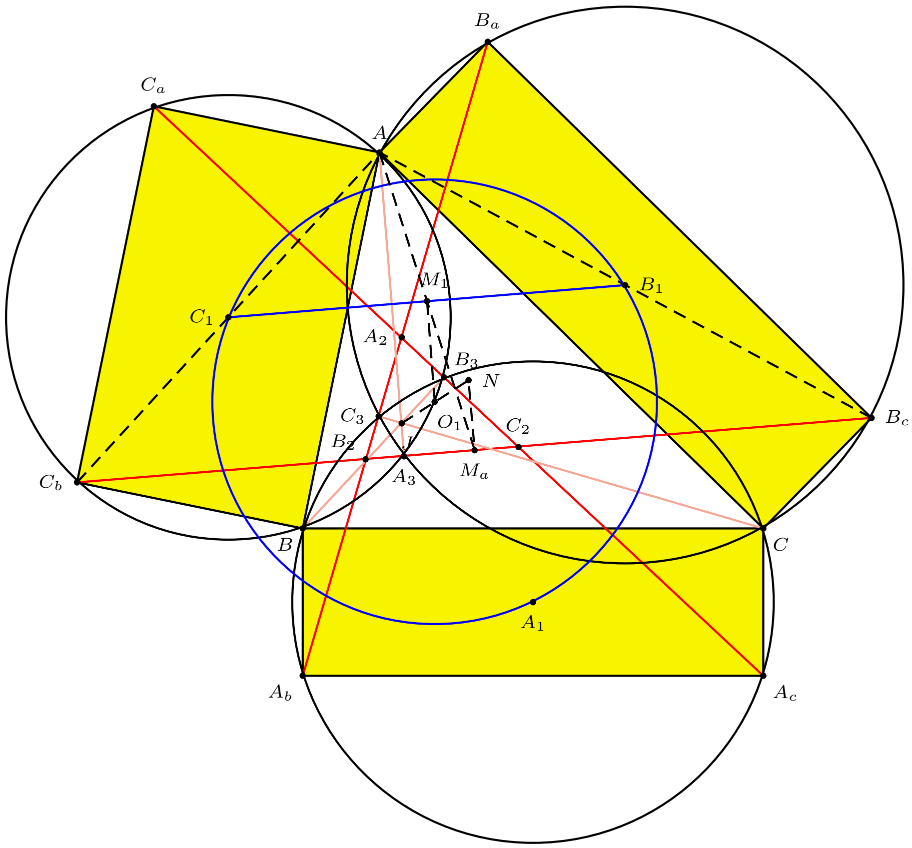

Figure 8

Proposition 4. *The triangles ABC and $A_2B_2C_2$ are orthologic. The perpendiculars from the vertices of one triangle to the corresponding lines of the other triangle concur at the point J.*

Proof. As C_1B_1 bisects AA_3, we see A_3 lies on B_cC_b and $AJ \perp B_cC_b$. Similarly, we have $BJ \perp C_aA_c$ and $CJ \perp A_bB_a$. The perpendiculars from A, B, C to the corresponding sides of $A_2B_2C_2$ concur at J.

On the other hand, the points B, C_3, B_3, C are concyclic and B_3C_3 is antiparallel to BC with respect to triangle JBC. The quadrilateral $JB_3A_2C_3$ is cyclic, with JA_2 as a diameter. It is known that every perpendicular to JA_2 is antiparallel to

B_3C_3 with respect to triangle JB_3C_3. Hence, $A_2J \perp BC$. Similarly, $B_2J \perp CA$ and $C_2J \perp AB$. □

It is clear that the perpendiculars from A_3, B_3, C_3 to the corresponding sides of triangle $A_2B_2C_2$ intersect at J. Hence, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are orthologic.

Proposition 5. *The perpendiculars from A_2, B_2, C_2 to the corresponding sides of $A_3B_3C_3$ meet at the reflection of J in the circumcenter O_3 of triangle $A_3B_3C_3$.*

Proof. Since triangle $A_3B_3C_3$ is the pedal triangle of J in $A_2B_2C_2$, and A_2J passes through the circumcenter of triangle $A_2B_3C_3$, the perpendicular from A_2 to B_3C_3 passes through the orthocenter of $A_2B_3C_3$ and is isogonal to A_2J in triangle $A_2B_2C_2$. This line therefore passes through the isogonal conjugate of J in $A_2B_2C_2$. We denote this point by $J^!$. Similarly, the perpendiculars from B_2, C_2 to the sides C_3A_3 and A_3B_3 pass through $J^!$. The circumcircle of $A_3B_3C_3$ is the pedal circle of J. Hence, its circumcenter O_3 is the midpoint of $JJ^!$. It follows that $J^!$ is the reflection of J in O_3. □

Remark. The point J and the circumcenters O and O_3 of triangles ABC and $A_3B_3C_3$ are collinear. This is because $|JA \cdot JA_3| = |JB \cdot JB_3| = |JC \cdot JC_3|$, say, $= d^2$, and an inversion in the circle (J, d) transforms ABC into $A_3B_3C_3$ or its reflection in J.

Theorem 6. *The perpendicular bisectors of B_cC_b, C_aA_c, A_bB_a are concurrent at a point which is the reflection of J in the circumcenter O_1 of triangle $A_1B_1C_1$.*

Proof. Let M_1 and M_a be the midpoints of B_1C_1 and B_cC_b respectively. Note that M_1 is also the midpoint of AM_a. Also, let O_1 be the circumcenter of $A_1B_1C_1$, and the perpendicular bisector of B_cC_b meet JO_1 at N. See Figure 8. Consider the trapezium AM_aNJ. Since O_1M_1 is parallel to AJ, we conclude that O_1 is the midpoint of JN. Similarly the perpendicular bisectors of C_aA_c, A_bB_a pass through N, which is the reflection of J in O_1. □

We record the coordinates of O_1:

$$\begin{aligned}&((c^2U^2V - a^2VW(V+W) + b^2WU(W+U)\\&\quad + UVW((S_A+3S_B)UV + (S_A+3S_C)UW))S\\&\quad + c^2S_BU^2V^2 + b^2S_CU^2W^2 - a^4V^2W^2\\&\quad + (S^2+S_{BC})U^2V^2W^2 + 4S^2U^2VW)\\&: \cdots : \cdots)\end{aligned}$$

In the Pythagorean case, the coordinates of O_1 are given in §2.1.

6. More triangles related to the attached rectangles

Write $U = \tan\alpha$, $V = \tan\beta$, and $W = \tan\gamma$ for angles α, β, γ in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. The point A_4 for which the swing angles CBA_4 and BCA_4 are β and γ

respectively has coordinates

$$(-a^2 : S_C + S \cdot \cot \gamma : S_B + S \cdot \cot \beta) = \left(-a^2 : S_C + \frac{S}{W} : S_B + \frac{S}{V}\right).$$

It is clear that this point lies on the line AJ. See Figure 9. If B_4 and C_4 are analogously defined, the triangles $A_4B_4C_4$ and ABC are perspective at J.

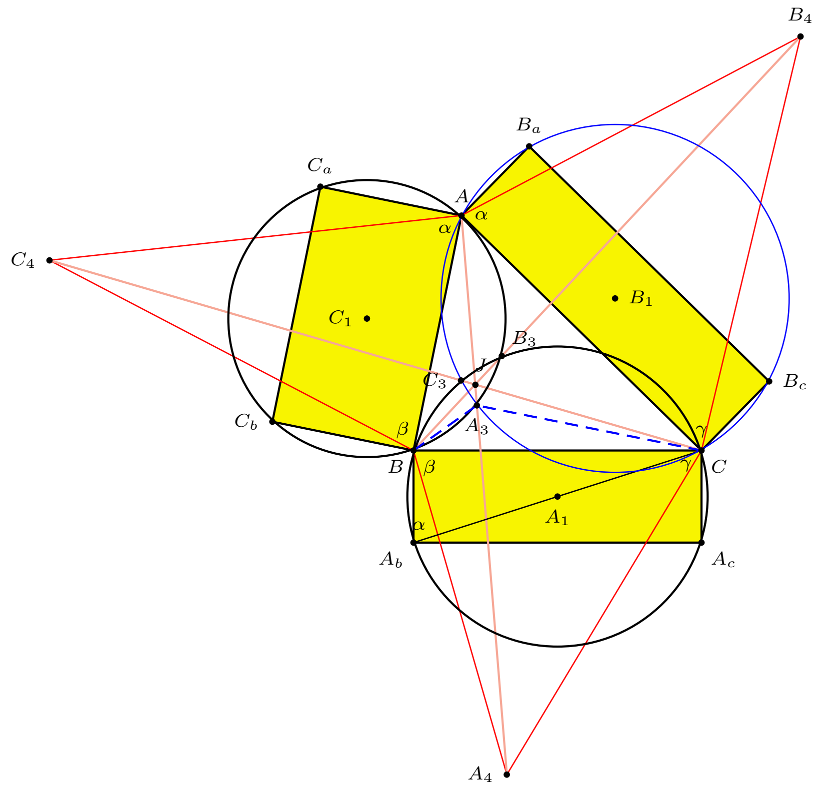

Figure 9

Note that A_3, B, A_4, C are concyclic since $\angle A_4BC = \beta = \angle AB_cV = \angle A_4A_3C$.

Let $d_1 = B_cC_b$, $d_2 = C_aA_c$, $d_3 = A_bB_a$, $d_1' = AA_4$, $d_2' = BB_4$, $d_3' = CC_4$.

Proposition 7. *The ratios* $\frac{d_i}{d_i'}$, $i = 1, 2, 3$, *are independent of triangle* ABC. *More precisely,*

$$\frac{d_1}{d_1'} = \frac{1}{V} + \frac{1}{W}, \qquad \frac{d_2}{d_2'} = \frac{1}{W} + \frac{1}{U}, \qquad \frac{d_3}{d_3'} = \frac{1}{U} + \frac{1}{V}.$$

Proof. Since $AA_4 \perp C_bB_c$, the circumcircle of the cyclic quadrilateral A_3BA_4C meets C_bB_c besides A_3 at the antipode A_5 of A_4. See Figure 10. Let f, g, h denote, for vectors, the compositions of a rotation by $\frac{\pi}{2}$, and homotheties of ratios

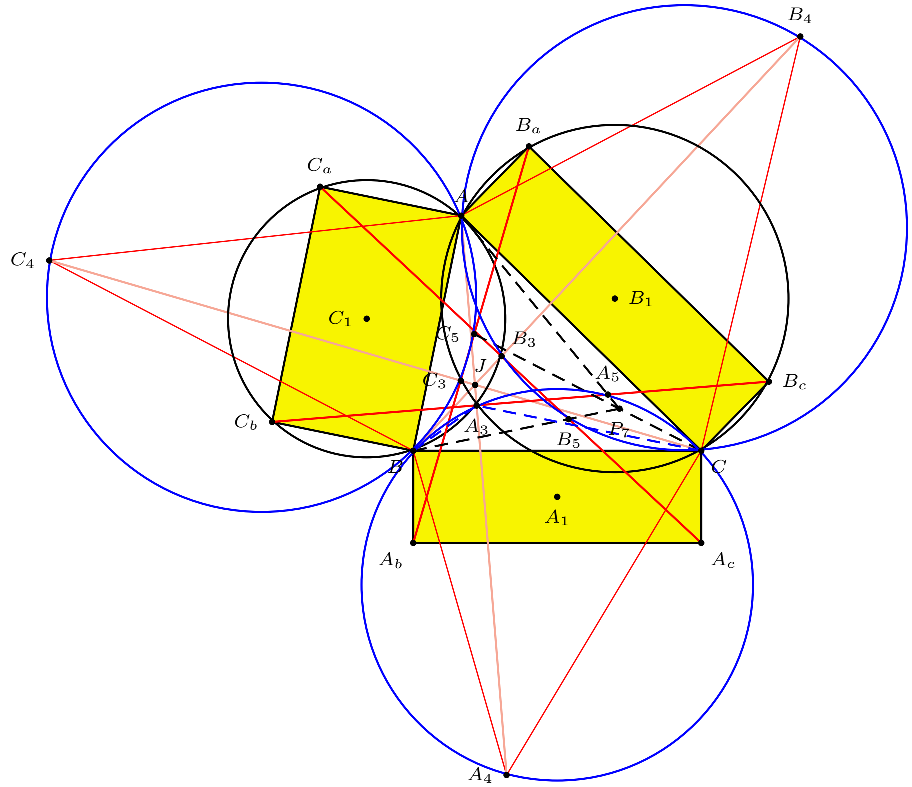

Figure 10

$\frac{1}{U}$, $\frac{1}{V}$, and $\frac{1}{W}$ respectively. Then

$$g(\overrightarrow{AA_4}) = g(\overrightarrow{AC}) + g(\overrightarrow{CA_4}) = \overrightarrow{CB_c} + \overrightarrow{A_5C} = \overrightarrow{A_5B_c},$$

and $\frac{A_5B_c}{AA_4} = \frac{1}{V}$. Similarly, $h(\overrightarrow{AA_4}) = \overrightarrow{C_bA_5}$, and $\frac{C_bA_5}{AA_4} = \frac{1}{W}$. It follows that $\frac{d_1}{d_1'} = \frac{1}{V} + \frac{1}{W}$. □

The coordinates of A_5 can be seen immediately: Since A_4A_5 is a diameter of the circle (A_4BC), we see that $\angle BCA_5 = -\frac{\pi}{2} + \angle BCA_4$, and

$$A_5 = (-a^2 : S_C - SW : S_B - SV).$$

Similarly, we have the coordinates of B_5 and C_5. From these, it is clear that $A_5B_5C_5$ and ABC are perspective at

$$P_7 = \left(\frac{1}{S_A - SU} : \frac{1}{S_B - SV} : \frac{1}{S_C - SW}\right) = \left(\frac{1}{\cot A - U} : \frac{1}{\cot B - V} : \frac{1}{\cot C - W}\right).$$

For example, in the similarity case it is obvious from the above proof that the points A_5, B_5, C_5 are the midpoints of B_cC_b, C_aA_c, A_bB_a. Clearly in the Pythagorean case, the points A_4, B_4, C_4 coincide with A_1, B_1, C_1 respectively.

In this case, J is the Vecten point and from the above proof we have $d_1 = 2d'_1$, $d_2 = 2d'_2$, $d_3 = 2d'_3$ and $P_7 = X_{486}$.

7. Another interesting special case

If $\alpha + \beta + \gamma = \pi$, then $U + V + W = UVW$. From Theorem 3 we conclude that $\triangle_2 = 0$, and the points A_2, B_2, C_2, A_3, B_3, C_3 coincide with J, which now is the common point of the circumcircles of the three rectangles. Also, the points A_4, B_4, C_4 lie on the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ respectively.

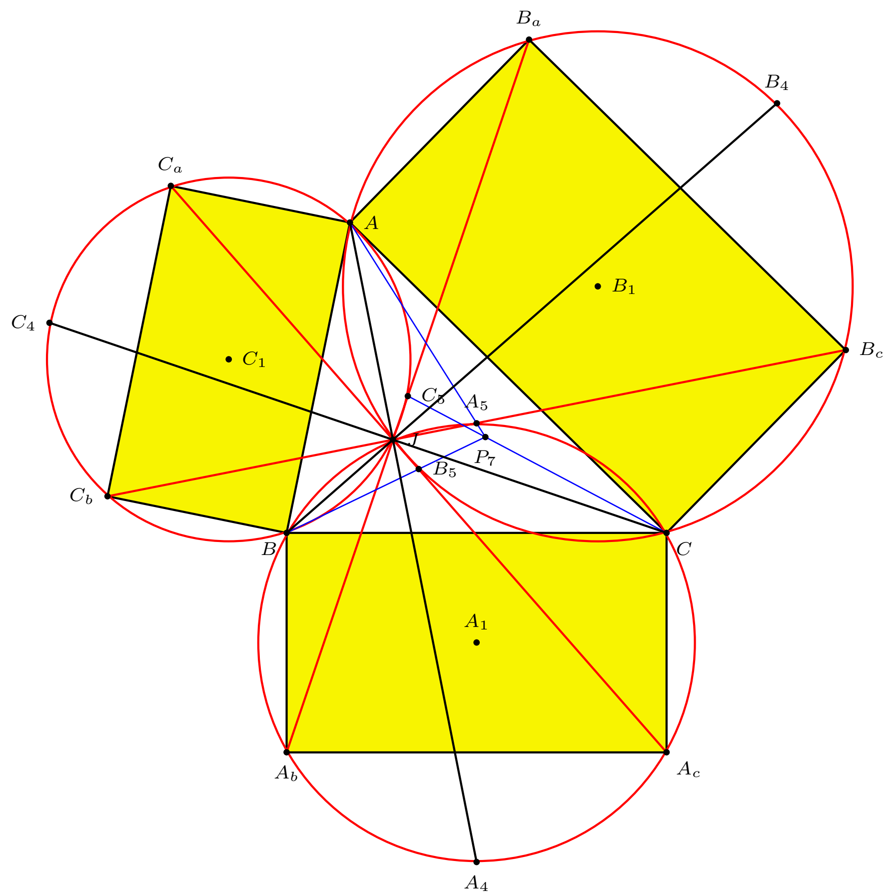

Figure 11

In Figure 11 we illustrate the case $\alpha = \beta = \gamma = \frac{\pi}{3}$. In this case, J is the Fermat point. The triangles BCA_4, CAB_4, ABC_4 are the Fermat equilateral triangles, and the angles of the lines AA_4, BB_4, CC_4, B_cC_b, C_aA_c, A_bB_a around J are $\frac{\pi}{6}$. The points A_5, B_5, C_5 are the mid points of B_cC_b, C_aA_c, A_bB_a. Also, $d'_1 = d'_2 = d'_3$, and $d_1 = d_2 = d_3 = \frac{2\sqrt{3}}{3}d'_1$. In this case, P_7 is the second Napoleon point, the point X_{18} in [7].

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