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A Generalization of the Lemoine Point

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Abstract. It is known that the Lemoine point K of a triangle in the Euclidean plane is the point of the plane where the sum of the squares of the distances d_1, d_2, and d_3 to the sides of the triangle takes its minimal value. There are several ways to generalize the Lemoine point. First, we can consider $n \geq 3$ lines u_1, ..., u_n instead of three in the Euclidean plane and search for the point which minimalizes the expression $d_1^2 + \cdots + d_n^2$, where d_i is the distance to the line u_i, $i = 1, \ldots, n$. Second, we can work in the Euclidean m-space R^m and consider n hyperplanes in R^m with $n \geq m+1$. In this paper a combination of these two generalizations is presented.

1. Introduction

Let us start with a triangle $A_1A_2A_3$ in the Euclidean plane R^2 and suppose that its sides $a_1 = A_2A_3$, $a_2 = A_3A_1$, and $a_3 = A_1A_2$ have length l_1, l_2, and l_3, respectively. The easiest way to deal with the Lemoine point K of the triangle is to work with trilinear coordinates with regard to $A_1A_2A_3$ (also called normal coordinates). See [1, 5, 6]. These are homogeneous projective coordinates (x_1, x_2, x_3) such that A_1, A_2, A_3, and the incenter I of the triangle, have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, respectively. If (a_i^1, a_i^2) are the non-homogeneous coordinates (x, y) of the point A_i with respect to an orthonormal coordinate system in R^2, $i = 1, 2, 3$, then the relationship between homogeneous cartesian coordinates (x, y, z) and trilinear coordinates (x_1, x_2, x_3) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1a_1^1 & l_2a_2^1 & l_3a_3^1 \\ l_1a_1^2 & l_2a_2^2 & l_3a_3^2 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This follows from the fact that the position vector of the incenter I of $A_1A_2A_3$ is given by

$$\vec{r} = \frac{l_1\vec{r}_1 + l_2\vec{r}_2 + l_3\vec{r}_3}{l_1 + l_2 + l_3},$$

with $\vec{r}_i$ the position vector of A_i. Remark also that $z = 0$ corresponds with $l_1x_1 + l_2x_2 + l_3x_3 = 0$, which is the equation in trilinear coordinates of the line at infinity

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of R^2. If (x_1, x_2, x_3) are normal coordinates of any point P of R^2 with regard to $A_1A_2A_3$, then the so-called absolute normal coordinates of P are

$$(d_1, d_2, d_3) = \left(\frac{2Fx_1}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_2}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_3}{l_1x_1 + l_2x_2 + l_3x_3} \right),$$

where F is the area of $A_1A_2A_3$. It is well known that d_i is the relative distance from P to the side a_i of the triangle (d_i is positive or negative, according as P lies at the same side or opposite side as A_i, with regard to a_i).

Next, consider the locus of the points of R^2 for which $d_1^2 + d_2^2 + d_3^2 = k$, with k a given value. In trilinear coordinates this locus is given by

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - k(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \tag{1}$$

For variable k, we get a pencil of homothetic ellipses (they all have the same points at infinity, the same asymptotes, the same center and the same axes), and the center of these ellipses is the Lemoine point K of the triangle $A_1A_2A_3$. A straightforward calculation gives that (l_1, l_2, l_3) are trilinear coordinates of K and the minimal value of $d_1^2 + d_2^2 + d_3^2$ reached at K is $\dfrac{4F^2}{l_1^2 + l_2^2 + l_3^2}$.

Remark also that K is the singular point of the degenerate ellipse of the pencil (1) corresponding with $k = \frac{1}{l_1^2+l_2^2+l_3^2}$ (set $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$).

More properties and constructions of the Lemoine point K can be found in [1]. And in [3] and [7] constructions for the axes of the ellipses (1) are given, while [7] contains a lot of generalizations.

Next, the foregoing can immediately be generalized to higher dimensions as follows. Consider in the Euclidean m-space R^m $(m \geq 2)$, $m+1$ hyperplanes not through a point and no two parallel; this determines an m-simplex with vertices $A_1, \ldots, A_{m+1}$. Let us denote the $(m-1)$-dimensional volume of the "face" a_i with vertices $A_1, \ldots, \hat{A}_i, \ldots, A_{m+1}$ by F_i, $i = 1, \ldots, m+1$. Then the position vector of the incenter I of $A_1A_2 \ldots A_{m+1}$ (= center of the hypersphere of R^m inscribed in $A_1 \ldots A_{m+1}$) is given by

$$\vec{r} = \frac{F_1\vec{r}_1 + F_2\vec{r}_2 + \cdots + F_{m+1}\vec{r}_{m+1}}{F_1 + F_2 + \cdots + F_{m+1}},$$

where $\vec{r}_i$ is the position vector of A_i, and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots, A_{m+1}$ are homogeneous projective coordinates such that $A_1, \ldots,$ A_{m+1}, and I, have coordinates $(1, 0, \ldots, 0)$, ..., $(0, \ldots, 0, 1)$, and $(1, 1, \ldots, 1)$, respectively. If $(a_i^1, a_i^2, \ldots, a_i^m)$ are cartesian coordinates (with respect to an orthonormal coordinate system) of A_i, $i = 1, \ldots, m+1$, the coordinate transformation between homogeneous cartesian coordinates $(z_1, \ldots, z_{m+1})$ and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots A_{m+1}$ is given by

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ z_{m+1} \end{pmatrix} = \begin{pmatrix} F_1 a_1^1 & F_2 a_2^1 & \dots & F_{m+1} a_{m+1}^1 \\ F_1 a_1^2 & F_2 a_2^2 & \dots & F_{m+1} a_{m+1}^2 \\ \vdots & \vdots & & \vdots \\ F_1 a_1^m & F_2 a_2^m & \dots & F_{m+1} a_{m+1}^m \\ F_1 & F_2 & \dots & F_{m+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \end{pmatrix}.$$

In normal coordinates the hyperplane at infinity of R^m has the equation $F_1 x_1 + \cdots + F_{m+1} x_{m+1} = 0$. Absolute normal coordinates of a point P of R^m with respect to $A_1, A_2, \dots, A_{m+1}$ are $d_i = \dfrac{mFx_i}{F_1 x_1 + \cdots + F_{m+1} x_{m+1}}$, $i = 1, \dots, m+1$, where F is the m-dimensional volume of $A_1 A_2 \dots A_{m+1}$ and d_i is the relative distance from P to the face a_i (d_i is positive or negative, according as P lies at the same side or at the opposite face as A_i, with regard to a_i). Remark that $F_1 d_1 + \cdots + F_{m+1} d_{m+1} = mF$.

The locus of the points of R^m for which $d_1^2 + \cdots + d_{m+1}^2 = k$ now determines a pencil of hyperquadrics (hyperellipsoids) with equation

$$x_1^2 + x_2^2 + \cdots + x_{m+1}^2 - k(F_1 x_1 + \cdots + F_{m+1} x_{m+1})^2 = 0 \qquad (2)$$

and all these (homothethic) hyperellipsoids have the same axes, the same points at infinity and the same center K, which we call the Lemoine point of $A_1 \dots A_{m+1}$ and which obviously has normal coordinates $(F_1, F_2, \dots, F_{m+1})$. The minimal value of $d_1^2 + \cdots + d_{m+1}^2$, reached at K is given by $\dfrac{m^2 F^2}{F_1^2 + \cdots + F_{m+1}^2}$. Remark that K is the singular point of the singular hyperquadric (hypercone) corresponding in the pencil (2) with the value $k = \frac{1}{F_1^2 + \cdots + F_{m+1}^2}$.

Remark. Some characterizations and constructions of the Lemoine point K of a triangle in the plane R^2 are no longer valid in higher dimensions. For instance, K is the perspective center of the triangle $A_1 A_2 A_3$ and the triangle $A_1' A_2' A_3'$ whose sides are the tangents of the circumscribed circle of $A_1 A_2 A_3$ at A_1, A_2, and A_3 (in trilinear coordinates the circumcircle has equation $l_1 x_2 x_3 + l_2 x_3 x_1 + l_3 x_1 x_2 = 0$). This construction is, in general, not correct in R^3 : a tetrahedron $A_1 A_2 A_3 A_4$ and its so called tangential tetrahedron, which is the tetrahedron $A_1' A_2' A_3' A_4'$ consisting of the tangent planes of the circumscribed sphere of $A_1 A_2 A_3 A_4$ at A_1, A_2, A_3, and A_4, are, in general, not perspective. If they are perspective, the tetrahedron is a special one, an *isodynamic* tetrahedron in which the three products of the three pairs of opposite edges are equal. The lines joining the vertices of an isodynamic tetrahedron to the Lemoine points of the respective opposite faces have a point in common and this common point is the perspective center of the isodynamic tetrahedron and its tangential tetrahedron (see [2]). It is not difficult to prove that this point of an isodynamic tetrahedron coincides with the Lemoine point K of the tetrahedron obtained with our definition of “Lemoine point”.

2. The main theorem

First we give some notations. Consider n hyperplanes, denoted by $u_1, \ldots, u_n$ in the Euclidean space R^m ($m \geq 2, n \geq m+1$), in general position (this means : no two are parallel and no $m+1$ are concurrent). The "figure" consisting of these n hyperplanes is called an n-hyperface (examples: for $m=2, n=3$ it determines a triangle in R^2, for $m=2$, $n=4$ it is an quadrilateral in R^2, and for $m=3$, $n=4$ it is a tetrahedron in R^3). The Lemoine point K of this n-hyperface is, by definition, the point of R^m for which the sum of the squares of the distances to the n hyperplanes $u_1, \ldots, u_n$ is minimal. The uniqueness of K follows from the proof of the next theorem.

Next, K^i is the Lemoine point of the $(n-1)$-hyperface $u_1 u_2 \ldots \hat{u}_i \ldots u_n$, $i = 1, \ldots, n$. And $K^{rs} = K^{sr}$ is the Lemoine point of the $(n-2)$-hyperface $u_1 u_2 \ldots \hat{u}_r \ldots \hat{u}_s \ldots u_n$, with $r, s = 1, \ldots, n$, $r \neq s$ (only defined if $n > m+1$).

Now, for an $(m+1)$-hyperface or m-simplex in R^m (a triangle in R^2, a tetrahedron in $R^3, \ldots$) we know the position (the normal coordinates) of the Lemoine point (see §1). The following theorem gives us a construction for the Lemoine point K of a general n-hyperface in R^m ($m \geq 2$ and $n > m+1$):

Theorem 1. *Working with an n-hyperface in R^m, we have, with the notations given above that $K^i K \cap u_j = K^j K^{ji} \cap u_j$, $i, j = 1, \ldots, n$ and $n > m+1$.*

Proof. In this proof, we work with cartesian coordinates $(x_1, \ldots, x_m)$ or homogeneous $(x_1, \ldots, x_{m+1})$ with respect to an orthonormal coordinate system in R^m. Suppose that the hyperplane u_r has equation $a_r^1 x_1 + a_r^2 x_2 + \cdots + a_r^m x_m + a_r^{m+1} = 0$, with $(a_r^1)^2 + (a_r^2)^2 + \cdots + (a_r^m)^2 = 1, r = 1, \ldots, n$. Then the Lemoine point K of the n-hyperface $u_1 u_2 \ldots u_n$ is the center of the hyperquadrics of the pencil with equation

$$\mathcal{F}(x_1, \ldots, x_m) = \sum_{r=1}^{n} (a_r^1 x_1 + a_r^2 x_2 + \cdots + a_r^m x_m + a_r^{m+1})^2 - k = 0, \qquad (3)$$

where k is a parameter. Indeed, since the coordinates of K minimize the expression $\sum_{r=1}^{n} \left(a_r^1 x_r + \cdots + a_r^{m+1}\right)^2$, they are a (the) solution of $\frac{\partial \mathcal{F}}{\partial x_1} = \frac{\partial \mathcal{F}}{\partial x_2} = \cdots = \frac{\partial \mathcal{F}}{\partial x_m} = 0$. In homogeneous coordinates, (3) becomes

$$\mathcal{F}(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (4)$$

Next, the Lemoine point K^i of $u_1 u_2 \ldots \hat{u}_i \ldots u_n$ is the center of the hyperquadrics of the pencil given by (we use the same notation k for the parameter)

$$\mathcal{F}^i(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r \neq i}}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (5)$$

The diameter of the hyperquadrics (5), conjugate with respect to the direction of the ith hyperplane u_i has the equations (consider the polar hyperplanes of the

$m-1$ points at infinity with coordinates $(a_i^2, -a_i^1, 0, \dots, 0)$, $(a_i^3, 0, -a_i^1, 0, \dots, 0)$, $(a_i^4, 0, 0, -a_i^1, 0, \dots, 0)$, ..., $(a_i^m, 0, \dots, 0, -a_i^1, 0)$ of the hyperplane u_i):

$$\begin{cases} \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0, \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^3 - a_r^3 a_i^1) = 0, \\ \quad \vdots \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{6}$$

But the first side of each of these equations becomes zero for $r = i$, and thus (6) gives us also the conjugate diameter with respect to the hyperplane u_i of the hyperquadrics of the pencil (5). It follows that (6) determines the line KK^i.

Next, the Lemoine point K^j is the center of the hyperquadrics of the pencil

$$\mathcal{F}^j(x_1, \dots, x_{m+1}) = \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0, \tag{7}$$

and K^{ji} is the center of the hyperquadrics:

$$\mathcal{F}^{ji}(x_1, \dots, x_{m+1}) = \sum_{\substack{r=1 \\ r\neq j,i}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \tag{8}$$

The diameter of the hyperquadrics (7), conjugate with respect to the direction of u_i is given by

$$\begin{cases} \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0 \\ \quad \vdots \\ \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{9}$$

And this gives us also the diameter of the hyperquadrics (8) conjugate with regard to the direction of u_i; in other words, (9) determines the line $K^j K^{ji}$.
Finally, the coordinates of the point $K^i K \cap u_j$ are the solutions of the linear system

$$\begin{cases} (6) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0, \end{cases}$$

while the point $K^j K^{ji} \cap u_j$ is given by

$$\begin{cases} (9) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0. \end{cases}$$

It is obvious that this gives the same point and the proof is complete. □

3. Applications

3.1. Let us first consider the easiest example for trying out our construction: the case where $m = 2$ and $n = 4$, or four lines u_1, u_2, u_3, u_4 in general position (they form a quadrilateral) in R^2. Using orthonormal coordinates (x, y, z) in R^2, the homogeneous equation of u_r is $a_r x + b_r y + c_r z = 0$ with $a_r^2 + b_r^2 = 1$,

$r = 1, 2, 3, 4$. Where lies the Lemoine point K of the quadrilateral $u_1u_2u_3u_4$? For instance K^1 is the Lemoine point of the triangle with sides (lines) u_2, u_3, u_4; K^2 of the triangle with sides u_1, u_3, u_4, and so on ... . We may assume that we can construct the Lemoine point of a triangle. But which point is, for instance, the point K^{12} : it is the Lemoine point of the 2-side u_3u_4, *i.e.*, it is the point $u_3 \cap u_4$.

Let us denote the six vertices of the quadrilateral as follows : $u_1 \cap u_2 = C, u_2 \cap u_3 = A, u_3 \cap u_4 = F, u_1 \cap u_4 = D, u_2 \cap u_4 = E$, and $u_1 \cap u_3 = B$, then $K^{12} = K^{21} = F$, $K^{23} = D, K^{34} = C, K^{14} = A, K^{24} = B$, and $K^{13} = E$. Now, from $K^iK \cap u_j = K^jK^{ji} \cap u_j$, we find, for instance for $i = 1$ and $j = 2$:

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 = K^2F \cap u_2$$

and for $i = 2$ and $j = 1$: $K^2K \cap u_1 = K^1K^{12} \cap u_1 = K^1F \cap u_1$, with K^1 (K^2, resp.) the Lemoine point of the triangle AFE (of the triangle BFD, resp.). This allows us to construct the point K.

In particular, we can construct the diameters KK^1, KK^2, KK^3, and KK^4 of the ellipses of the pencil $\sum_{r=1}^{4}(a_rx + b_ry + c_rz)^2 = kz^2$, which are conjugate to the directions of the lines u_1, u_2, u_3, and u_4, respectively. In other words, we have four pairs of conjugate diameters of these ellipses : (KK^i, KI^i_∞), where I^i_∞ is the point at infinity of the line $u_i, i = 1, \dots, 4$. From this, we can construct the axes of the conics of this bundle (in fact, two pairs of conjugate diameters are sufficient): consider any circle $\mathcal{C}$ through K and project the involution of conjugate diameters onto $\mathcal{C}$; if S is the center of this involution on $\mathcal{C}$ and if the diameter of $\mathcal{C}$ through S intersects $\mathcal{C}$ at the points S_1 and S_2, then KS_1 and KS_2 are the axes.

In the case of a triangle in R^2, constructions of the common axes of the ellipses determined by $d_1^2 + d_2^2 + d_3^2 = k$ with center the Lemoine point of the triangle, are given in [3] and [7]. In [3], J. Bilo proved that the axes are the perpendicular lines through K on the Simson lines of the common points of the Euler line and the circumscribed circle of the triangle. And in [7], we proved that these axes are the orthogonal lines through K which cut the sides of the triangle in pairs of points whose midpoints are three collinear points. Moreover [7] contains a lot of generalizations for pencils whose conics have any point P of the plane as common center and whose common axes are constructed in the same way.

3.2. In the case $m = 2$ and $n \geq 4$, we can construct the n diameters $KK^1, \dots, KK^n$ of the ellipses $\sum_{r=1}^{n}(a_rx + b_ry + c_rz)^2 = kz^2$ which are conjugate to the directions of the n lines $u_1, \dots, u_n$.

3.3. The easiest example in space is the case where $m = 3$ and $n = 5$, or five planes in R^3. Assume that the planes have equations $a_rx + b_ry + c_rz + d_ru = 0$, with $a_r^2 + b_r^2 + c_r^2 = 1, r = 1, 2, \dots, 5$. We look for the Lemoine point K of the “5-plane” $u_1u_2u_3u_4u_5$ in R^3 and assume that we know the position of the Lemoine point of any tetrahedron in R^3 (we know its normal coordinates). The points $K^1, \dots, K^5$ are the Lemoine points of the tetrahedra $u_2u_3u_4u_5, \dots, u_1u_2u_3u_4$, respectively. And, for instance K^{12} is the Lemoine point of the “3-plane” $u_3u_4u_5$, *i.e.*, it is the common point of these three planes u_3, u_4, and u_5. Now, for instance from

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 \quad \text{and} \quad K^2K \cap u_1 = K^1K^{12} \cap u_1,$$

we can construct the lines K^1K and K^2K, and thus the point K. In fact, we can construct the diameters $KK^1, \ldots, KK^5$ conjugate to the plane directions of $u_1, \ldots, u_5$, respectively, of the quadrics with center K of the pencil given by $d_1^2 + \cdots + d_5^2 = k$ or

$$\sum_{r=1}^{5}(a_r x + b_r y + c_r z + d_r u)^2 = ku^2.$$

Finally, the construction of the point K in the general case $n > m + 1$, $m \geq 2$ is obvious.

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