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Two Triangle Centers Associated with the Excircles

Milorad R. Stevanović

Abstract. The triangle formed by the second intersections of the bisectors of a triangle and the respective excircles is perspective to each of the medial and intouch triangles. We identify the perspectors. In the former case, the perspector is closely related to the Yff center of congruence.

1. Introduction

In this note we construct two triangle centers associated with the excircles. Given a triangle ABC, let A' be the "second" intersection of the bisector of angle A with the A-excircle, which is outside the segment AI_a, I_a being the A-excenter. Similarly, define B' and C'.

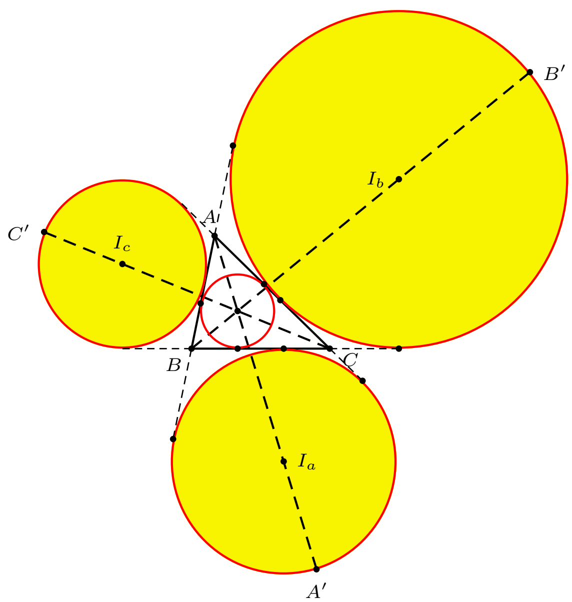

Figure 1

Theorem 1. *Triangle $A'B'C'$ is perspective with the medial triangle at the Yff center of congruence of the latter triangle, namely, the point P with homogeneous barycentric coordinates*

$$\left(\sin\frac{B}{2}+\sin\frac{C}{2}:\sin\frac{C}{2}+\sin\frac{A}{2}:\sin\frac{A}{2}+\sin\frac{B}{2}\right)$$

with respect to ABC.

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Theorem 2. *Triangle $A'B'C'$ is perspective with the intouch triangle at the point Q with homogeneous barycentric coordinates*

$$\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).$$

Remark. These triangle centers now appear as X_{2090} and X_{2091} in [2].

2. Notations and preliminaries

We shall make use of the following notations. In a triangle ABC of sidelengths a, b, c, circumradius R, inradius r, and semiperimeter s, let

$$s_a=\sin\tfrac{A}{2},\quad s_b=\sin\tfrac{B}{2},\quad s_c=\sin\tfrac{C}{2};$$

$$c_a=\cos\tfrac{A}{2},\quad c_b=\cos\tfrac{B}{2},\quad c_c=\cos\tfrac{C}{2}.$$

The following formulae can be found, for example, in [1].

$$\begin{array}{lll} r=4Rs_as_bs_c, & s=4Rc_ac_bc_c; & \\ s-a=4Rc_as_bs_c, & s-b=4Rs_ac_bs_c, & s-c=4Rs_as_bc_c. \end{array}$$

2.1. *The medial triangle.* The medial triangle $A_1B_1C_1$ has vertices the midpoints of the sides BC, CA, AB of triangle ABC. From

$$\mathbf{A_1}=\frac{\mathbf{B}+\mathbf{C}}{2},\qquad \mathbf{B}_1=\frac{\mathbf{C}+\mathbf{A}}{2},\qquad \mathbf{C}_1=\frac{\mathbf{A}+\mathbf{B}}{2},$$

we have

$$\mathbf{A}=\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1,\qquad \mathbf{B}=\mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1,\qquad \mathbf{C}=\mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1. \quad (1)$$

Lemma 3. *The barycentric coordinates of the excenters with respect to the medial triangle are*

$$\begin{aligned} \mathbf{I}_a &= \frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a},\\ \mathbf{I}_b &= \frac{-(s-c)\mathbf{A}_1+s\cdot\mathbf{B}_1-(s-a)\mathbf{C}_1}{s-b},\\ \mathbf{I}_c &= \frac{-(s-b)\mathbf{A}_1-(s-a)\mathbf{B}_1+s\cdot\mathbf{C}_1}{s-c}. \end{aligned}$$

Proof. It is enough to compute the coordinates of the excenter I_a:

$$\begin{aligned} \mathbf{I}_a &= \frac{-a\cdot\mathbf{A}+b\cdot\mathbf{B}+c\cdot\mathbf{C}}{b+c-a}\\ &= \frac{-a(\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1)+b(\mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1)+c(\mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1)}{b+c-a}\\ &= \frac{(a+b+c)\mathbf{A}_1-(a+b-c)\mathbf{B}_1-(c+a-b)\mathbf{C}_1}{b+c-a}\\ &= \frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a}. \end{aligned}$$

□

2.2. *The intouch triangle.* The vertices of the intouch triangle are the points of tangency of the incircle with the sides. These are

$$\mathbf{X} = \frac{(s-c)\mathbf{B} + (s-b)\mathbf{C}}{a}, \quad \mathbf{Y} = \frac{(s-c)\mathbf{A} + (s-a)\mathbf{C}}{b}, \quad \mathbf{Z} = \frac{(s-b)\mathbf{A} + (s-a)\mathbf{B}}{c}.$$

Equivalently,

$$\begin{aligned}
\mathbf{A} =& \frac{-a(s-a)\mathbf{X} + b(s-b)\mathbf{Y} + c(s-c)\mathbf{Z}}{2(s-b)(s-c)},\\
\mathbf{B} =& \frac{a(s-a)\mathbf{X} - b(s-b)\mathbf{Y} + c(s-c)\mathbf{Z}}{2(s-c)(s-a)},\\
\mathbf{C} =& \frac{a(s-a)\mathbf{X} + b(s-b)\mathbf{Y} - c(s-c)\mathbf{Z}}{2(s-a)(s-b)}.
\end{aligned} \tag{2}$$

Lemma 4. *The barycentric coordinates of the excenters with respect to the intouch triangle are*

$$\begin{aligned}
\mathbf{I}_a =& \frac{a(bc-(s-a)^2)\mathbf{X} - b(s-b)^2\mathbf{Y} - c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\
\mathbf{I}_b =& \frac{-a(s-a)^2\mathbf{X} + b(ca-(s-b)^2)\mathbf{Y} - c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\
\mathbf{I}_c =& \frac{-a(s-a)^2\mathbf{X} - b(s-b)^2\mathbf{Y} + c(ab-(s-c)^2)\mathbf{Z}}{2(s-a)(s-b)(s-c)}.
\end{aligned}$$

3. Proof of Theorem 1

We compute the barycentric coordinates of A' with respect to the medial triangle. Note that A' divides AI_a externally in the ratio $AA' : A'I_a = 1 + s_a : -s_a$. It follows that

$$\begin{aligned}
\mathbf{A}' =& (1+s_a)\mathbf{I}_a - s_a \cdot \mathbf{A}\\
=& \frac{1+s_a}{s-a}(s \cdot \mathbf{A}_1 - (s-c)\mathbf{B}_1 - (s-b)\mathbf{C}_1) - s_a(\mathbf{B}_1 + \mathbf{C}_1 - \mathbf{A}_1).
\end{aligned}$$

From this, the homogeneous barycentric coordinates of A' with respect to $A_1B_1C_1$ are

$$\begin{aligned}
&(1+s_a)s + s_a(s-a) : -(1+s_a)(s-c) - s_a(s-a)\\
&\qquad : -(1+s_a)(s-b) - s_a(s-a)\\
=& s + s_a(b+c) : -((s-c) + s_a b) : -((s-b) + s_a c)\\
=& 4Rc_ac_bc_c + 4Rs_a(s_bc_b + s_cc_c) : -4R(s_as_bc_c + s_as_bc_b) : -4R(s_ac_bs_c + s_as_cc_c)\\
=& -\frac{c_ac_bc_c + s_a(s_bc_b + s_cc_c)}{s_a(c_b + c_c)} : s_b : s_c.
\end{aligned}$$

Similarly,

$$B' = \left(s_a : -\frac{c_a c_b c_c + s_b(s_c c_c + s_a c_a)}{s_b(c_c + c_a)} : s_c \right),$$

$$C' = \left(s_a : s_b : -\frac{c_a c_b c_c + s_c(s_a c_a + s_b c_b)}{s_c(c_a + c_b)} \right).$$

From these, it is clear that $A'B'C'$ and the medial triangle are perspective at the point with coordinates $(s_a : s_b : s_c)$ relative to $A_1B_1C_1$. This is clearly the Yff center of congruence of the medial triangle. See Figure 2. Its coordinates with respect to ABC are

$$(s_b + s_c : s_c + s_a : s_a + s_b).$$

This completes the proof of Theorem 1.

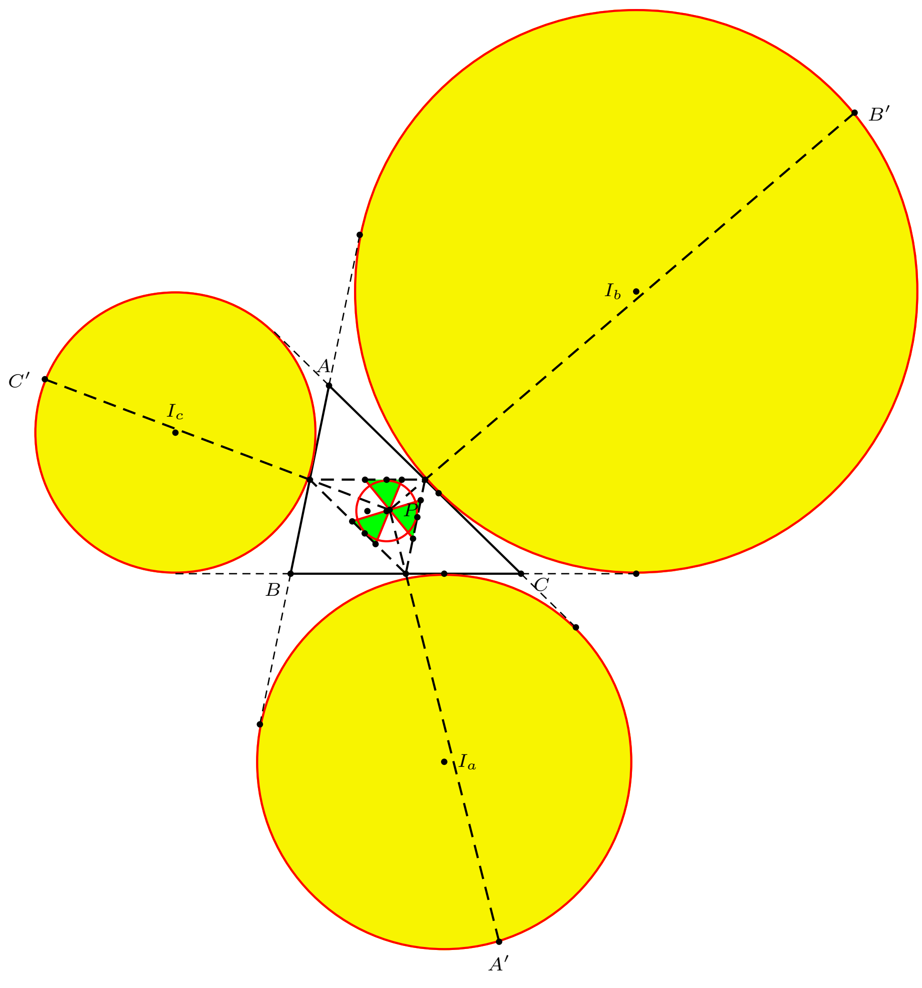

Figure 2

Remark. In triangle ABC, let A'', B'', C'' be the feet of the bisectors of angles BIC, CIA, AIB respectively on sides BC, CA, AB. Triangles $A''B''C''$ and ABC are perspective at the Yff center of congruence X_{174}, *i.e.*, if the perpendiculars from X_{174} to the bisectors of the angles of ABC intersect the sides of triangle ABC at X_b, X_c, Y_a, Y_c, Z_a, Z_b (see Figure 3), then the triangles $X_{174}X_bX_c$, $Y_aX_{174}Y_c$ and $Z_aZ_bX_{174}$ are congruent. See [3].

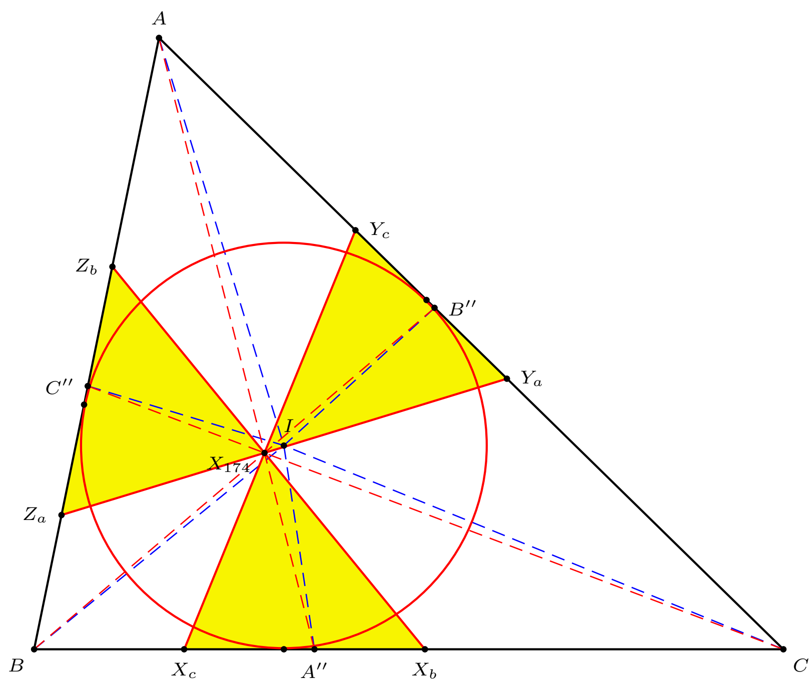

Figure 3

4. Proof of Theorem 2

Consider the coordinates of $\mathbf{A}' = (1+s_a)\mathbf{I}_a - s_a \cdot \mathbf{A}$ with respect to the intouch triangle XYZ. By Lemma 3, the Y-coordinate is

$$\begin{aligned}
&\frac{-(1+s_a)b(s-b)^2 - s_a b(s-a)(s-b)}{2(s-a)(s-b)(s-c)} \\
=&\frac{-b(s-b)((1+s_a)(s-b)+s_a(s-a))}{2(s-a)(s-b)(s-c)} \\
=&\frac{-b(s-b)(s-b+s_a\cdot c)}{2(s-a)(s-b)(s-c)} \\
=&\frac{-(c_b+c_c)}{2c_ac_bc_c}\cdot\frac{c_b^2}{s_b}.
\end{aligned}$$

Similarly for the Z-coordinate is $\frac{-(c_b+c_c)}{2c_ac_bc_c}\cdot\frac{c_c^2}{s_c}$. Therefore, $A'B'C'$ is perspective with XYZ at

$$Q=\left(\frac{c_a^2}{s_a}:\frac{c_b^2}{s_b}:\frac{c_c^2}{s_c}\right).$$

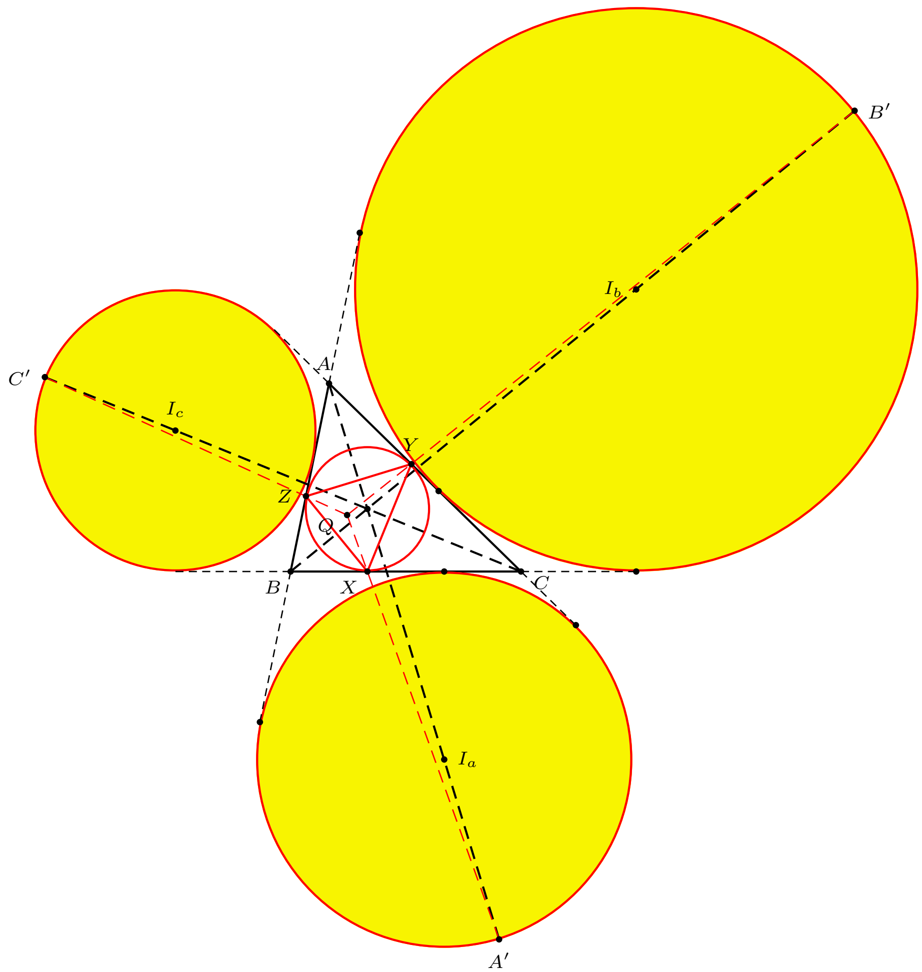

Figure 4

Note that the angles of the intouch triangles are $X=\frac{B+C}{2}$, $Y=\frac{C+A}{2}$, and $Z=\frac{A+B}{2}$. This means

$$s_a=\cos\frac{B+C}{2}=\cos X,\qquad c_a=\sin\frac{B+C}{2}=\sin X,$$

etc. It follows that Q has homogeneous barycentric coordinates

$$\left(\frac{\sin^2 X}{\cos X}:\frac{\sin^2 Y}{\cos Y}:\frac{\sin^2 Z}{\cos Z}\right)$$

and is the Clawson point of the intouch triangle XYZ. With respect to triangle ABC, this perspector Q has coordinates given by

$$\begin{aligned}
&\left(\frac{a(s-a)}{s_a}+\frac{b(s-b)}{s_b}+\frac{c(s-c)}{s_c}\right)\mathbf{Q}\\
=&\frac{a(s-a)\mathbf{X}}{s_a}+\frac{b(s-b)\mathbf{Y}}{s_b}+\frac{c(s-c)\mathbf{Z}}{s_c}\\
=&\frac{(s-b)(s-c)(s_b+s_c)}{s_bs_c}\mathbf{A}+\frac{(s-c)(s-a)(s_c+s_a)}{s_cs_a}\mathbf{B}+\frac{(s-a)(s-b)(s_a+s_b)}{s_as_b}\mathbf{C}\\
=&(4R)^2s_a^2c_bc_c(s_b+s_c)\mathbf{A}+(4R)^2s_b^2c_cc_a(s_c+s_a)\mathbf{B}+(4R)^2s_c^2c_ac_b(s_a+s_b)\mathbf{C}\\
=&(4R)^2c_ac_bc_c\left(\frac{s_a^2(s_b+s_c)}{c_a}\cdot\mathbf{A}+\frac{s_b^2(s_c+s_a)}{c_b}\cdot\mathbf{B}+\frac{s_c^2(s_a+s_b)}{c_c}\cdot\mathbf{C}\right).
\end{aligned}$$

Therefore, the homogeneous barycentric coordinates of Q with respect to ABC are

$$\begin{aligned}
&\left(\frac{s_a^2(s_b+s_c)}{c_a}:\frac{s_b^2(s_c+s_a)}{c_b}:\frac{s_c^2(s_a+s_b)}{c_c}\right)\\
=&\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Inasmuch as Q is the Clawson point of the intouch triangle, it is interesting to point out that the congruent isoscelizers point X_{173}, a point closely related to the Yff center of congruence X_{174} and with coordinates

$$(a(-c_a+c_b+c_c):b(c_a-c_b+c_c):c(c_a+c_b-c_c)),$$

is the Clawson point of the excentral triangle $I_aI_bI_c$ (which is homothetic to the intouch triangle at X_{57}). This fact was stated in an earlier edition of [2], and can be easily proved by the method of this paper.

References

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Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia and Montenegro

E-mail address: milmath@ptt.yu