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Circumcenters of Residual Triangles

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Abstract. This paper is an extension of Mario Dalcín's work on isotomic inscribed triangles and their residuals [1]. Considering the circumcircles of residual triangles with respect to isotomic inscribed triangles there are two congruent triangles of circumcenters. We show that there is a rotation mapping these triangles to each other. The center and angle of rotation depend on the Miquel points. Furthermore we give an interesting generalization of Dalcin's definitive example.

1. Introduction

If X, Y, Z are points on the sides of a triangle ABC, there are three residual triangles AZY, BXZ, CYX. The circumcenters of these triangles form a triangle $O_aO_bO_c$ similar to the reference triangle ABC [2]. The circumcircles have a common point M by Miquel's theorem. The lines MX, MY, MZ and the corresponding side lines have the same angle of intersection $\mu = (AY, YM) = (BZ, ZM) = (CX, XM)$. The angles are directed angles measured between 0 and π.

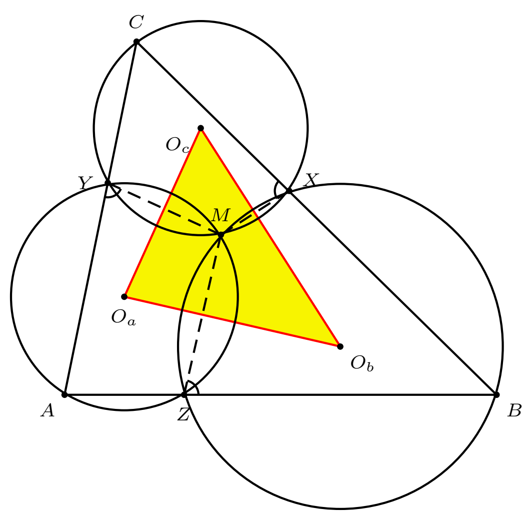

Figure 1

Dalcín considers isotomic inscribed triangles XYZ and $X'Y'Z'$. Here, X', Y', Z' are the reflections of X, Y, Z in the midpoints of the respective sides. The triangle XYZ may or may not be cevian. If it is the cevian triangle of a point P, then $X'Y'Z'$ is the cevian triangle of the isotomic conjugate of P. The

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corresponding Miquel point M' of X', Y', Z' has Miquel angle $\mu' = \pi - \mu$. The circumcircles of the residual triangles $AZ'Y'$, $BX'Z'$, $CY'X'$ give further points of intersection. The intersections A' of the circles AZY and $AZ'Y'$, B' of BXZ and $BX'Z'$, and C' of CYX and $CY'X'$ form a triangle $A'B'C'$ perspective to the reference triangle ABC with the center of perspectivity Q. See Figure 2. It can be shown that the points M, M', A', B', C', Q and the circumcenter O of the reference triangle lie on a circle with the diameter OQ.

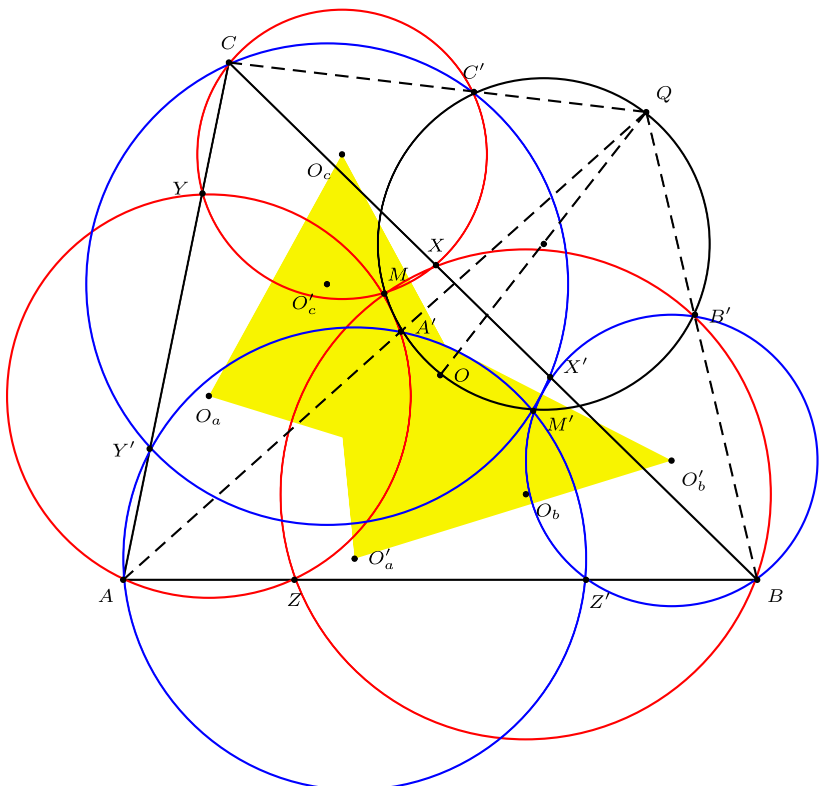

Figure 2

These results can be proved by analytical calculations. We make use of homogeneous barycentric coordinates. Let X, Y, Z divide the sides BC, CA, AB respectively in the ratios

$$BX : XC = x : 1, \qquad CY : YA = y : 1, \qquad AZ : ZB = z : 1.$$

These points have coordinates

$$\begin{aligned} X &= (0 : 1 : x), \quad & Y &= (y : 0 : 1), \quad & Z &= (1 : z : 0); \\ X' &= (0 : x : 1), \quad & Y' &= (1 : 0 : y), \quad & Z' &= (z : 1 : 0). \end{aligned}$$

The circumcenter, the Miquel points, and the center of perspectivity are the points

$$\begin{aligned}
O =&(a^2(b^2+c^2-a^2):b^2(c^2+a^2-b^2):c^2(a^2+b^2-c^2)),\\
M =&(a^2x(1+y)(1+z)-b^2xy(1+x)(1+z)-c^2(1+x)(1+y):\cdots:\cdots),\\
M' =&(a^2x(1+y)(1+z)-b^2(1+x)(1+z)-c^2xz(1+x)(1+y):\cdots:\cdots),\\
Q =&\left(\frac{(1-x)a^2}{1+x}:\frac{(1-y)b^2}{1+y}:\frac{(1-z)c^2}{1+z}\right).
\end{aligned}$$

The Miquel angle μ is given by

$$\cot\mu=\frac{1-yz}{(1+y)(1+z)}\cot A+\frac{1-zx}{(1+z)(1+x)}\cot B+\frac{1-xy}{(1+x)(1+y)}\cot C.$$

For example, let X, Y, Z divide the sides in the same ratio k, *i.e.*, $x=y=z=k$, then we have

$$\begin{aligned}
M =&(a^2(-c^2+a^2k-b^2k^2):b^2(-a^2+b^2k-c^2k^2):c^2(-b^2+c^2k-a^2k^2)),\\
M' =&(a^2(-b^2+a^2k-c^2k^2):b^2(-c^2+b^2k-a^2k^2):c^2(-a^2+c^2k-b^2k^2)),\\
Q =&(a^2:b^2:c^2)=X_6(\text{ Lemoine point});\\
\cot\mu =&\frac{1-k}{1+k}\cot\omega,
\end{aligned}$$

where ω is the Brocard angle.

2. Two triangles of circumcenters

Considering the circumcenters of the residual triangles for XYZ and $X'Y'Z'$, Dalcín ([1, Theorem 10]) has shown that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. We show that there is a rotation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$. This rotation also maps the Miquel point M to the circumcenter O, and O to the other Miquel point M'. See Figure 3. The center of rotation is therefore the midpoint of OQ. This center of rotation is situated with respect to $O_aO_bO_c$ and $O'_aO'_bO'_c$ as the center of perspectivity with respect to the reference triangle ABC. The angle φ of rotation is given by

$$\varphi=\pi-2\mu.$$

The similarity ratio of triangles $O_aO_bO_c$ and ABC is

$$\frac{1}{2\cos\frac{\varphi}{2}}=\frac{1}{2\sin\mu},$$

similarly for triangle $O'_aO'_bO'_c$.

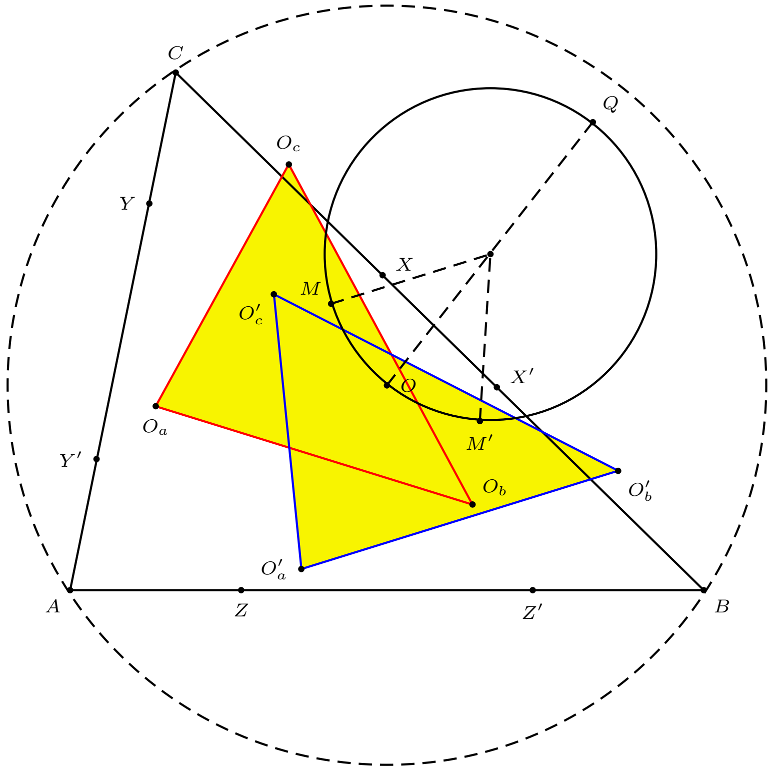

Figure 3

3. Dalcín's example

If we choose X, Y, Z as the points of tangency of the incircle with the sides, XYZ is the cevian triangle of the Gergonne point G_e and $X'Y'Z'$ is the cevian triangle of the Nagel point N_a. The Miquel point M is the incenter I and the Miquel point M' is the reflection of I in O, *i.e.*,

$$X_{40} = (a(a^3 - b^3 - c^3 + (a-b)(a-c)(b+c)) : \cdots : \cdots).$$

In this case, $O_aO_bO_c$ is homothetic to ABC at M, with factor $\frac{1}{2}$. This is also the case when XYZ is the cevian triangle of the Nagel point, with $M = X_{40}$.

Therefore, the circle described in §2, degenerates into a line. The center of perspectivity $Q(a(b-c) : b(c-a) : c(a-b))$ is a point of infinity. The triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are homothetic to the triangle ABC at the Miquel points M and M' with factor $\frac{1}{2}$. There is a parallel translation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$.

The fact that ABC is homothetic to $OaObOc$ with the factor $\frac{1}{2}$ does not only hold for the Gergonne and Nagel points. Here are further examples.

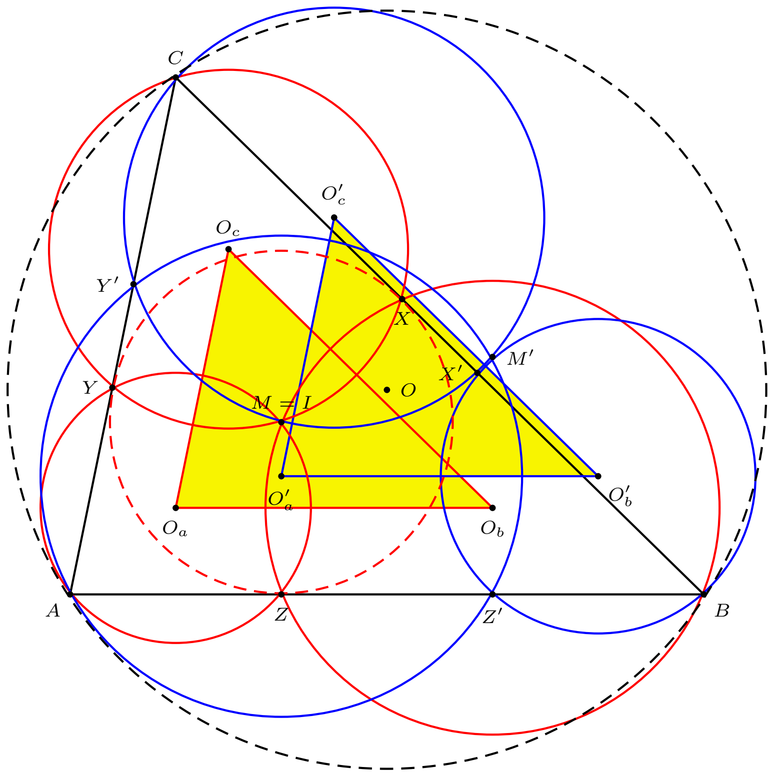

Figure 4

P	Homothetic center and Miquel point M
centroid G	circumcenter O
orthocenter H	H
X_{69}	X_{20}
X_{189}	X_{84}
X_{253}	X_{64}
X_{329}	X_{1490}

These points $P(u : v : w)$, whose cevian triangle is also the pedal triangle of the point M, lie on the Lucas cubic [1]

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

The points M lie on the Darboux cubic. [2] Isotomic points P and $P^{\wedge}$ on the Lucas cubic have corresponding points M and M' on the Darboux cubic symmetric with respect to the circumcenter. Isogonal points M and M^* on the Darboux cubic have

[1]The Lucas cubic is invariant under the isotomic conjugation and the isotomic conjugate X_{69} of the orthocenter is the pivot point.

[2]The Darboux cubic is invariant under the isogonal conjugation and the pivot point is the DeLongchamps point X_{20}, the reflection of the orthocenter in the circumcenter. It is symmetric with respect to the circumcenter.

corresponding points P and P' on the Lucas cubic with $P' = P^{\wedge * \wedge}$. Here, $()^*$ is the isogonal conjugation with respect to the anticomplementary triangle of ABC. The line PM and MM^* all correspond with the DeLongchamps point X_{20} and so the points P, $P^{\wedge * \wedge}$, M, M^* and X_{20} are collinear. For example, for $P = N_a$, the five points N_a, X_{189}, X_{40}, X_{84}, X_{20} are collinear.

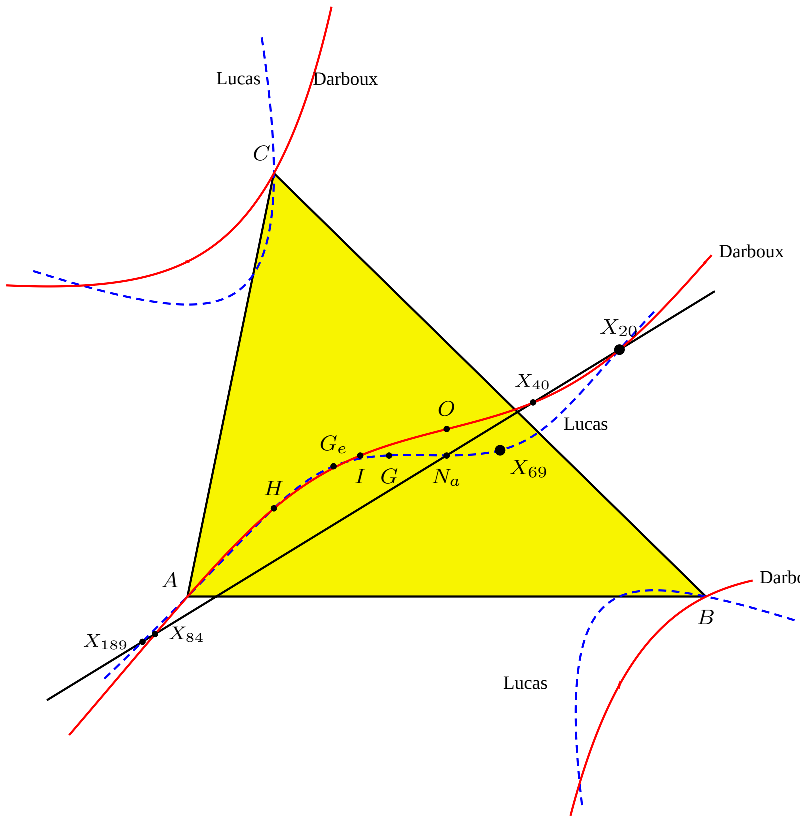

Figure 5. The Darboux and Lucas cubics

4. Further results

Dalcín's example can be extended. The cevian triangle of the Gergonne point G_e is the triangle of tangency of the incircle, the cevian triangle of the Nagel point N_a is the triangle of the inner points of tangency of the excircles. Consider the points of tangency of the excircles with the sidelines:

$A-$excircle	$B_a = (-a+b-c : 0 : a+b+c)$ $C_a = (-a-b+c : a+b+c : 0)$	with CA with AB
$B-$excircle	$A_b = (0 : a-b-c : a+b+c)$ $C_b = (a+b+c : -a-b+c : 0)$	with BC with AB
$C-$excircle	$A_c = (0 : a+b+c : a-b-c)$ $B_c = (a+b+c : 0 : -a+b-c)$	with BC with CA

The point pairs (A_b, A_c), (B_c, B_a) and (C_a, C_b) are symmetric with respect to the corresponding midpoints of the sides. If $XYZ = A_bB_cC_a$, then $X'Y'Z' = A_cB_aC_b$. See Figure 6.

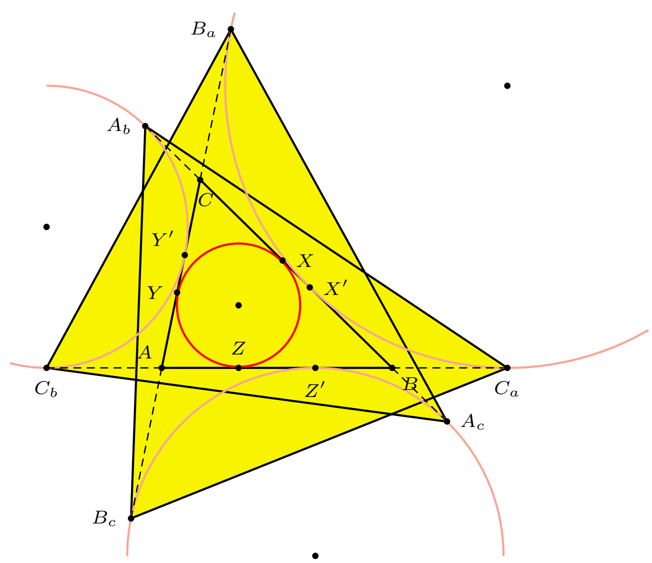

Figure 6

Consider the residual triangles of $A_bB_cC_a$ and those of $A_cB_aC_b$, with the circumcenters. The two congruent triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ have a common area

$$\frac{\triangle}{4} + \frac{(ab+bc+ca)^2}{16\triangle}.$$

The center of perspectivity is

$$Q = (a(b+c) : b(c+a) : c(a+b)) = X_{37}.$$

The center of rotation which maps $O_aO_bO_c$ to $O'_aO'_bO'_c$ is the midpoint of OQ. The point X_{37} of a triangle is the complement of the isotomic conjugate of the incenter. The center of rotation is the common point X_{37} of $O_aO_bO_c$ and $O'_aO'_bO'_c$. The angle of rotation is given by

$$\tan\frac{\varphi}{2} = \frac{ab+bc+ca}{2\triangle} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}.$$

References

[1] M. Dalcín, Isotomic inscribed triangles and their residuals, *Forum Geom.*, 3 (2003) 125–134.
[2] E. Donath: *Die merkwürdigen Punkte und Linien des ebenen Dreiecks*, VEB Deutscher Verlag der Wissenschaften, Berlin 1976.
[3] G. M. Pinkernell, Cubic curves in the triangle plane, *Journal of Geometry*, 55 (1996), 144–161.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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