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Antiorthocorrespondents of Circumconics

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Abstract. This note is a complement to our previous paper [3]. We study how circumconics are transformed under antiorthocorrespondence. This leads us to consider a pencil of pivotal circular cubics which contains in particular the Neuberg cubic of the orthic triangle.

1. Introduction

This paper is a complement to our previous paper [3] on the orthocorrespondence. Recall that in the plane of a given triangle ABC, the orthocorrespondent of a point M is the point $M^{\perp}$ whose trilinear polar intersects the sidelines of ABC at the orthotraces of M. If $M = (p : q : r)$ in homogeneous barycentric coordinates, then [1]

$$M^{\perp} = (p(-pS_A + qS_B + rS_C) + a^2qr : \cdots : \cdots). \tag{1}$$

The antiorthocorrespondents of M consists of the two points M_1 and M_2, not necessarily real, for which $M_1^{\perp} = M = M_2^{\perp}$. We write $M^{\top} = \{M_1, M_2\}$, and say that M_1 and M_2 are orthoassociates. We shall make use of the following basic results.

Lemma 1. *Let $M = (p : q : r)$ and $M^{\top} = \{M_1, M_2\}$.*

(1) *The line M_1M_2* [2] *has equation*

$$S_A(q-r)x + S_B(r-p)y + S_C(p-q)z = 0.$$

It always passes through the orthocenter H, and intersects the line GM at the point

$$\big((b^2-c^2)/(q-r) : \cdots : \cdots\big)$$

on the Kiepert hyperbola.

(2) *The perpendicular bisector ℓ_M of the segment M_1M_2 is the trilinear polar of the isotomic conjugate of the anticomplement of M*, i.e.,

$$(q+r-p)x + (r+p-q)y + (p+q-r)z = 0.$$

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[1]Throughout this paper, we use the same notations in [3]. All coordinates are barycentric coordinates with respect to the reference triangle ABC.

[2]M_1M_2 is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isotomic conjugate of M.

We study how circumconics are transformed under antiorthocorrespondence. Let $P = (u : v : w)$ be a point not lying on the sidelines of ABC. Denote by Γ_P the circumconic with perspector P, namely,

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.$$

This has center [3]

$$G/P = (u(-u+v+w) : v(-v+w+u) : w(u+v-w)),$$

and is the locus of trilinear poles of lines passing through P.

A point $(x : y : z)$ is the orthocorrespondent of a point on Γ_P if and only if

$$\sum_{\text{cyclic}} \frac{u}{x(-xS_A + yS_B + zS_C) + a^2yz} = 0. \tag{2}$$

The antiorthocorrespondent of Γ_P is therefore in general a quartic $\mathcal{Q}_P$. It is easy to check that $\mathcal{Q}_P$ passes through the vertices of the orthic triangle and the pedal triangle of P. It is obviously invariant under orthoassociation, *i.e.*, inversion with respect to the polar circle. See [3, §2]. It is therefore a special case of anallagmatic fourth degree curve.

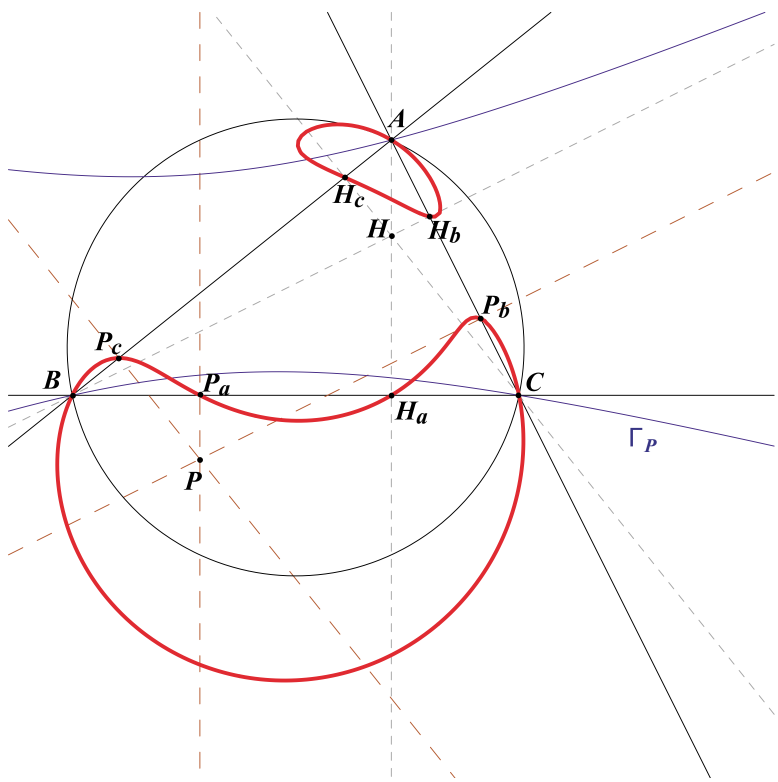

Figure 1. The bicircular circum-quartic $\mathcal{Q}_P$

[3]This is the perspector of the medial triangle and anticevian triangle of P.

The equation of $\mathcal{Q}_P$ can be rewritten as

$$(u+v+w)\mathcal{C}^2 - \left(\sum_{\text{cyclic}} (v+w)S_A\, x\right)\mathcal{L}\mathcal{C} - \left(\sum_{\text{cyclic}} uS_BS_C\, yz\right)\mathcal{L}^2 = 0, \quad (3)$$

with

$$\mathcal{C} = a^2yz + b^2zx + c^2xy, \qquad \mathcal{L} = x+y+z.$$

From this it is clear that $\mathcal{Q}_P$ is a bicircular quartic if and only if $u+v+w \neq 0$; equivalently, Γ_P does not contain the centroid G. We shall study this case in §3 below, and the case $G \in \Gamma_P$ in §4.

2. The conic γ_P

A generic point on the conic Γ_P is

$$M = M(t) = \left(\frac{u}{(v-w)(u+t)} : \frac{v}{(w-u)(v+t)} : \frac{w}{(u-v)(w+t)}\right).$$

As M varies on the circumconic Γ_P, the perpendicular bisector ℓ_M of M_1M_2 envelopes the conic γ_P:

$$\sum((u+v+w)^2 - 4vw)x^2 - 2(u+v+w)(v+w-u)yz = 0.$$

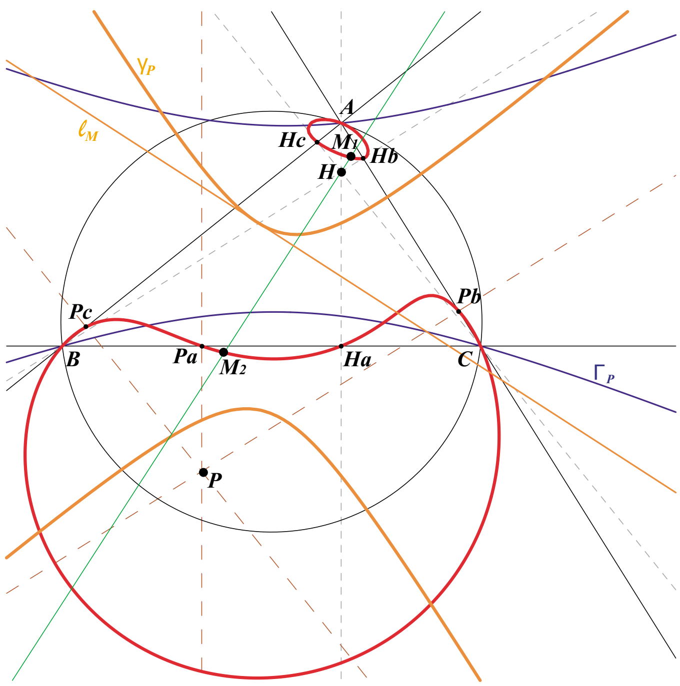

Figure 2. The conic γ_P

The point of tangency of γ_P and the perpendicular bisector of M_1M_2 is

$$T_M = (v(u-v)^2(w+t)^2 + w(u-w)^2(v+t)^2 : \cdots : \cdots).$$

The conic γ_P is called the déférente of Γ_P in [1]. It has center $\omega_P = (2u + v + w : \cdots : \cdots)$, and is homothetic to the circumconic with perspector $((v + w)^2 : (w + u)^2 : (u + v)^2)$. [4] It is therefore a circle when P is the Nagel point or one of its extraversions. This circle is the Spieker circle. We shall see in §3.5 below that $\mathcal{Q}_P$ is an oval of Descartes.

It is clear that γ_P is a parabola if and only if ω_P and therefore P are at infinity. In this case, Γ_P contains the centroid G. See §4 below.

3. Antiorthocorrespondent of a circumconic not containing the centroid

Throughout this section we assume P a finite point so that the circumconic Γ_P does not contain the centroid G.

Proposition 2. *Let ℓ be a line through G intersecting Γ_P at two points M and N. The antiorthocorrespondents of M and N are four collinear points on $\mathcal{Q}_P$.*

Proof. Let M_1, M_2 be the antiorthocorrespondents of M, and N_1, N_2 those of N. By Lemma 1, each of the lines M_1M_2 and N_1N_2 intersects ℓ at the same point on the Kiepert hyperbola. Since they both contain H, M_1M_2 and N_1N_2 are the same line. □

Corollary 3. *Let the medians of ABC meet Γ_P again at A_g, B_g, C_g. The antiorthocorrespondents of these points are the third and fourth intersections of $\mathcal{Q}_P$ with the altitudes of ABC.* [5]

Proof. The antiorthocorrespondents of A are A and H_a. □

In this case, the third and fourth points on AH are symmetric about the second tangent to γ_P which is parallel to BC. The first tangent is the perpendicular bisector of AH_a with contact $(v + w : v : w)$, the contact with this second tangent is $(u(v + w) : uw + (v + w)^2 : uv + (v + w)^2)$ while $A_g = (-u : v + w : v + w)$.

For distinct points P_1 and P_2, the circumconics Γ_{P_1} and Γ_{P_2} have a "fourth" common point T, which is the trilinear pole of the line P_1P_2. Let $T^\top = \{T_1, T_2\}$. The conics Γ_{P_1} and Γ_{P_2} generate a pencil $\mathcal{F}$ consisting of Γ_P for P on the line P_1P_2. The antiorthocorrespondent of every conic $\Gamma_P \in \mathcal{F}$ contains the following 16 points:

(i) the vertices of ABC and the orthic triangle $H_aH_bH_c$,
(ii) the circular points at infinity with multiplicity 4,[6]
(iii) the antiorthocorrespondents $T^\top = \{T_1, T_2\}$.

Proposition 4. *Apart from the circular points at infinity and the vertices of ABC and the orthic triangle, the common points of the quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$ are the antiorthocorrespondents of the trilinear pole of the line P_1P_2.*

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[4] It is inscribed in the medial triangle; its anticomplement is the circumconic with center the complement of P, with perspector the isotomic conjugate of P.

[5] They are not always real when ABC is obtuse angle.

[6] Think of $\mathcal{Q}_{P_1}$ as the union of two circles and $\mathcal{Q}_{P_2}$ likewise. These have at most 8 real finite points and the remaining 8 are the circular points at infinity, each counted with multiplicity 4.

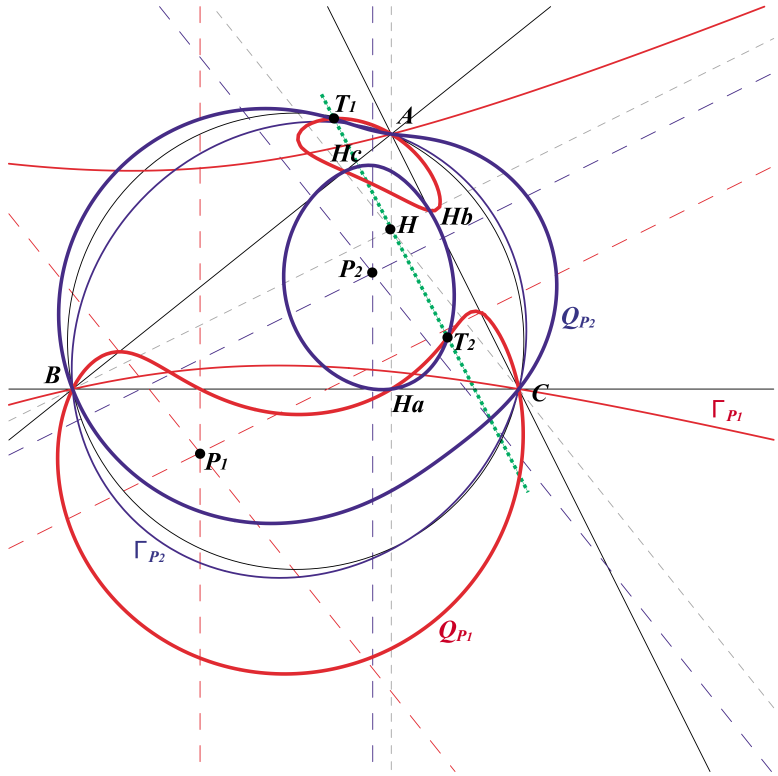

Figure 3. The bicircular quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$

Remarks. 1. T_1 and T_2 lie on the line through H which is the orthocorrespondent of the line GT. See [3, §2.4]. This line T_1T_2 is the directrix of the inscribed (in ABC) parabola tangent to the line P_1P_2.

2. The pencil $\mathcal{F}$ contains three degenerate conics $BC \cup AT$, $CA \cup BT$, and $AB \cup CT$. The antiorthocorrespondent of $BC \cup AT$, for example, degenerates into the circle with diameter BC and another circle through A, H_a, T_1 and T_2 (see [3, Proposition 2]).

3.1. *The points* S_1 *and* S_2. Since $\mathcal{Q}_P$ and the circumcircle have already seven common points, the vertices A, B, C, and the circular points at infinity, each of multiplicity 2, they must have an eighth common point, namely,

$$S_1 = \left(\frac{a^2}{\frac{v}{b^2 S_B} - \frac{w}{c^2 S_C}} : \cdots : \cdots \right), \tag{4}$$

which is the isogonal conjugate of the infinite point of the line

$$\frac{u}{a^2 S_A}x + \frac{v}{b^2 S_B}y + \frac{w}{c^2 S_C}z = 0.$$

Similarly, $\mathcal{Q}_P$ and the nine-point circle also have a real eighth common point

$$S_2 = ((S_B(u-v+w) - S_C(u+v-w))(c^2 S_C v - b^2 S_B w) : \cdots : \cdots), \quad (5)$$

which is the inferior of

$$\left(\frac{a^2}{S_B(u-v+w) - S_C(u+v-w)} : \cdots : \cdots \right)$$

on the circumcircle.

We know that the orthocorrespondent of the circumcircle is the circum-ellipse Γ_O, with center K, the Lemoine point, ([3, §2.6]). If $P \neq O$, this ellipse meets Γ_P at A, B, C and a fourth point

$$S = S(P) = \left(\frac{1}{c^2 S_C v - b^2 S_B w} : \cdots : \cdots \right), \quad (6)$$

which is the trilinear pole of the line OP. The point S lies on the circumcircle if and only if P is on the Brocard axis OK.

Proposition 5. $S^\top = \{S_1, S_2\}$.

Corollary 6. *$S(P) = S(P')$ if and only if P, P' and O are collinear.*

Remark. When $P = O$ (circumcenter), Γ_P is the circum-ellipse with center K. In this case $\mathcal{Q}_P$ decomposes into the union of the circumcircle and the nine point circle.

3.2. *Bitangents.*

Proposition 7. *The points of tangency of the two bitangents to $\mathcal{Q}_P$ passing through H are the antiorthocorrespondents of the points where the polar line of G in Γ_P meets Γ_P.*

Proof. Consider a line ℓ_H through H which is supposed to be tangent to $\mathcal{Q}_P$ at two (orthoassociate) points M and N. The orthocorrespondents of M and N must lie on Γ_P and on the orthocorrespondent of ℓ_H which is a line through G. Since M and N are double points, the line through G must be tangent to Γ_P and MN is the polar of G in Γ_P. □

Remark. M and N are not necessarily real. If $M^\top = \{M_1, M_2\}$ and $N^\top = \{N_1, N_2\}$, the perpendicular bisectors of M_1M_2 and N_1N_2 are the asymptotes of γ_P.[7] The four points M_1, M_2, N_1, N_2 are concyclic and the circle passing through them is centered at ω_P.

Denote by H_1, H_2, H_3 the vertices of the triangle which is self polar in both the polar circle and γ_P. The orthocenter of this triangle is obviously H. For $i = 1,\ 2,\ 3$, let $\mathcal{C}_i$ be the circle centered at H_i orthogonal to the polar circle and Γ_i the circle centered at ω_P orthogonal to $\mathcal{C}_i$. The circle Γ_i intersects $\mathcal{Q}_P$ at the circular points at infinity (with multiplicity 2) and four other points two by two homologous in the inversion with respect to $\mathcal{C}_i$ which are the points of tangency of the (not

[7] The union of the line at infinity and a bitangent is a degenerate circle which is bitangent to $\mathcal{Q}_P$. Its center must be an infinite point of γ_P.

always real) bitangents drawn from H_i to $\mathcal{Q}_P$. The orthocorrespondent of Γ_i is a conic (see [3, §2.6]) intersecting Γ_P at four points whose antiorthocorrespondents are eight points, two by two orthoassociate. Four of them lie on Γ_i and are the required points of tangency. The remaining four are their orthoassociates and they lie on the circle which is the orthoassociate of Γ_i. Figure 4 below shows an example of $\mathcal{Q}_P$ with three pairs of real bitangents.

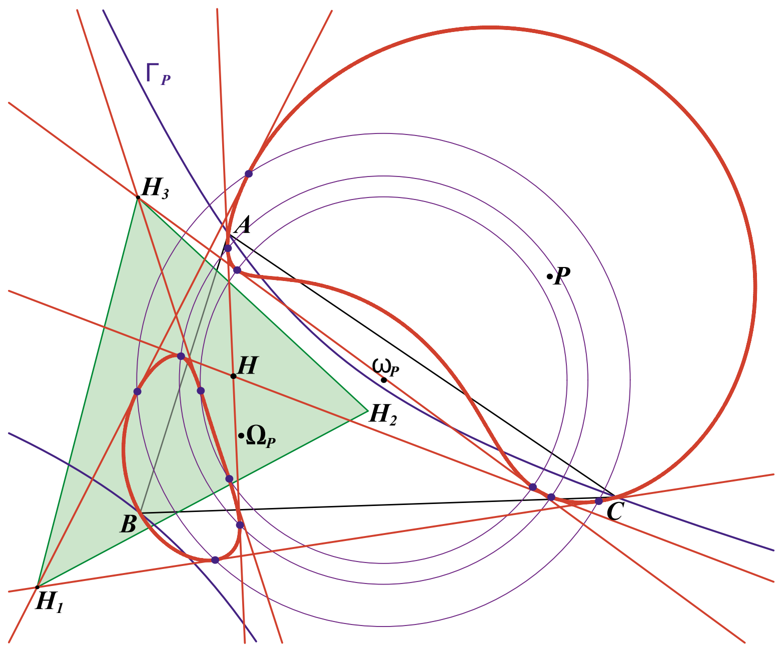

Figure 4. Bitangents to $\mathcal{Q}_P$

Proposition 8. *$\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to BC, CA, AB if and only if $P = H$.*

3.3. *$\mathcal{Q}_P$ as an envelope of circles.*

Theorem 9. *The circle $\mathcal{C}_M$ centered at T_M passing through M_1 and M_2 is bitangent to $\mathcal{Q}_P$ at those points and orthogonal to the polar circle.*

This is a consequence of the following result from [1, tome 3, p.170]. A bicircular quartic is a special case of "plane cyclic curve". Such a curve always can be considered in four different ways as the envelope of circles centered on a conic (déférente) cutting orthogonally a fixed circle. Here the fixed circle is the polar circle with center H, and since M_1 and M_2 are anallagmatic (inverse in the polar circle) and collinear with H, there is a circle passing through M_1, M_2, centered on the déférente, which must be bitangent to the quartic.

Corollary 10. *$\mathcal{Q}_P$ is the envelope of circles $\mathcal{C}_M$, $M \in \Gamma_P$, centered on γ_P and orthogonal to the polar circle.*

Construction. It is easy to draw γ_P since we know its center ω_P. For m on γ_P, draw the tangent t_m at m to γ_P. The perpendicular at m to Hm meets the perpendicular bisector of AH_a at a point which is the center of a circle through A (and H_a). This circle intersects Hm at two points which lie on the circle centered at m and orthogonal to the polar circle. This circle intersects the perpendicular at H to t_m at two points of $\mathcal{Q}_P$.

Corollary 11. *The tangents at M_1 and M_2 to $\mathcal{Q}_P$ are the tangents to the circle $\mathcal{C}_M$ at these points.*

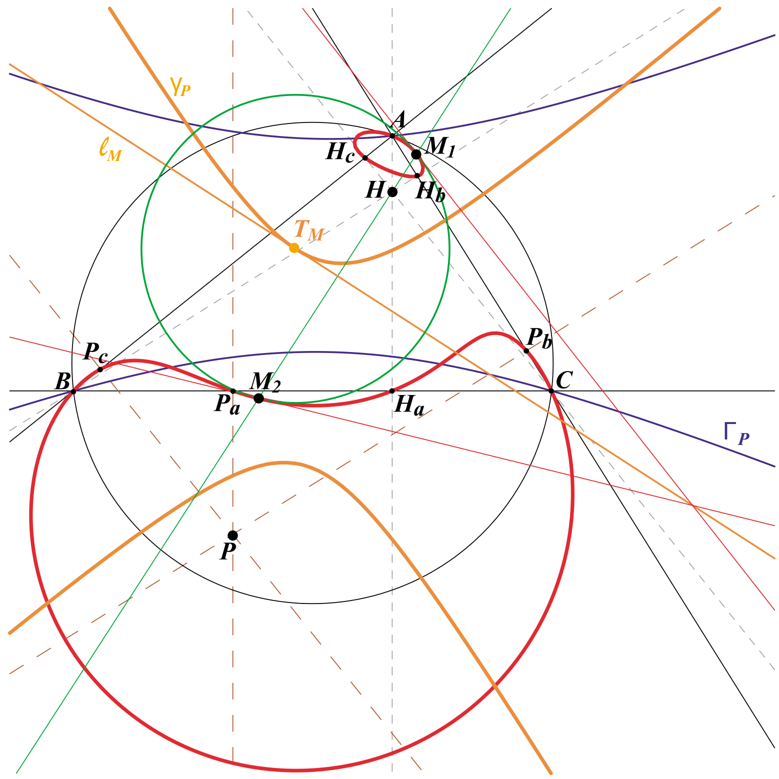

Figure 5. $\mathcal{Q}_P$ as an envelope of circles

3.4. *Inversions leaving $\mathcal{Q}_P$ invariant.*

Theorem 12. *$\mathcal{Q}_P$ is invariant under three other inversions whose poles are the vertices of the triangle which is self-polar in both the polar circle and γ_P.*

Proof. This is a consequence of [1, tome 3, p.172].

Construction: Consider the transformation ϕ which maps any point M of the plane to the intersection M' of the polars of M in both the polar circle and γ_P. Let Σ_a, Σ_b, Σ_c be the conics which are the images of the altitudes AH, BH, CH under ϕ. The conic Σ_a is entirely defined by the following five points:

(1) the point at infinity of BC.
(2) the point at infinity of the polar of H in γ_P.
(3) the foot on BC of the polar of A in γ_P.
(4) the intersection of the polar of H_a in γ_P with the parallel at A to BC.
(5) the pole of AH in γ_P.

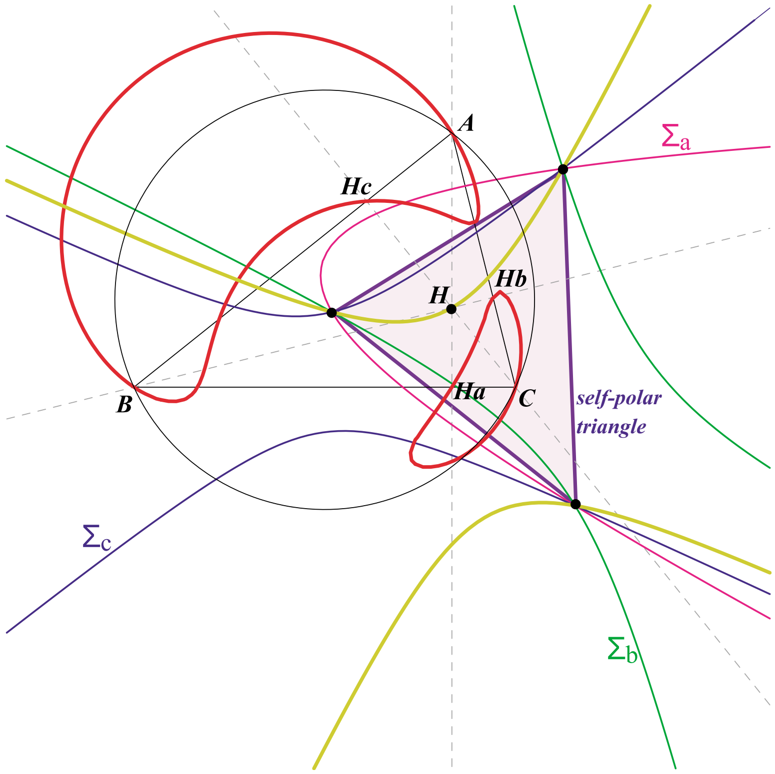

Figure 6. The conics Σ_a, Σ_b, Σ_c

Similarly, we define the conics Σ_b and Σ_c. These conics are in the same pencil and meet at four points: one of them is the point at infinity of the polar of H in γ_P and the three others are the required poles. The circles of inversion are centered at those points and are orthogonal to the polar circle. Their radical axes with the polar circle are the sidelines of the self-polar triangle. □

Another construction is possible : the transformation of the sidelines of triangle ABC under ϕ gives three other conics σ_a, σ_b, σ_c but not defining a pencil since the three lines are not now concurrent. σ_a passes through A, the two points where the trilinear polar of P^+ (anticomplement of P) meets AB and AC, the pole of the line BC in γ_P, the intersection of the parallel at A to BC with the polar of H_a in γ_P. See Figure 7.

Remark. The Jacobian of σ_a, σ_b, σ_c is a degenerate cubic consisting of the union of the sidelines of the self-polar triangle.

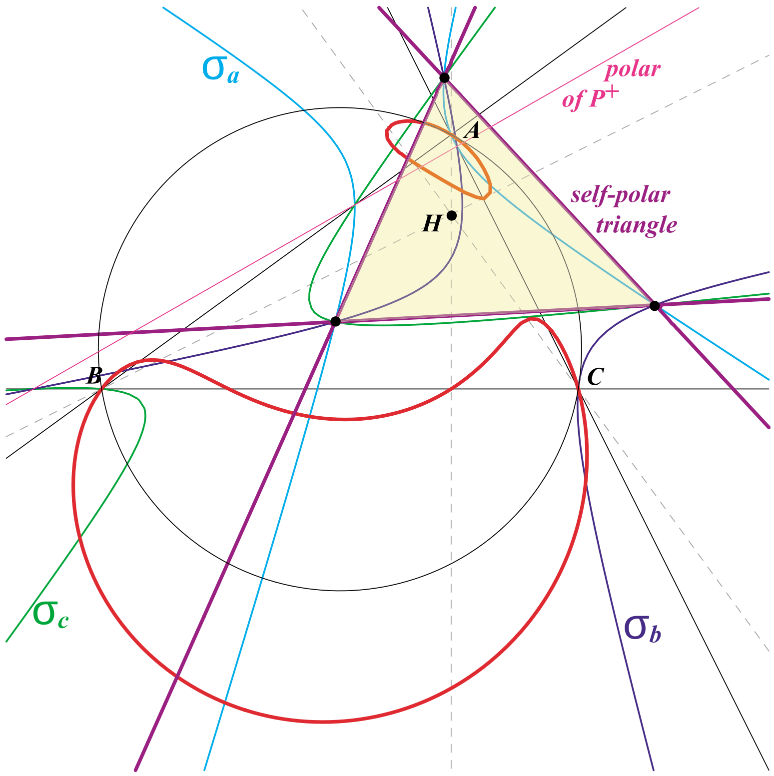

Figure 7. The conics σ_a, σ_b, σ_c

3.5. *Examples.* We provide some examples related to common centers of ABC.

P	S	S_1	S_2	Γ_P	Remark
H	X_{648}	X_{107}	X_{125}		see Figure 8
K	X_{110}	X_{112}	X_{115}	circumcircle	
G	X_{648}	X_{107}	S_{125}	Steiner circum – ellipse	
X_{647}				Jerabek hyperbola	

Remarks. 1. For $P = H$, $\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to the sidelines of ABC. See Figure 8.

2. $P = X_{647}$, the isogonal conjugate of the tripole of the Euler line: Γ_P is the Jerabek hyperbola.

3. $\mathcal{Q}_P$ has two axes of symmetry if and only if P is the point such that $\overrightarrow{OP} = 3\,\overrightarrow{OH}$ (this is a consequence of [1, tome 3, p.172, §15]. Those axes are the parallels at H to the asymptotes of the Kiepert hyperbola. See Figure 9.

4. When $P = X_8$ (Nagel point), γ_P is the incircle of the medial triangle (its center is X_{10} = Spieker center) and Γ_P the circum-conic centered at $\Omega_P = ((b + c - a)(b + c - 3a) : \cdots : \cdots)$. Since the déférente is a circle, $\mathcal{Q}_P$ is now an oval

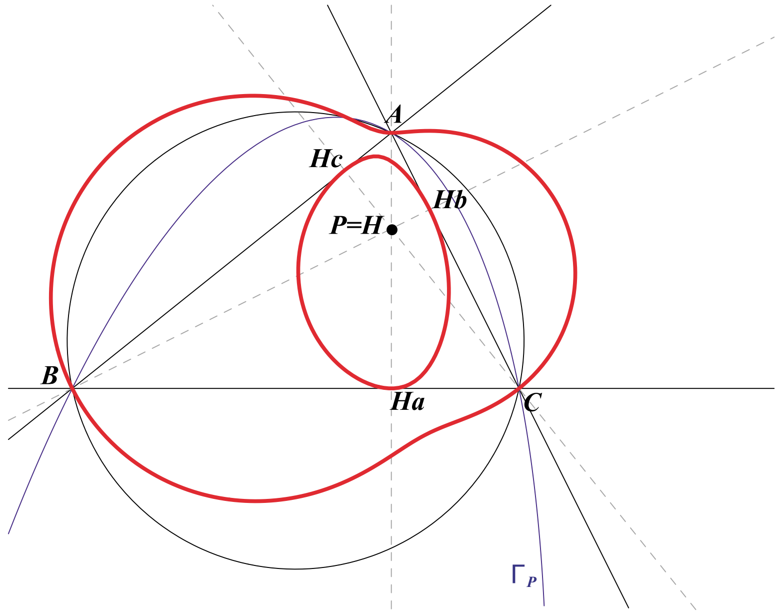

Figure 8. The quartic $\mathcal{Q}_H$

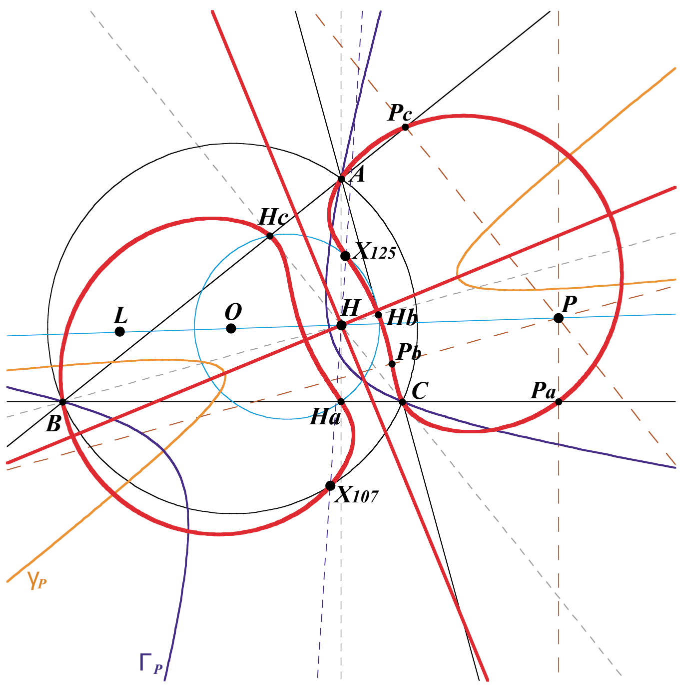

Figure 9. $\mathcal{Q}_P$ with two axes of symmetry

of Descartes (see [1, tome 1, p.8]) with axis the line HX_{10}. We obtain three more ovals of Descartes if X_8 is replaced by one of its extraversions. See Figure 10.

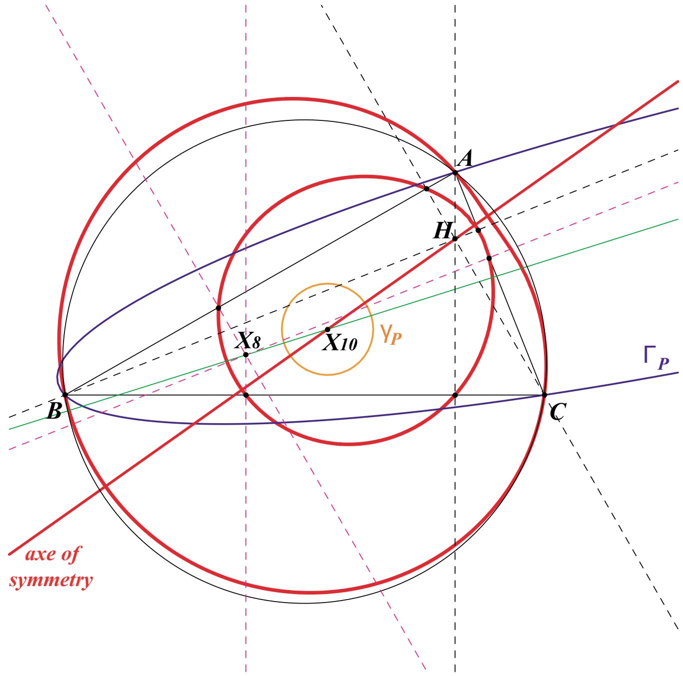

Figure 10. $\mathcal{Q}_P$ as an oval of Descartes

4. Antiorthocorrespondent of a circum-conic passing through G

We consider the case when the circumconic Γ_P contains the centroid G; equivalently, $P = (u : v : w)$ is an infinite point. In this case, Γ_P has center $(u^2 : v^2 : w^2)$ on the inscribed Steiner ellipse. The trilinear polar of points $Q \neq G$ on Γ_P are all parallel, and have infinite point P. It is clear from (3) that the curve $\mathcal{Q}_P$ decomposes into the union of the line at infinity $\mathcal{L}^\infty : x + y + z = 0$ and the cubic $\mathcal{K}_P$

$$\sum x(S_B(S_Au - S_Bv)y^2 - S_C(S_Cw - S_Au)z^2) = 0. \tag{7}$$

This is the pivotal isocubic p$\mathcal{K}(\Omega_P, H)$, with pivot H and pole

$$\Omega_P = \left(\frac{S_Bv - S_Cw}{S_A} : \frac{S_Cw - S_Au}{S_B} : \frac{S_Au - S_Bv}{S_C}\right).$$

Since the orthocorrespondent of the line at infinity is the centroid G, we shall simply say that the antiorthocorrespondent of Γ_P is the cubic $\mathcal{K}_P$. The orthocenter H is the only finite point whose orthocorrespondent is G. We know that $\mathcal{Q}_P$ has already the circular points (counted twice) on $\mathcal{L}^\infty$. This means that the cubic $\mathcal{K}_P$ is also a circular cubic. In fact, equation (7) can be rewritten as

$$\begin{aligned}&(uS_Ax + vS_By + wS_Cz)(a^2yz + b^2zx + c^2xy)\\&+(x + y + z)(uS_{BC}yz + vS_{CA}zx + wS_{AB}xy) = 0.\end{aligned} \tag{8}$$

As P traverses $\mathcal{L}^\infty$, these cubics $\mathcal{K}_P$ form a pencil of circular pivotal isocubics since they all contain A, B, C, H, H_a, H_b, H_c and the circular points at infinity. The poles of these isocubics all lie on the orthic axis.

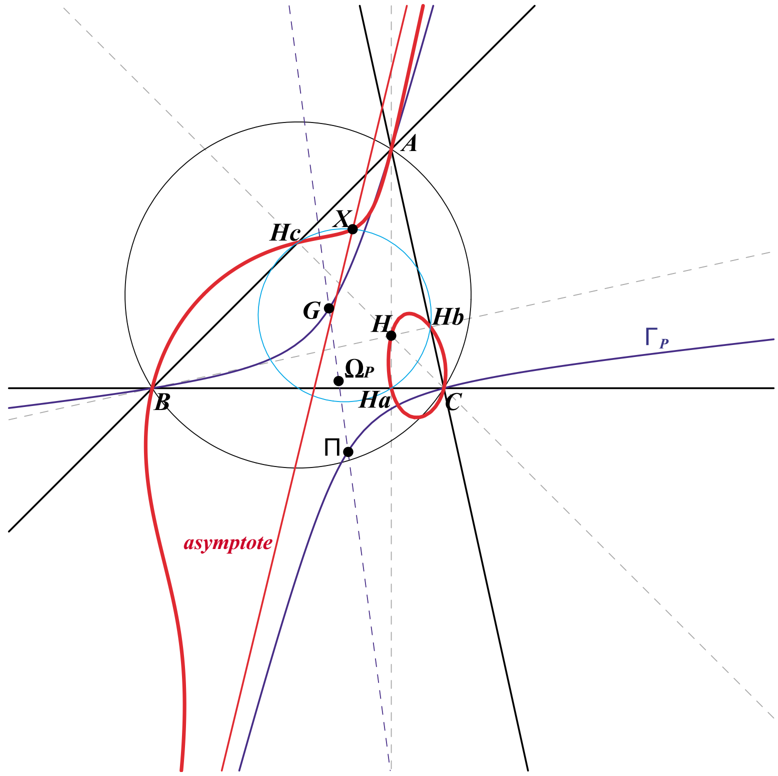

Figure 11. The circular pivotal cubic $\mathcal{K}_P$

4.1. *Properties of* $\mathcal{K}_P$.

(1) $\mathcal{K}_P$ is invariant under orthoassociation: the line through H and M on $\mathcal{K}_P$ meets $\mathcal{K}_P$ again at M' simultaneously the Ω_P-isoconjugate and orthoassociate of M. $\mathcal{K}_P$ is also invariant under the three inversions with poles A, B, C which swap H and H_a, H_b, H_c respectively. [8] See Figure 11.

(2) The real asymptote of $\mathcal{K}_P$ is the line ℓ_P

$$\frac{u}{S_Bv - S_Cw}x + \frac{v}{S_Cw - S_Au}y + \frac{w}{S_Au - S_Bv}z = 0. \tag{9}$$

It has infinite point

$$P' = (S_Bv - S_Cw : S_Cw - S_Au : S_Au - S_Bv),$$

[8] H, H_a, H_b, H_c are often called the centers of anallagmaty of the circular cubic.

and is parallel to the tangents at A, B, C, and H.[9] It is indeed the Simson line of the isogonal conjugate of P. It is therefore tangent to the Steiner deltoid.

(3) The tangents to $\mathcal{K}_P$ at H_a, H_b, H_c are the reflections of those at A, B, C, about the perpendicular bisectors of AH_a, BH_b, CH_c respectively.[10] They concur on the cubic at the point

$$X = \left(\frac{S_Bv - S_Cw}{u}\left(\frac{b^2S_B}{v} - \frac{c^2S_C}{w}\right) : \cdots : \cdots\right),$$

which is also the intersection of ℓ_P and the nine point circle. This is the inferior of the isogonal conjugate of P'. It is also the image of P^*, the isogonal conjugate of P, under the homothety $\mathsf{h}(H, \frac{1}{2})$.

(4) The antipode F of X on the nine point circle is the singular focus of $\mathcal{K}_P$:

$$F = (u(b^2v - c^2w) : v(c^2w - a^2u) : w(a^2u - b^2v)).$$

(5) The orthoassociate Y of X is the "last" intersection of $\mathcal{K}_P$ with the circumcircle, apart from the vertices and the circular points at infinity.

(6) The second intersection of the line XY with the circumcircle is $Z = P^*$.

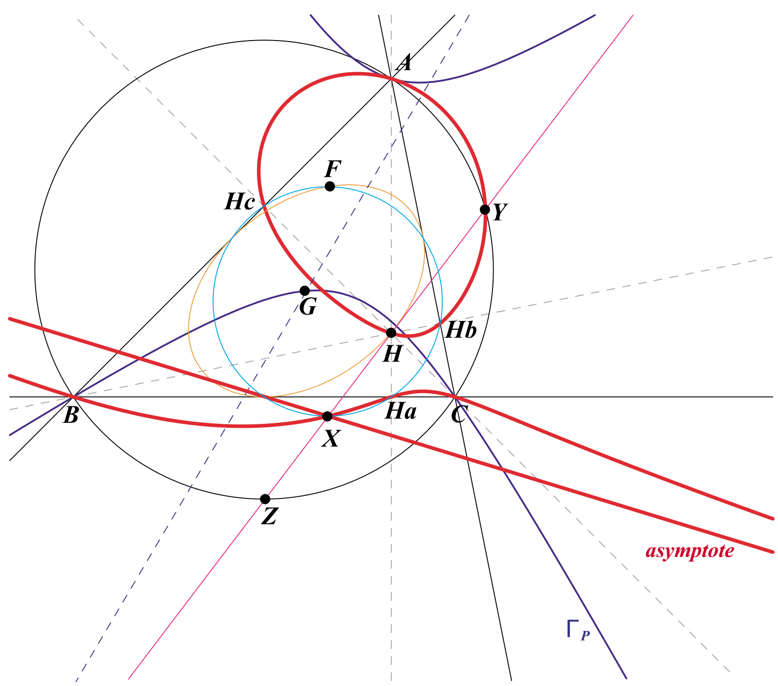

Figure 12. The points X, Y, Z and $\mathcal{K}_P$ for $P = X_{512}$

[9]The latter is the line $uS_Ax + vS_By + wS_Cz = 0$.

[10]These are the lines $S^2ux - (S_Bv - S_Cw)(S_By - S_Cz) = 0$ etc.

(7) $\mathcal{K}_P$ intersects the sidelines of the orthic triangle at three points lying on the cevian lines of Y in ABC.

(8) $\mathcal{K}_P$ is the envelope of circles centered on the parabola $\mathcal{P}_P$ (focus F, directrix the parallel at O to the Simson line of Z) and orthogonal to the polar circle. See Figure 13.

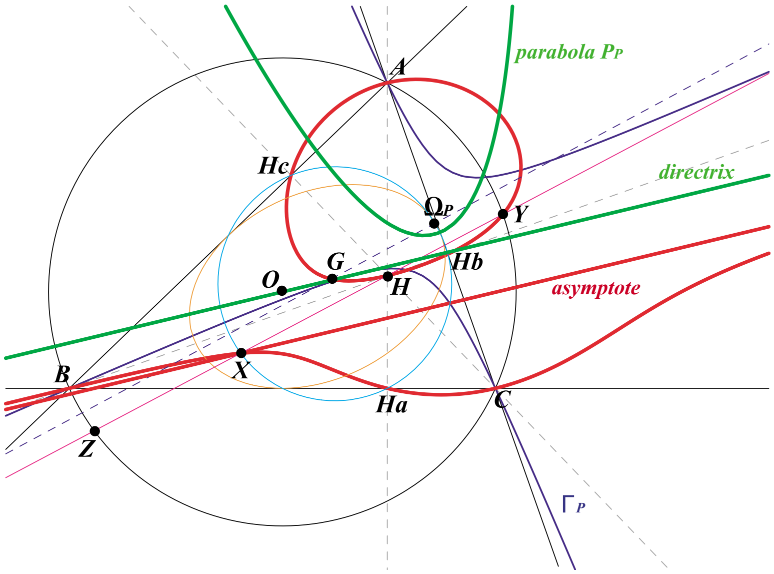

Figure 13. $\mathcal{K}_P$ and the parabola $\mathcal{P}_P$

(9) Γ_P meets the circumcircle again at

$$S = \left(\frac{1}{b^2v - c^2w} : \frac{1}{c^2w - a^2u} : \frac{1}{a^2u - b^2v}\right)$$

and the Steiner circum-ellipse again at

$$R = \left(\frac{1}{v - w} : \frac{1}{w - u} : \frac{1}{u - v}\right).$$

The antiorthocorrespondents of these two points S are four points on $\mathcal{K}_P$. They lie on a same circle orthogonal to the polar circle. See [3, §2.5] and Figure 14.

4.2. *$\mathcal{K}_P$ passing through a given point.* Since all the cubics form a pencil, there is a unique $\mathcal{K}_P$ passing through a given point Q which is not a base-point of the pencil. The circumconic Γ_P clearly contains G and $Q^{\perp}$, the orthocorrespondent of Q. It follows that P is the infinite point of the tripolar of $Q^{\perp}$.

Here is another construction of P. The circumconic through G and $Q^{\perp}$ intersects the Steiner circum-ellipse at a fourth point R. The midpoint M of GR is the center of Γ_P. The anticevian triangle of M is perspective to the medial triangle at P. The lines through their corresponding vertices are parallel to the tangents to

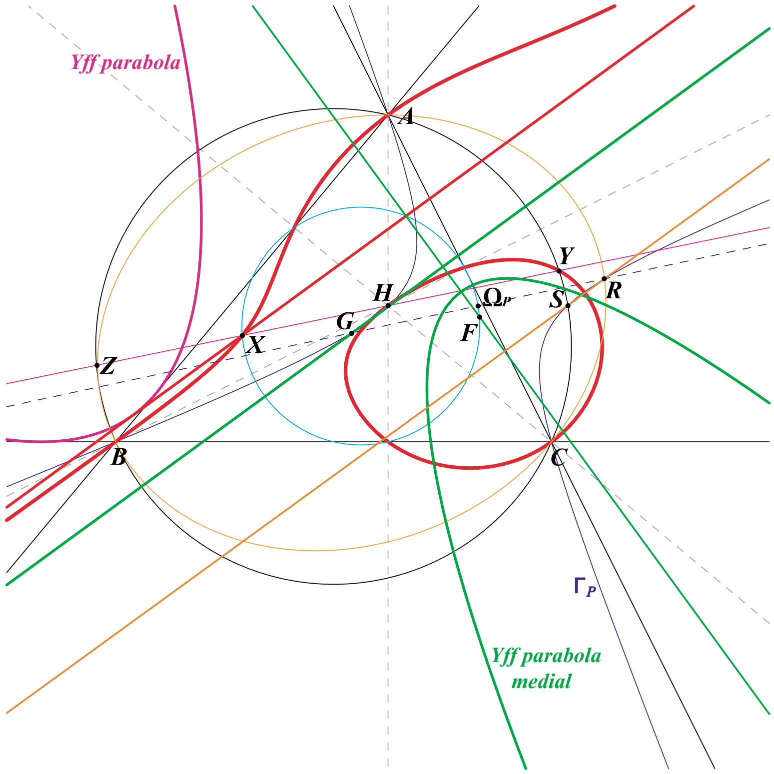

Figure 14. The points R, S and $\mathcal{K}_P$ for $P = X_{514}$

$\mathcal{K}_P$ at A, B, C. The point at infinity of these parallel lines is the point P for which $\mathcal{K}_P$ contains Q.

In particular, if Q is a point on the circumcircle, P is simply the isogonal conjugate of the second intersection of the line HQ with the circumcircle.

4.3. *Some examples and special cases.*

(1) The most remarkable circum-conic through G is probably the Kiepert rectangular hyperbola with perspector $P = X_{523}$, point at infinity of the orthic axis. Its antiorthocorrespondent is p$\mathcal{K}(X_{1990}, H)$, identified as the orthopivotal cubic $\mathcal{O}(H)$ in [3, §6.2.1]. See Figure 15.

(2) With $P =$ isogonal conjugate of X_{930} [11], $\mathcal{K}_P$ is the Neuberg cubic of the orthic triangle. We have $F = X_{137}$, $X = X_{128}$, $Y =$ isogonal conjugate of X_{539}, $Z = X_{930}$. The cubic contains X_5, X_{15}, X_{16}, X_{52}, X_{186}, X_{1154} (at infinity). See Figure 16.

(3) $\mathcal{K}_P$ degenerates when P is the point at infinity of one altitude. For example, with the altitude AH, $\mathcal{K}_P$ is the union of the sideline BC and the circle through A, H, H_b, H_c.

[11] $P = (a^2(b^2 - c^2)(4S_A^2 - 3b^2c^2) : \cdots : \cdots)$. The point X_{930} is the anticomplement of X_{137} which is X_{110} of the orthic triangle.

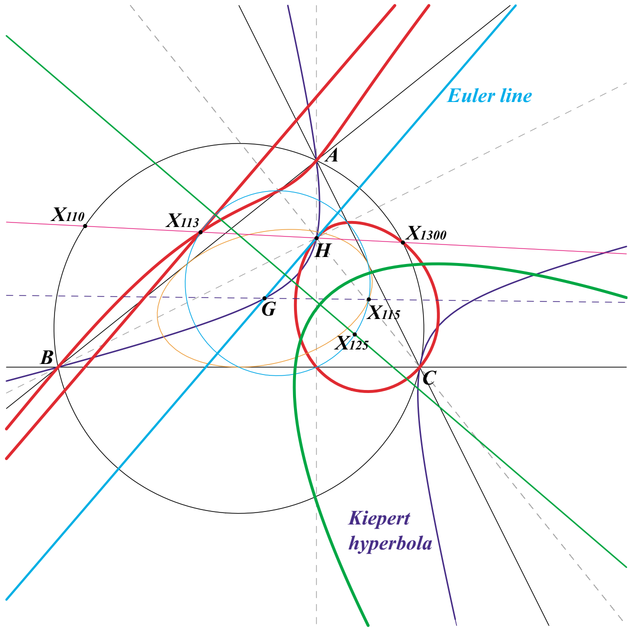

Figure 15. $\mathcal{O}(H)$ or $\mathcal{K}_P$ for $P = X_{523}$

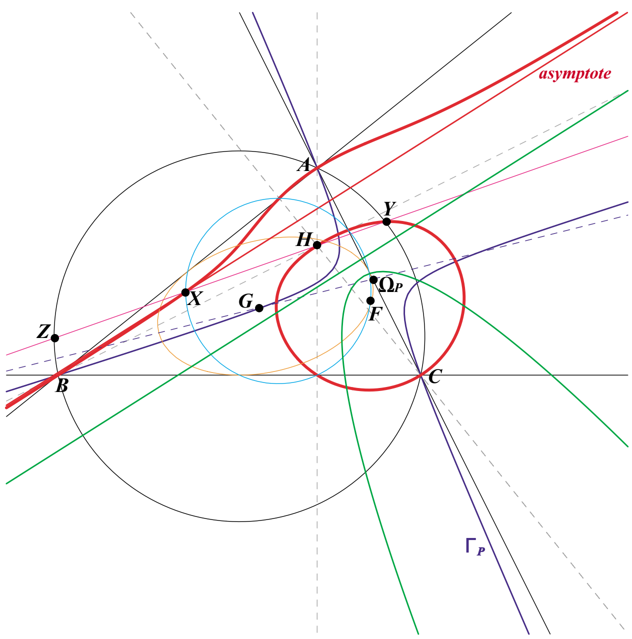

Figure 16. $\mathcal{K}_P$ as the Neuberg cubic of the orthic triangle

(4) $\mathcal{K}_P$ is a focal cubic if and only if P is the point at infinity of one tangent to the circumcircle at A, B, C. For example, with A, $\mathcal{K}_P$ is the focal cubic

denoted $\mathcal{K}_a$ with singular focus H_a and pole the intersection of the orthic axis with the symmedian AK. The tangents at A, B, C, H are parallel to the line OA. Γ_P is the isogonal conjugate of the line passing through K and the midpoint of BC. $\mathcal{P}_P$ is the parabola with focus H_a and directrix the line OA.

$\mathcal{K}_a$ is the locus of point M from which the segments BH_b, CH_c are seen under equal or supplementary angles. It is also the locus of contacts of tangents drawn from H_a to the circles centered on H_bH_c and orthogonal to the circle with diameter H_bH_c. See Figure 17.

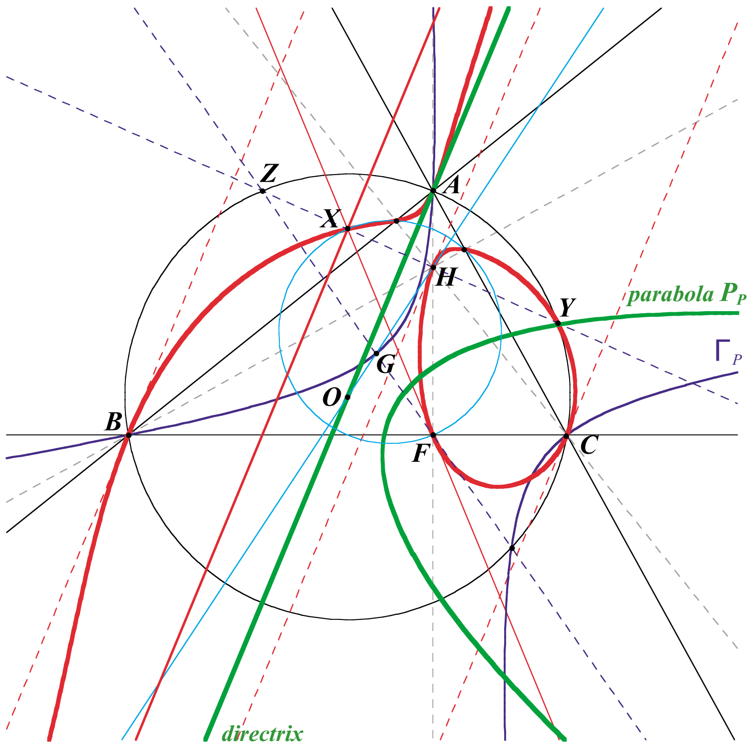

Figure 17. The focal cubic $\mathcal{K}_a$

4.4. *Conclusion.* We conclude with the following table showing the repartition of the points we met in the study above in some particular situations. Recall that P', X, Y always lie on $\mathcal{K}_P$, Y, Z, S on the circumcircle, X, F on the nine point circle, R on the Steiner circum-ellipse. When the point is not mentioned in [6], its first barycentric coordinate is given, as far as it is not too complicated. M^* denotes the isogonal conjugate of M, and $M^{\#}$ denotes the isotomic conjugate of M.

P	P'	X	Y	Z	F	S	R	Remark
X_{30}	X_{523}	X_{125}	X_{107}	X_{74}	X_{113}	X_{1302}	X_{648}	
X_{523}	X_{30}	X_{113}	X_{1300}	X_{110}	X_{125}	X_{98}	X_{671}	(1)
X_{514}	X_{516}	X_{118}	X_{917}	X_{101}	X_{116}	X_{675}	X_{903}	(2)
X_{511}	X_{512}	X_{115}	X_{112}	X_{98}	X_{114}	X_{110}	M_1	
X_{512}	X_{511}	X_{114}	M_2	X_{99}	X_{115}	X_{111}	$X^{\#}_{538}$	(3)
X_{513}	X_{517}	X_{119}	X_{915}	X_{100}	X_{11}	X_{105}	$X^{\#}_{536}$	(4)
X_{524}	X_{1499}	M_3	M_4	X_{111}	X_{126}	X_{99}	X_{99}	
X_{520}	X^{*}_{1294}	X_{133}	X_{74}	X_{107}	X_{122}	X_{1297}		
X_{525}	X_{1503}	X_{132}	X_{98}	X_{112}	X_{127}	$X^{\#}_{858}$	$X^{\#}_{30}$	
X^{*}_{930}	X_{1154}	X_{128}	X^{*}_{539}	X_{930}	X_{137}			
X_{515}	X_{522}	X_{124}	M_5	X_{102}	X_{117}			
X_{516}	X_{514}	X_{116}	M_6	X_{103}	X_{118}		M_7	

Remarks. (1) $\Omega_P = X_{115}$. Γ_P is the Kiepert hyperbola. $\mathcal{P}_P$ is the Kiepert parabola of the medial triangle with directrix the Euler line. See Figure 15.

(2) $\Omega_P = X_{1086}$. $\mathcal{P}_P$ is the Yff parabola of the medial triangle. See Figure 14.

(3) $\Omega_P = X_{1084}$. The directrix of $\mathcal{P}_P$ is the Brocard line.

(4) $\Omega_P = X_{1015}$. The directrix of $\mathcal{P}_P$ is the line OI.

The points $M_1, \dots, M_7$ are defined by their first barycentric coordinates as follows.

M_1	$1/[(b^2-c^2)(a^2S_A+b^2c^2)]$
M_2	$a^2/[S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))]$
M_3	$(b^2-c^2)^2(b^2+c^2-5a^2)(b^4+c^4-a^4-4b^2c^2)$
M_4	$1/[S_A(b^2-c^2)(b^4+c^4-a^4-4b^2c^2)]$
M_5	$S_A(b-c)(b^3+c^3-a^2b-a^2c+abc)$
M_6	$1/[S_A(b-c)(b^2+c^2-ab-ac+bc)]$
M_7	$1/[(b-c)(3b^2+3c^2-a^2-2ab-2ac+2bc)]$

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