Editorial Board

Advisors:
John H. Conway Princeton, New Jersey, USA
Julio Gonzalez Cabillon Montevideo, Uruguay
Richard Guy Calgary, Alberta, Canada
George Kapetis Thessaloniki, Greece
Clark Kimberling Evansville, Indiana, USA
Kee Yuen Lam Vancouver, British Columbia, Canada
Tsit Yuen Lam Berkeley, California, USA
Fred Richman Boca Raton, Florida, USA

Editor-in-chief:
Paul Yiu Boca Raton, Florida, USA

Editors:
Clayton Dodge Orono, Maine, USA
Roland Eddy St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann Paris, France
Lawrence Evans La Grange, Illinois, USA
Chris Fisher Regina, Saskatchewan, Canada
Rudolf Fritsch Munich, Germany
Bernard Gibert St Etiene, France
Antreas P. Hatzipolakis Athens, Greece
Michael Lambrou Crete, Greece
Floor van Lamoen Goes, Netherlands
Fred Pui Fai Leung Singapore, Singapore
Daniel B. Shapiro Columbus, Ohio, USA
Steve Sigur Atlanta, Georgia, USA
Man Keung Siu Hong Kong, China
Peter Woo La Mirada, California, USA

Technical Editors:
Yuandan Lin Boca Raton, Florida, USA
Aaron Meyerowitz Boca Raton, Florida, USA
Xiao-Dong Zhang Boca Raton, Florida, USA

Consultants:
Frederick Hoffman Boca Raton, Florida, USA
Stephen Locke Boca Raton, Florida, USA
Heinrich Niederhausen Boca Raton, Florida, USA
Table of Contents

Bernard Gibert, Orthocorrespondence and orthopivotal cubics, 1
Alexei Myakishev, On the procircumcenter and related points, 29
Clark Kimberling, Bicentric pairs of points and related triangle centers, 35
Lawrence Evans, Some configurations of triangle centers, 49
Alexei Myakishev and Peter Woo, On the circumcenters of cevasix configurations, 57
Floor van Lamoen, Napoleon triangles and Kiepert perspectors, 65
Paul Yiu, On the Fermat lines, 73
Milorad Stevanović, Triangle centers associated with the Malfatti circles, 83
Wilfred Reyes, The Lucas circles and the Descartes formula, 95
Jean-Pierre Ehrmann, Similar pedal and cevian triangles, 101
Darij Grinberg, On the Kosnita point and the reflection triangle, 105
Lev Emelyanov and Tatiana Emelyanova, A note on the Schiffler point, 113
Nikolaos Dergiades and Juan Carlos Salazar, Harcourt’s theorem, 117
Mario Dalcín, Isotomic inscribed triangles and their residuals, 125
Alexei Myakishev, The M-configuration of a triangle, 135
Nikolaos Dergiades and Floor van Lamoen, Rectangles attached to the sides of a triangle, 145
Charles Thas, A generalization of the Lemoine point, 161
Bernard Gibert and Floor van Lamoen, The parasix configuration and orthocorrespondence, 169
Lawrence Evans, A tetrahedral arrangement of triangle centers, 181
Milorad Stevanović, The Apollonius circle and related triangle centers, 187
Milorad Stevanović, Two triangle centers associated with the excircles, 197
Kurt Hofstetter, A 5-step division of a segment in the golden section, 205
Eckart Schmidt, Circumcenters of residual triangles, 207
Floor van Lamoen, Circumrhombi, 215
Jean-Louis Ayme, Sawayama and Thébault’s theorem, 225
Bernard Gibert, Antiorthocorrespondents of Circumconics, 231
Author Index, 251
Orthocorrespondence and Orthopivotal Cubics

Bernard Gibert

Abstract. We define and study a transformation in the triangle plane called the orthocorrespondence. This transformation leads to the consideration of a family of circular circumcubics containing the Neuberg cubic and several hitherto unknown ones.

1. The orthocorrespondence

Let $P$ be a point in the plane of triangle $ABC$ with barycentric coordinates $(u : v : w)$. The perpendicular lines at $P$ to $AP$, $BP$, $CP$ intersect $BC$, $CA$, $AB$ respectively at $P_a$, $P_b$, $P_c$, which we call the orthotraces of $P$. These orthotraces lie on a line $L_P$, which we call the orthotransversal of $P$.\(^1\) We denote the trilinear pole of $L_P$ by $P^\perp$, and call it the orthocorrespondent of $P$.

![Figure 1. The orthotransversal and orthocorrespondent](image)

In barycentric coordinates,\(^2\)

$$P^\perp = (u(-uS_A + vS_B + wS_C) + a^2vw : \cdots : \cdots), \quad (1)$$

\(^1\)The homography on the pencil of lines through $P$ which swaps a line and its perpendicular at $P$ is an involution. According to a Desargues theorem, the points are collinear.

\(^2\)All coordinates in this paper are homogeneous barycentric coordinates. Often for triangle centers, we list only the first coordinate. The remaining two can be easily obtained by cyclically permuting $a$, $b$, $c$, and corresponding quantities. Thus, for example, in (1), the second and third coordinates are $v(-vS_B + wS_C + uS_A) + b^2wu$ and $w(-wS_C + uS_A + vS_B) + c^2uv$ respectively.
where, \(a, b, c\) are respectively the lengths of the sides \(BC, CA, AB\) of triangle \(ABC\), and, in J.H. Conway’s notations,
\[
S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2).
\]
(2)
The mapping \(\Phi : P \mapsto P^\perp\) is called the orthocorrespondence (with respect to triangle \(ABC\)).

Here are some examples. We adopt the notations of [5] for triangle centers, except for a few commonest ones. Triangle centers without an explicit identification as \(X_n\) are not in the current edition of [5].

(1) \(I^\perp = X_{57}\), the isogonal conjugate of the Mittenpunkt \(X_9\).
(2) \(G^\perp = (b^2 + c^2 - 5a^2 : \cdots : \cdots)\) is the reflection of \(G\) about \(K\), and the orthotransversal is perpendicular to \(GK\).
(3) \(H^\perp = G\).
(4) \(O^\perp = (\cos^2 A : \cos^2 B : \cos^2 C)\) on the line \(GK\).
(5) More generally, the orthocorrespondent of the Euler line is the line \(GK\).
The orthotransversal envelopes the Kiepert parabola.
(6) \(K^\perp = (a^2(b^4 + c^4 - a^4 - 4b^2c^2) : \cdots : \cdots)\) on the Euler line.
(7) \(X_{15}^{115} = X_{62}\) and \(X_{16}^{115} = X_{61}\).
(8) \(X_{112}^\perp = X_{115}^{11} = X_{110}\).

See §2.3 for points on the circumcircle and the nine-point circle with orthocorrespondents having simple barycentric coordinates.

Remarks. (1) While the geometric definition above of \(P^\perp\) is not valid when \(P\) is a vertex of triangle \(ABC\), by (1) we extend the orthocorrespondence \(\Phi\) to cover these points. Thus, \(A^\perp = A, B^\perp = B,\) and \(C^\perp = C\).

(2) The orthocorrespondent of \(P\) is not defined if and only if the three coordinates of \(P^\perp\) given in (1) are simultaneously zero. This is the case when \(P\) belongs to the three circles with diameters \(BC, CA, AB\).\(^3\) There are only two such points, namely, the circular points at infinity.

(3) We denote by \(P^*\) the isogonal conjugate of \(P\) and by \(H/P\) the cevian quotient of \(H\) and \(P\).\(^4\) It is known that
\[
H/P = (u(-uS_A + vS_B + wS_C) : \cdots : \cdots).
\]
This shows that \(P^\perp\) lies on the line through \(P^*\) and \(H/P\). In fact,
\[
(H/P)P^\perp : (H/P)P^* = a^2vw + b^2wu + c^2uv : S_Au^2 + S_Bv^2 + S_Cw^2.
\]
In [6], Jim Parish claimed that this line also contains the isogonal conjugate of \(P\) with respect to its anticevian triangle. We add that this point is in fact the harmonic conjugate of \(P^\perp\) with respect to \(P^*\) and \(H/P\). Note also that the line through \(P\) and \(H/P\) is perpendicular to the orthotransversal \(L_P\).

(4) The orthocorrespondent of any (real) point on the line at infinity \(L^\infty\) is \(G\).

\(^3\)See Proposition 2 below.
\(^4\)\(H/P\) is the perspector of the cevian triangle of \(H\) (orthic triangle) and the anticevian triangle of \(P\).
A straightforward computation shows that the orthocorrespondence $\Phi$ has exactly five fixed points. These are the vertices $A$, $B$, $C$, and the two Fermat points $X_{13}$, $X_{14}$. Jim Parish [7] and Aad Goddijn [2] have given nice synthetic proofs of this in answering a question of Floor van Lamoen [3]. In other words, $X_{13}$ and $X_{14}$ are the only points whose orthotransversal and trilinear polar coincide.

**Theorem 1.** The orthocorrespondent $P_{\perp}$ is a point at infinity if and only if $P$ lies on the Monge (orthoptic) circle of the inscribed Steiner ellipse.

**Proof.** From (1), $P_{\perp}$ is a point at infinity if and only if
\[\sum_{\text{cyclic}} S_A x^2 - 2a^2 y z = 0.\] (3)
This is a circle in the pencil generated by the circumcircle and the nine-point circle, and is readily identified as the Monge circle of the inscribed Steiner ellipse. $\square$

It is obvious that $P_{\perp}$ is at infinity if and only if $L_P$ is tangent to the inscribed Steiner ellipse. $\square$

**Proposition 2.** The orthocorrespondent $P_{\perp}$ lies on the sideline $BC$ if and only if $P$ lies on the circle $\Gamma_{BC}$ with diameter $BC$. The perpendicular at $P$ to $AP$ intersects $BC$ at the harmonic conjugate of $P_{\perp}$ with respect to $B$ and $C$.

**Proof.** $P_{\perp}$ lies on $BC$ if and only if its first barycentric coordinate is 0, i.e., if and only if $u(-uS_A + vS_B + wS_C) + a^2 vw = 0$ which shows that $P$ must lie on $\Gamma_{BC}$. $\square$

2. Orthoassociates and the critical conic

2.1. Orthoassociates and antiorthocorrespondents.

**Theorem 3.** Let $Q$ be a finite point. There are exactly two points $P_1$ and $P_2$ (not necessarily real nor distinct) such that $Q = P_{\perp_1} = P_{\perp_2}$.

**Proof.** Let $Q$ be a finite point. The trilinear polar $\ell_Q$ of $Q$ intersects the sidelines of triangle $ABC$ at $Q_a$, $Q_b$, $Q_c$. The circles $\Gamma_a$, $\Gamma_b$, $\Gamma_c$ with diameters $AQ_a$, $BQ_b$, $CQ_c$ are in the same pencil of circles since their centers $O_a$, $O_b$, $O_c$ are collinear (on the Newton line of the quadrilateral formed by the sidelines of $ABC$ and $\ell_Q$), and since they are all orthogonal to the polar circle. Thus, they have two points $P_1$ and $P_2$ in common. These points, if real, satisfy $P_{\perp_1} = Q = P_{\perp_2}$. $\square$

We call $P_1$ and $P_2$ the antiorthocorrespondents of $Q$ and write $Q^T = \{P_1, P_2\}$. We also say that $P_1$ and $P_2$ are orthoassociates, since they share the same orthocorrespondent and the same orthotransversal. Note that $P_1$ and $P_2$ are homologous.

---

5 The Monge (orthopt) circle of a conic is the locus of points whose two tangents to the conic are perpendicular to each other. It has the same center of the conic. For the inscribed Steiner ellipse, the radius of the Monge circle is $\sqrt{2} \sqrt{a^2 + b^2 + c^2}$.  

6 The trilinear polar of a point at infinity is tangent to the in-Steiner ellipse since it is the in-conic with perspector $G$.  

7 $P_1$ and $P_2$ are not always real when $ABC$ is obtuse angled, see §2.2 below.
under the inversion $\iota_H$ with pole $H$ which swaps the circumcircle and the nine-point circle.

**Proposition 4.** The orthoassociate $\mathcal{T}$ of $P(u : v : w)$ has coordinates

$$
\left( \frac{S_B v^2 + S_C w^2 - S_A u(v + w)}{S_A}, \frac{S_C w^2 + S_A u^2 - S_B v(w + u)}{S_B}, \frac{S_A u^2 + S_B v^2 - S_C w(u + v)}{S_C} \right). 
$$

Let $S$ denote twice of the area of triangle $ABC$. In terms of $S_A$, $S_B$, $S_C$ in (2), we have

$$
S^2 = S_A S_B + S_B S_C + S_C S_A.
$$

**Proposition 5.** Let

$$
K(u, v, w) = S^2(u + v + w)^2 - 4(a^2S_Avw + b^2S_Bwu + c^2S_Cuv).
$$
The antiorthocorrespondents of $Q = (u : v : w)$ are the points with barycentric coordinates

$((u-w)(u+v-w)S_B + (u-v)(u-v+w)S_C \pm \sqrt{K(u,v,w)}}{S}((u-w)S_B + (u-v)S_C) : \cdots : \cdots)$. (5)

These are real points if and only if $K(u, v, w) \geq 0$.

2.2. The critical conic $C$. Consider the critical conic $C$ with equation

$$S^2(x + y + z)^2 - 4 \sum_{\text{cyclic}} a^2 S_Ayz = 0,$$

which is degenerate, real, imaginary according as triangle $ABC$ is right-, obtuse-, or acute-angled. It has center the Lemoine point $K$, and the same infinite points as the circumconic

$$a^2 S_Ayz + b^2 S_Bzx + c^2 S_Cxy = 0,$$

which is the isogonal conjugate of the orthic axis $S_Ax + S_By + S_Cz = 0$, and has the same center $K$. This critical conic is a hyperbola when it is real. Clearly, if $Q$ lies on the critical conic, its two real antiorthocorrespondents coincide.

Figure 3. The critical conic
Proposition 6. The antiorthocorrespondents of Q are real if and only if one of the following conditions holds.
(1) Triangle ABC is acute-angled.
(2) Triangle ABC is obtuse-angled and Q lies in the component of the critical hyperbola not containing the center K.

Proposition 7. The critical conic is the orthocorrespondent of the polar circle. When it is real, it intersects each sideline of ABC at two points symmetric about the corresponding midpoint. These points are the orthocorrespondents of the intersections of the polar circle and the circles Γ_{BC}, Γ_{CA}, Γ_{AB} with diameters BC, CA, AB.

2.3. Orthocorrespondent of the circumcircle. Let P be a point on the circumcircle. Its orthotransversal passes through O, and P⊥ lies on the circumconic centered at K.

<table>
<thead>
<tr>
<th>P</th>
<th>P*</th>
<th>P⊥</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_{74}</td>
<td>X_{30}</td>
<td>X_{133}</td>
</tr>
<tr>
<td>X_{98}</td>
<td>X_{511}</td>
<td>X_{132}</td>
</tr>
<tr>
<td>X_{99}</td>
<td>X_{512}</td>
<td>(b^2 - c^2)(S_A - a^2)/S_A</td>
</tr>
<tr>
<td>X_{100}</td>
<td>X_{513}</td>
<td>aS_A/(b - c)</td>
</tr>
<tr>
<td>X_{101}</td>
<td>X_{514}</td>
<td>a^2S_A/(b - c)</td>
</tr>
<tr>
<td>X_{105}</td>
<td>X_{518}</td>
<td>a^2S_A/(b + c^2 - ab - ac)</td>
</tr>
<tr>
<td>X_{106}</td>
<td>X_{519}</td>
<td>a^2S_A/(b + c - 2a)</td>
</tr>
<tr>
<td>X_{107}</td>
<td>X_{520}</td>
<td>X_{125}</td>
</tr>
<tr>
<td>X_{108}</td>
<td>X_{521}</td>
<td>X_{11}</td>
</tr>
<tr>
<td>X_{109}</td>
<td>X_{522}</td>
<td>X_{110}</td>
</tr>
<tr>
<td>X_{110}</td>
<td>X_{523}</td>
<td>X_{111}</td>
</tr>
<tr>
<td>X_{111}</td>
<td>X_{524}</td>
<td>X_{112}</td>
</tr>
<tr>
<td>X_{112}</td>
<td>X_{525}</td>
<td>X_{113}</td>
</tr>
<tr>
<td>X_{113}</td>
<td>X_{526}</td>
<td>X_{114}</td>
</tr>
<tr>
<td>X_{114}</td>
<td>X_{527}</td>
<td>X_{230}</td>
</tr>
</tbody>
</table>

Remark. The coordinates of P_1 can be obtained from those of X_{230} by making use of the fact that X_{230} is the barycentric product of P_1 and X_{69}. Thus,

\[
P_1 = \left( \frac{a^2}{S_A((b^2 - c^2)^2 - a^2(b^2 + c^2 - 2a^2))}, \ldots, \ldots \right).
\]

8If P = (u : v : w) lies on the circumcircle, then P⊥ = (uS_A : vS_B : wS_C) is the barycentric product of P and X_{69}. See [9]. The orthotransversal is the line \( \frac{1}{uS_A} + \frac{1}{vS_B} + \frac{1}{wS_C} = 0 \) which contains O.

9The isogonal conjugates are trivially infinite points.
2.4. The orthocorrespondent of a line. The orthocorrespondent of a sideline, say $BC$, is the circumconic through $G$ and its projection on the corresponding altitude. The orthoassociate is the circle with the segment $AH$ as diameter.

Consider a line $\ell$ intersecting $BC, CA, AB$ at $X, Y, Z$ respectively. The orthocorrespondent $\ell^\perp$ of $\ell$ is a conic containing the centroid $G$ (the orthocorrespondent of the infinite point of $\ell$) and the points $X^\perp, Y^\perp, Z^\perp$. A fifth point can be constructed as $P^\perp$, where $P$ is the pedal of $G$ on $\ell$. These five points entirely determine the conic. According to Proposition 2, $\ell^\perp$ meets $BC$ at the orthocorrespondents of the points where $\ell$ intersects the circle $\Gamma_{BC}$. It is also the orthocorrespondent of the circle through $H$ which is the orthoassociate of $\ell$.

If the line $\ell$ contains $H$, the conic $\ell^\perp$ degenerates into a double line containing $G$. If $\ell$ also contains $P = (u : v : w)$ other than $H$, then this line has equation

$$(S_B v - S_C w)x + (S_C w - S_A u)y + (S_A u - S_B v)z = 0.$$ 

This double line passes through the second intersection of $\ell$ with the Kiepert hyperbola. It also contains the point $((uS_A : vS_B : wS_C))$. The two lines intersect at the point

$$\left(\frac{S_B - S_C}{S_B v - S_C w}, \frac{S_C - S_A}{S_C w - S_A u}, \frac{S_A - S_B}{S_A u - S_B v}\right).$$

The orthotransversals of points on $\ell$ envelope the inscribed parabola with directrix $\ell$ and focus the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of $\ell$.

2.5. The antiorthocorrespondent of a line. Let $\ell$ be the line with equation $lx + my + nz = 0$.

When $ABC$ is acute angled, the antiorthocorrespondent $\ell^\top$ of $\ell$ is the circle centered at $\Omega_\ell = (m + n : n + l : l + m)$ and orthogonal to the polar circle. It has square radius

$$\frac{S_A(m + n)^2 + S_B(n + l)^2 + S_C(l + m)^2}{4(l + m + n)^2}$$

and equation

$$(x + y + z) \left( \sum_{\text{cyclic}} S_A lx \right) - (l + m + n) \left( \sum_{\text{cyclic}} a^2 yz \right) = 0.$$ 

When $ABC$ is obtuse angled, $\ell^\top$ is only a part of this circle according to its position with respect to the critical hyperbola $\mathcal{C}$. This circle clearly degenerates

---

10 These points can be easily constructed. For example, $X^\perp$ is the trilinear pole of the perpendicular at $X$ to $BC$.

11 $P^\perp$ is the antipode of $G$ on the conic.

12 These points can be real or imaginary, distinct or equal.

13 In particular, the orthocorrespondent of the tangent at $H$ to the Kiepert hyperbola, i.e., the line $HK$, is the Euler line.

14 $\Omega_\ell$ is the complement of the isotomic conjugate of the trilinear pole of $\ell$. 
into the union of $\mathcal{L}\infty$ and a line through $H$ when $G$ lies on $\ell$. This line is the directrix of the inscribed conic which is now a parabola.

Conversely, any circle centered at $\Omega$ (proper or degenerate) orthogonal to the polar circle is the orthoptic circle of the inscribed conic whose perspector $P$ is the isotomic conjugate of the anticomplement of the center of the circle. The ortho-correspondent of this circle is the trilinear polar $\ell_P$ of $P$. The table below shows a selection of usual lines and inscribed conics.\(^{15}\)

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\Omega$</th>
<th>$\ell$</th>
<th>inscribed conic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$X_{37}$</td>
<td>antiorthic axis</td>
<td>ellipse, center $I$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$X_2$</td>
<td>$\mathcal{L}\infty$</td>
<td>Steiner in-ellipse</td>
</tr>
<tr>
<td>$X_4$</td>
<td>$X_6$</td>
<td>orthic axis</td>
<td>ellipse, center $K$</td>
</tr>
<tr>
<td>$X_6$</td>
<td>$X_{39}$</td>
<td>Lemoine axis</td>
<td>Brocard ellipse</td>
</tr>
<tr>
<td>$X_7$</td>
<td>$X_1$</td>
<td>Gergonne axis</td>
<td>incircle</td>
</tr>
<tr>
<td>$X_8$</td>
<td>$X_9$</td>
<td></td>
<td>Mandart ellipse</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>$X_{306}$</td>
<td></td>
<td>Simmons conic</td>
</tr>
<tr>
<td>$X_{76}$</td>
<td>$X_{141}$</td>
<td>de Longchamps axis</td>
<td></td>
</tr>
<tr>
<td>$X_{110}$</td>
<td>$X_{647}$</td>
<td>Brocard axis</td>
<td></td>
</tr>
<tr>
<td>$X_{508}$</td>
<td>$X_{597}$</td>
<td></td>
<td>Lemoine ellipse</td>
</tr>
</tbody>
</table>

2.6. Orthocorrespondent and antiorthocorrespondent of a circle. In general, the orthocorrespondent of a circle is a conic. More precisely, two orthoassociate circles share the same orthocorrespondent conic, or the part of it outside the critical conic $\mathcal{C}$ when $ABC$ is obtuse-angled. For example, the circumcircle and the nine-point circle have the same orthocorrespondent which is the circumconic centered at $K$. The orthocorrespondent of each circle (and its orthoassociate) of the pencil generated by circumcircle and the nine-point circle is another conic also centered at $K$ and homothetic of the previous one. The axis of these conics are the parallels at $K$ to the asymptotes of the Kiepert hyperbola. The critical conic is one of them since the polar circle belongs to the pencil.

This conic degenerates into a double line (or part of it) if and only if the circle is orthogonal to the polar circle. If the radical axis of the circumcircle and this circle is $lx + my + nz = 0$, this double line has equation $\frac{1}{S_A}x + \frac{m}{S_B}y + \frac{n}{S_C}z = 0$. This is the trilinear polar of the barycentric product $X_{69}$ and the trilinear pole of the radical axis.

The antiorthocorrespondent of a circle is in general a bicircular quartic.

\(^{15}\)The conics in this table are entirely defined either by their center or their perspector in the table. See [1]. In fact, there are two Simmons conics (and not ellipses as Brocard and Lemoyne wrote) with perspectors (and foci) $X_{13}$ and $X_{14}$.
3. Orthopivotal cubics

For a given a point \( P \) with barycentric coordinates \((u : v : w)\), the locus of point \( M \) such that \( P, M, M^\perp \) are collinear is the cubic curve \( \mathcal{O}(P) \):

\[
\sum_{\text{cyclic}} x \left( (c^2 u - 2S_B v) y^2 - (b^2 u - 2S_C v) z^2 \right) = 0. \tag{7}
\]

Equivalently, \( \mathcal{O}(P) \) is the locus of the intersections of a line through \( P \) with the circle which is its antiorthocorrespondent. See §2.5. We shall say that \( \mathcal{O}(P) \) is an orthopivotal cubic, and call \( P \) its orthopivot.

Equation (7) can be rewritten as

\[
\sum_{\text{cyclic}} u \left( x(c^2 y^2 - b^2 z^2) + 2yz(S_B y - S_C z) \right) = 0. \tag{8}
\]

Accordingly, we consider the cubic curves

\[
\begin{align*}
\Sigma_a : & \quad x(c^2 y^2 - b^2 z^2) + 2yz(S_B y - S_C z) = 0, \\
\Sigma_b : & \quad y(a^2 z^2 - c^2 x^2) + 2zx(S_C z - S_A x) = 0, \\
\Sigma_c : & \quad z(b^2 x^2 - a^2 y^2) + 2xy(S_A x - S_B y) = 0,
\end{align*} \tag{9}
\]

and very loosely write (8) in the form

\[
u \Sigma_a + v \Sigma_b + w \Sigma_c = 0. \tag{10}\]

We study the cubics \( \Sigma_a, \Sigma_b, \Sigma_c \) in §6.5 below, where we shall see that they are strophoids. We list some basic properties of the \( \mathcal{O}(P) \).

**Proposition 8.**

1. The orthopivotal cubic \( \mathcal{O}(P) \) is a circular circumcubic,\(^{16} \) passing through the Fermat points, \( P \), the infinite point of the line \( GP \), and

\[
P' = \left( \frac{b^2 - c^2}{v - w} : \frac{c^2 - a^2}{w - u} : \frac{a^2 - b^2}{u - v} \right), \tag{11}\]

which is the second intersection of the line \( GP \) and the Kiepert hyperbola.\(^{17} \)

2. The “third” intersection of \( \mathcal{O}(P) \) and the Fermat line \( X_{13}X_{14} \) is on the line \( PX_{110} \).

3. The tangent to \( \mathcal{O}(P) \) at \( P \) is the line \( PP^\perp \).

4. \( \mathcal{O}(P) \) intersects the sidelines \( BC, CA, AB \) at \( U, V, W \) respectively given by

\[
\begin{align*}
U = & \ (0 : 2S_C u - a^2 v : 2S_B u - a^2 w), \\
V = & \ (2S_C v - b^2 u : 0 : 2S_A v - b^2 w), \\
W = & \ (2S_B w - c^2 u : 2S_A w - c^2 v : 0).
\end{align*}
\]

5. \( \mathcal{O}(P) \) also contains the (not always real) antiorthocorrespondents \( P_1 \) and \( P_2 \) of \( P \).

\(^{16}\)This means that the cubic passes through the two circular points at infinity common to all circles, and the three vertices of the reference triangle.

\(^{17}\)This is therefore the sixth intersection of \( \mathcal{O}(P) \) with the Kiepert hyperbola.
Here is a simple construction of the intersection $U$ in (4) above. If the parallel at $G$ to $BC$ intersects the altitude $AH$ at $H_a$, then $U$ is the intersection of $PH_a$ and $BC$.  

4. Construction of $\mathcal{O}(P)$ and other points

Let the trilinear polar of $P$ intersect the sidelines $BC, CA, AB$ at $X, Y, Z$ respectively. Denote by $\Gamma_a, \Gamma_b, \Gamma_c$ the circles with diameters $AX, BY, CZ$ and centers $O_a, O_b, O_c$. They are in the same pencil $\mathbb{F}$ whose radical axis is the perpendicular at $H$ to the line $\Gamma$ passing through $O_a, O_b, O_c$, and the points $P_1$ and $P_2$ seen above. 

For an arbitrary point $M$ on $\mathcal{L}$, let $\Gamma$ be the circle of $\mathbb{F}$ passing through $M$. The line $PM \perp$ intersects $\Gamma$ at two points $N_1$ and $N_2$ on $\mathcal{O}(P)$. From these we note the following.

1. $\mathcal{O}(P)$ contains the second intersections $A_2, B_2, C_2$ of the lines $AP, BP, CP$ with the circles $\Gamma_a, \Gamma_b, \Gamma_c$.
2. The point $P'$ in (11) lies on the radical axis of $\mathbb{F}$.
3. The circle of $\mathbb{F}$ passing through $P$ meets the line $PP' \perp$ at $\tilde{P}$, tangential of $P$.
4. The perpendicular bisector of $N_1N_2$ envelopes the parabola with focus $F_P$. (see §5 below) and directrix the line $GP$. This parabola is tangent to $\mathcal{L}$ and to the two axes of the inscribed Steiner ellipse.

This yields another construction of $\mathcal{O}(P)$: a tangent to the parabola meets $\mathcal{L}$ at $\omega$. The perpendicular at $P$ to this tangent intersects the circle of $\mathbb{F}$ centered at $\omega$ at two points on $\mathcal{O}(P)$.

5. Singular focus and an involutive transformation

The singular focus of a circular cubic is the intersection of the two tangents to the curve at the circular points at infinity. When this singular focus lies on the curve, the cubic is said to be a focal cubic. The singular focus of $\mathcal{O}(P)$ is the point $F_P = (a^2(v^2 + w^2 - u^2 - vw) + b^2u(u + v - 2w) + c^2u(u + w - 2v) : \cdots : \cdots )$.

If we denote by $F_1$ and $F_2$ the foci of the inscribed Steiner ellipse, then $F_P$ is the inverse of the reflection of $P$ in the line $F_1F_2$ with respect to the circle with diameter $F_1F_2$.

Consider the mapping $\Psi : P \mapsto F_P$ in the affine plane (without the centroid $G$) which transforms a point $P$ into the singular focus $F_P$ of $\mathcal{O}(P)$. This is clearly an involution: $F_P$ is the singular focus of $\mathcal{O}(P)$ if and only if $P$ is the singular focus of $\mathcal{O}(F_P)$. It has exactly two fixed points, i.e., $F_1$ and $F_2$.

---

18 $H_a$ is the “third” intersection of $AH$ with the Napoleon cubic, the isogonal cubic with pivot $X_5$.
19 This line $\mathcal{L}$ is the trilinear polar of the isotomic conjugate of the anticomplement of $P$.
20 The two cubics $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are central foci with centers at $F_1$ and $F_2$ respectively, with inflexional tangents through $K$, sharing the same real asymptote $F_1F_2$. 
The table below shows a selection of homologous points under $\Psi$, most of which we shall meet in the sequel. When $P$ is at infinity, $F_P = G$, i.e., all $O(P)$ with orthopivot at infinity have $G$ as singular focus.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_6$</th>
<th>$X_{13}$</th>
<th>$X_{15}$</th>
<th>$X_{23}$</th>
<th>$X_{69}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_P$</td>
<td>$X_{1054}$</td>
<td>$X_{110}$</td>
<td>$X_{125}$</td>
<td>$X_{111}$</td>
<td>$X_{14}$</td>
<td>$X_{16}$</td>
<td>$X_{182}$</td>
<td>$X_{216}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_{100}$</th>
<th>$X_{184}$</th>
<th>$X_{187}$</th>
<th>$X_{352}$</th>
<th>$X_{616}$</th>
<th>$X_{617}$</th>
<th>$X_{621}$</th>
<th>$X_{622}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_P$</td>
<td>$X_{1083}$</td>
<td>$X_{186}$</td>
<td>$X_{353}$</td>
<td>$X_{574}$</td>
<td>$X_{619}$</td>
<td>$X_{618}$</td>
<td>$X_{624}$</td>
<td>$X_{623}$</td>
</tr>
</tbody>
</table>

The involutive transformation $\Psi$ swaps

1. the Euler line and the line through $GX_{110}$, \(^{21}\)
2. more generally, any line $GP$ and its reflection in $F_1F_2$,
3. the Brocard axis $OK$ and the Parry circle.
4. more generally, any line $OP$ (which is not the Euler line) and the circle through $G$, $X_{110}$, and $F_P$.
5. the circumcircle and the Brocard circle.
6. more generally, any circle not through $G$ and another circle not through $G$.

\(^{21}\)The nine-point center is swapped into the anticomplement of $X_{110}$.
The involutive transformation $\Psi$ leaves the second Brocard cubic $B_2^{22}$
\[ \sum_{\text{cyclic}} (b^2 - c^2)x(c^2y^2 + b^2z^2) = 0 \]
globally invariant. See §6.4 below. More generally, $\Psi$ leaves invariant the pencil of circular circumcubics through the vertices of the second Brocard triangle (they all pass through $G$).\textsuperscript{23} There is another cubic from this pencil which is also globally invariant, namely,
\[ (a^2b^2c^2 - 8SA_SB_SC)x y z + \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x(c^2S_Cy^2 + b^2S_Bz^2) = 0. \]
We call this cubic $B_6$. It passes through $X_3$, $X_{110}$, and $X_{525}$.

If $O(P)$ is nondegenerate, then its real asymptote is the homothetic image of the line $GP$ under the homothety $h(F_P, 2)$.

6. Special orthopivotal cubics

6.1. Degenerate orthopivotal cubics. There are only two situations where we find a degenerate $O(P)$. A cubic can only degenerate into the union of a line and a conic. If the line is $L_\infty$, we find only one such cubic. It is $O(G)$, the union of $L_\infty$ and the Kiepert hyperbola. If the line is not $L_\infty$, there are ten different possibilities depending on the number of vertices of triangle $ABC$ lying on the conic above which now must be a circle.

1. $O(X_{110})$ is the union of the circumcircle and the Fermat line.\textsuperscript{24}
2. $O(P)$ is the union of one sideline of triangle $ABC$ and the circle through the remaining vertex and the two Fermat points when $P$ is the “third” intersection of an altitude of $ABC$ with the Napoleon cubic.\textsuperscript{25}
3. $O(P)$ is the union of a circle through two vertices of $ABC$ and one Fermat point and a line through the remaining vertex and Fermat point when $P$ is a vertex of one of the two Napoleon triangles. See [4, §6.31].

6.2. Isocubics $O(P)$. We denote by $pK$ a pivotal isocubic and by $nK$ a non-pivotal isocubic. Consider an orthopivotal circumcubic $O(P)$ intersecting the sidelines of triangle $ABC$ at $U$, $V$, $W$ respectively. The cubic $O(P)$ is an isocubic in the two following cases.

\textsuperscript{22} The second Brocard cubic $B_2$ is the locus of foci of inscribed conics centered on the line $GK$. It is also the locus of $M$ for which the line $MM^\perp$ contains the Lemoine point $K$.

\textsuperscript{23} The involutive image of a circular cubic with respect to one of its points is another circular cubic through the same point. Here, $\Psi$ swaps $ABC$ and the second Brocard triangle $A_2B_2C_2$. Hence, each circular cubic through $A$, $B$, $C$, $A_2$, $B_2$, $C_2$ and $G$ has an involutive image through the same points.

\textsuperscript{24} $X_{110}$ is the focus of the Kiepert parabola.

\textsuperscript{25} The Napoleon cubic is the isogonal cubic with pivot $X_5$. These third intersections are the intersections of the altitudes with the parallel through $G$ to the corresponding sidelines.
6.2.1. Pivotal $O(P)$.

**Proposition 9.** An orthopivotal cubic $O(P)$ is a pivotal circumcubic $pK$ if and only if the triangles $ABC$ and $UVW$ are perspective, i.e., if and only if $P$ lies on the Napoleon cubic (isogonal $pK$ with pivot $X_5$). In this case,

1. the pivot $Q$ of $O(P)$ lies on the cubic $K_n$; \(^{26}\) it is the perspector of $ABC$ and the $(-2)$-pedal triangle of $P$, \(^{27}\) and lies on the line $PX_5$;
2. the pole $\Omega$ of the isogonal conjugation lies on the cubic

$$C_0 : \sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(b^2z - c^2y) = 0.$$ 

The $\Omega$-isogonal $Q^*$ of $Q$ lies on the Neuberg cubic and is the inverse in the circumcircle of the isogonal conjugate of $Q$. The $\Omega$-isogonal conjugate $P^*$ of $P$ lies on $K_n$ and is the third intersection with the line $QX_5$.

Here are several examples of such cubics.

1. $O(O) = O(X_3)$ is the Neuberg cubic.
2. $O(X_5)$ is $K_n$.
3. $O(I) = O(X_1)$ has pivot $X_{80} = ((2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, pole \((a(2S_C - ab)(2S_B - ac) : \cdots : \cdots)\), and singular focus \((a(2S_A + ab + ac - 3bc) : \cdots : \cdots)\).
4. $O(H) = O(X_4)$ has pivot $H$, pole $M_o$ the intersection of $HK$ and the orthic axis, with coordinates

$$\left(\frac{a^2(b^2 + c^2 - 2a^2) + (b^2 - c^2)^2}{S_A} : \cdots : \cdots\right),$$

and singular focus $X_{125}$, center of the Jerabek hyperbola.

$O(H)$ is a very remarkable cubic since every point on it has orthocorrespondent on the Kiepert hyperbola. It is invariant under the inversion with respect to the conjugated polar circle and is also invariant under the isogonal transformation with respect to the orthic triangle. It is an isogonal $pK$ with pivot $X_{80}$ with respect to this triangle.

6.2.2. Non-pivotal $O(P)$.

**Proposition 10.** An orthopivotal cubic $O(P)$ is a non-pivotal circumcubic $nK$ if and only if its “third” intersections with the sidelines \(^{28}\) are collinear, i.e., if and only if $P$ lies on the isogonal $nK$ with root $X_{30}$: \(^{29}\)

$$\sum_{\text{cyclic}} ((b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2))x(c^2y^2 + b^2z^2) + 2(8S_As_BSC - a^2b^2c^2)xyz = 0.$$ 

We give two examples of such cubics.

\(^{26}\) $K_n$ is the 2-cevian cubic associated with the Neuberg and the Napoleon cubics. See [8].

\(^{27}\) For any non-zero real number $t$, the $t$-pedal triangle of $P$ is the image of its pedal triangle under the homothety $h(P, t)$.

\(^{28}\) These are the points $U, V, W$ in Proposition 8(4).

\(^{29}\) This passes through $G, K, X_{110}$, and $X_{523}$. 

(1) $\mathcal{O}(K) = \mathcal{O}(X_6)$ is the second Brocard cubic $B_2$.

(2) $\mathcal{O}(X_{523})$ is a $nK$ with pole and root both at the isogonal conjugate of $X_{523}$, and singular focus $G$:

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(y + z) = 0$$

6.3. *Isogonal* $\mathcal{O}(P)$. There are only two $\mathcal{O}(P)$ which are isogonal cubics, one pivotal and one non-pivotal:

(i) $\mathcal{O}(X_3)$ is the Neuberg cubic (pivotal),

(ii) $\mathcal{O}(X_6)$ is $B_2$ (nonpivotal).

---

$\mathcal{O}(X_{523})$ meets the circumcircle at the Tixier point $X_{476}$. 

---
6.4. Orthopivotal focals. Recall that a focal is a circular cubic containing its own singular focus.  

**Proposition 11.** An orthopivotal cubic $O(P)$ is a focal if and only if $P$ lies on $B_2$.

This is the case of $B_2$ itself, which is an isogonal focal cubic passing through the following points: $A$, $B$, $C$, $G$, $K$, $X_{13}$, $X_{14}$, $X_{15}$, $X_{16}$, $X_{111}$ (the singular focus), $X_{368}$, $X_{524}$, the vertices of the second Brocard triangle and their isogonal conjugates. All those points are orthopivots of orthopivotal focals. When the orthopivot is a fixed point of the orthocorrespondence, we shall see in §6.5 below that $O(P)$ is a strophoid.

We have seen in §5 that $F_1$ and $F_2$ are invariant under $\Psi$. These two points lie on $B_2$ (and also on the Thomson cubic). The singular focus of an orthopivotal focal $O(P)$ always lies on $B_2$; it is the “third” point of $B_2$ and the line $KP$.

---

31 Typically, a focal is the locus of foci of conics inscribed in a quadrilateral. The only focals having double points (nodes) are the strophoids.
One remarkable cubic is $O(X_{524})$: it is another central cubic with center and singular focus at $G$ and the line $GK$ as real asymptote. This cubic passes through $X_{67}$ and obviously the symmetrics of $A, B, C, X_{13}, X_{14}, X_{67}$ about $G$. Its equation is

$$\sum_{\text{cyclic}} x \left( (b^2 + c^2 - a^4 - c^4(a^2 + 2b^2 - 2c^2)) y^2 - (b^4 + c^4 - a^4 - b^2(a^2 - 2b^2 + 2c^2)) z^2 \right) = 0.$$

Another interesting cubic is $O(X_{111})$ with $K$ as singular focus. Its equation is

$$\sum_{\text{cyclic}} (b^2 + c^2 - 2a^2) x^2 \left( c^2(a^4 - b^4 c^2 + 3b^4 - c^4 - 2a^2 b^2) y - b^2(a^4 - b^4 c^2 + 3c^4 - b^4 - 2a^2 c^2) z \right) = 0.$$

The sixth intersection with the Kiepert hyperbola is $X_{671}$, a point on the Steiner circumellipse and on the line through $X_{99}$ and $X_{111}$.

6.5. Orthopivotal strophoids. It is easy to see that $O(P)$ is a strophoid if and only if $P$ is one of the five real fixed points of the orthocorrespondence, namely, $A, B, C, X_{13}, X_{14}$, the fixed point being the double point of the curve. This means that the mesh of orthopivotal cubics contains five strophoids denoted by $O(A)$, $O(B)$, $O(C)$, $O(X_{13})$, $O(X_{14})$. 
6.5.1. The strophoids $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$. These are the cubics $\Sigma_a$, $\Sigma_b$, $\Sigma_c$ with equations given in (9). It is enough to consider $\mathcal{O}(A) = \Sigma_a$. The bisectors of angle $A$ are the tangents at the double point $A$. The singular focus is the corresponding vertex of the second Brocard triangle, namely, the point $A_2 = (a^2 : b^2 : c^2)$. The real asymptote is parallel to the median $AG$, being the homothetic image of $AG$ under $h(A_2, 2)$.

Here are some interesting properties of $\mathcal{O}(A) = \Sigma_a$.

1. $\Sigma_a$ is the isogonal conjugate of the Apollonian $A$-circle

$$\mathcal{C}_A : \quad a^2(b^2 z^2 - c^2 y^2) + 2x(b^2 S_B z - c^2 S_C y) = 0,$$

which passes through $A$ and the two isodynamic points $X_{15}$ and $X_{16}$.

2. The isogonal conjugate of $A_2$ is the point $A_4 = (a^2 : 2S_A : 2S_A)$ on the Apollonian circle $\mathcal{C}_A$, which is the projection of $H$ on $AG$. The isogonal conjugate of the antipode of $A_4$ on $\mathcal{C}_A$ is the intersection of $\Sigma_a$ with its real asymptote.\(^{32}\)

3. $\mathcal{O}(A) = \Sigma_a$ is the pedal curve with respect to $A$ of the parabola with focus at the second intersection of $\mathcal{C}_A$ and the circumcircle and with directrix the median $AG$.

\(^{32}\)This is the projection of $O$ on the symmedian $AK$, the tangent at $A_2$ being the reflection about $OA_2$ of the parallel at $A_2$ to $AG$.

\(^{33}\)This isogonal conjugate is on the perpendicular at $A$ to $AK$, and on the tangent at $A_2$ to $\Sigma_a$. 
6.5.2. The strophoids $\mathcal{O}(X_{13})$ and $\mathcal{O}(X_{14})$. The strophoid $\mathcal{O}(X_{13})$ has singular focus $X_{14}$, real asymptote the parallel at $X_{99}$ to the line $GX_{13}$. The circle centered at $X_{14}$ passing through $X_{13}$ intersects the parallel at $X_{14}$ to $GX_{13}$ at $D_1$ and $D_2$ which lie on the nodal tangents. The perpendicular at $X_{14}$ to the Fermat line meets the bisectors of the nodal tangents at $E_1$ and $E_2$ which are the points where the tangents are parallel to the asymptote and therefore the centers of anallagmaty of the curve.

$\mathcal{O}(X_{13})$ is the pedal curve with respect to $X_{13}$ of the parabola with directrix the line $GX_{13}$ and focus $X'_{13}$, the symmetric of $X_{13}$ about $X_{14}$.

---

The “third intersection” of this asymptote with the cubic lies on the perpendicular at $X_{13}$ to the Fermat line. The intersection of the perpendicular at $X_{13}$ to $GX_{13}$ and the parallel at $X_{14}$ to $GX_{13}$ is another point on the curve.

This means that $E_1$ and $E_2$ are the centers of two circles through $X_{13}$ and the two inversions with respect to those circles leave $\mathcal{O}(X_{13})$ unchanged.
The construction of \( O(X_{13}) \) is easy to realize. Draw the parallel \( \ell \) at \( X_{14} \) to \( G X_{13} \) and take a variable point \( M \) on it. The perpendicular at \( M \) to \( M X_{13}' \) and the parallel at \( X_{13} \) to \( M X_{13}' \) intersect at a point on the strophoid.

We can easily adapt all these to \( O(X_{14}) \).

6.6. Other remarkable \( O(P) \). The following table gives a list triangle centers \( P \) with \( O(P) \) passing through the Fermat points \( X_{13}, X_{14} \), and at least four more triangle centers of [5]. Some of them are already known and some others will be detailed in the next section. The very frequent appearance of \( X_{15}, X_{16} \) is explained in §7.3 below.
7. Pencils of $O(P)$

7.1. Generalities. The orthopivotal cubics with orthopivots on a given line $\ell$ form a pencil $F_\ell$ generated by any two of them. Apart from the vertices, the Fermat points, and two circular points at infinity, all the cubics in the pencil pass through two fixed points depending on the line $\ell$. Consequently, all the orthopivotal cubics passing through a given point $Q$ have their orthopivots on the tangent at $Q$ to $O(Q)$, namely, the line $QQ^2$. They all pass through another point $Q'$ on this line which is its second intersection with the circle which is its antiothcorrespondent. For example, $O(Q)$ passes through $G$, $O$, or $H$ if and only if $Q$ lies on $GK$, $OX_{54}$, or the Euler line respectively.

7.2. Pencils with orthopivot on a line passing through $G$. If $\ell$ contains the centroid $G$, every orthopivotal cubic in the pencil $F_\ell$ passes through its infinite point and second intersection with the Kiepert hyperbola. As $P$ traverses $\ell$, the singular focus of $O(P)$ traverses its reflection about $F_1F_2$ (see §5).

The most remarkable pencil is the one with $\ell$ the Euler line. In this case, the two fixed points are the infinite point $X_{30}$ and the orthocenter $H$. In other words, all the cubics in this pencil have their asymptote parallel to the Euler line. In this pencil, we find the Neuberg cubic and $K_n$. The singular focus traverses the line $GX_{98}$, $X_{98}$ being the Tarry point.

Another worth noticing pencil is obtained when $\ell$ is the line $GX_{98}$. In this case, the two fixed points are the infinite point $X_{542}$ and $X_{98}$. The singular focus traverses the Euler line. This pencil contains the two degenerate cubics $O(G)$ and $O(X_{110})$ seen in §6.1.

When $\ell$ is the line $GK$, the two fixed points are the infinite point $X_{521}$ and the centroid $G$. The singular focus lies on the line $GX_{99}$, $X_{99}$ being the Steiner point. This pencil contains $B_2$ and the central cubic seen in §6.4.
7.3. **Pencils with orthopivots on a line not passing through G.** If \( \ell \) is a line not through \( G \), the orthopivotal cubics in the pencil \( F_\ell \) pass through the two (not necessarily real nor distinct) intersections of \( \ell \) with the circle which is its antiothocorrespondent of. See §2.5 and §3. The singular focus lies on a circle through \( G \), and the real asymptote envelopes a deltoid tangent to the line \( F_1 F_2 \) and tritangent to the reflection of this circle about \( G \).

According to §6.2.1, §6.2.2, §6.4, this pencil contains at least one, at most three \( pK, nK \), focal(s) depending of the number of intersections of \( \ell \) with the cubics met in those paragraphs respectively.

Consider, for example, the Brocard axis \( OK \). We have seen in §6.3 that there are two and only two isogonal \( O(P) \), the Neuberg cubic and the second Brocard cubic \( B_2 \) obtained when the orthopivots are \( O \) and \( K \) respectively. The two fixed points of the pencil are the isodynamic points.\(^3^6\)

The singular focus lies on the Parry circle (see §5) and the asymptote envelopes a deltoid tritangent to the reflection of the Parry circle about \( G \).

The pencil \( F_{OK} \) is invariant under isogonal conjugation, the isogonal conjugate of \( O(P) \) being \( O(Q) \), where \( Q \) is the harmonic conjugate of \( P \) with respect to

\(^{36}\)The antiothocorrespondent of the Brocard axis is a circle centered at \( X_{647} \), the isogonal conjugate of the trilinear pole of the Euler line.
$O$ and $K$. It is obvious that the Neuberg cubic and $B_2$ are the only cubic which are “self-isogonal” and all the others correspond two by two. Since $OK$ intersects the Napoleon cubic at $O, X_{61}$ and $X_{62}$, there are only three $pK$ in this pencil, the Neuberg cubic and $O(X_{61}), O(X_{62})$.\footnote{\(O(X_{61})\) and $O(X_{62})$ are isogonal conjugates of each other. Their pivots are $X_{14}$ and $X_{13}$ respectively and their poles are quite complicated and unknown in [5].}

$O(X_{61})$ passes though $X_{18}, X_{533}, X_{618}$, and the isogonal conjugates of $X_{532}$ and $X_{619}$.

$O(X_{62})$ passes though $X_{17}, X_{532}, X_{619}$, and the isogonal conjugates of $X_{533}$ and $X_{618}$. There are only three focals in the pencil $\mathbb{P}_{OK}$, namely, $B_2$ and $O(X_{15}), O(X_{16})$ (with singular foci $X_{16}, X_{15}$ respectively).

![Figure 12. The Brocard pencil](image)

An interesting situation is found when $P = X_{182}$, the midpoint of $OK$. Its harmonic conjugate with respect to $OK$ is the infinite point $Q = X_{511}$. $O(X_{511})$ passes through $X_{262}$ which is its intersection with its real asymptote parallel at $G$. 
to $OK$. Its singular focus is $G$. The third intersection with the Fermat line is $U_1$ on $X_{23}X_{110}$ and the last intersection with the circumcircle is $X_{842} = X_{542}^*$.\[38\]

$O(X_{182})$ is the isogonal conjugate of $O(X_{511})$ and passes through $X_{98}$, $X_{182}$. Its singular focus is $X_{23}$, inverse of $G$ in the circumcircle. Its real asymptote is parallel to the Fermat line at $X_{323}$ and the intersection is the isogonal conjugate of $U_1$.

The following table gives several pairs of harmonic conjugates $P$ and $Q$ on $OK$. Each column gives two cubics $O(P)$ and $O(Q)$, each one being the isogonal conjugate of the other.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_{32}$</th>
<th>$X_{50}$</th>
<th>$X_{52}$</th>
<th>$X_{58}$</th>
<th>$X_{187}$</th>
<th>$X_{216}$</th>
<th>$X_{284}$</th>
<th>$X_{371}$</th>
<th>$X_{389}$</th>
<th>$X_{500}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>$X_{39}$</td>
<td>$X_{566}$</td>
<td>$X_{569}$</td>
<td>$X_{386}$</td>
<td>$X_{574}$</td>
<td>$X_{577}$</td>
<td>$X_{579}$</td>
<td>$X_{372}$</td>
<td>$X_{578}$</td>
<td>$X_{582}$</td>
</tr>
</tbody>
</table>

8. A quintic and a quartic

We present a pair of interesting higher degree curves associated with the ortho-correspondence.

**Theorem 12.** The locus of point $P$ whose orthotransversal $L_P$ and trilinear polar $\ell_P$ are parallel is the circular quintic

$$Q_1 : \sum_{\text{cyclic}} a^2 y^2 z^2 (S_B y - S_C z) = 0.$$ 

Equivalently, $Q_1$ is the locus of point $P$ for which

(1) the lines $PP^*$ and $\ell_P$ (or $L_P$) are perpendicular,
(2) $P$ lies on the Euler line of the pedal triangle of $P^*$,
(3) $P, P^*, H/P$ (and $P^\perp$) are collinear,
(4) $P$ lies on $O(P^*)$.

Note that $L_P$ and $\ell_P$ coincide when $P$ is one of the Fermat points.\[39\]

**Theorem 13.** The isogonal transform of the quintic $Q_1$ is the circular quartic

$$Q_2 : \sum_{\text{cyclic}} a^4 S_A y z (c^2 y^2 - b^2 z^2) = 0,$$

which is also the locus of point $P$ such that

(1) the lines $PP^*$ and $\ell_{P^*}$ (or $L_{P^*}$) are perpendicular,
(2) $P$ lies on the Euler line of its pedal triangle,
(3) $P, P^*, H/P^*$ are collinear,
(4) $P^*$ lies on $O(P)$.

These two curves $Q_1$ and $Q_2$ contain a large number of interesting points, which we enumerate below.

**Proposition 14.** The quintic $Q_1$ contains the 58 following points:

\[38\] This is on $X_{23}X_{110}$ too. It is the reflection of the Tarry point $X_{98}$ about the Euler line and the reflection of $X_{74}$ about the Brocard line.

\[39\] See §1, Remark (5).
Figure 13. The quintic $Q_1$

(1) the vertices $A$, $B$, $C$, which are singular points with the bisectors as tangents,
(2) the circular points at infinity and the singular focus $G$,\textsuperscript{40}
(3) the three infinite points of the Thomson cubic,\textsuperscript{41}
(4) the in/excenters $I, I_a, I_b, I_c$, with tangents passing through $O$, and the isogonal conjugates of the intersections of these tangents with the trilinear polars of the corresponding in/excenters,
(5) $H$, with tangent the Euler line,

\textsuperscript{40}The tangent at $G$ passes through the isotomic conjugate of $G^\perp$, the point with coordinates $(\tfrac{1}{a+b+c-2b} : \ldots : \ldots)$. \textsuperscript{41}In other words, $Q_1$ has three real asymptotes parallel to those of the Thomson cubic.
(6) the six points where a circle with diameter a side of $ABC$ intersects the corresponding median,\(^{42}\)
(7) the feet of the altitudes, the tangents being the altitudes,
(8) the Fermat points $X_{13}$ and $X_{14}$,
(9) the points $X_{1113}$ and $X_{1114}$ where the Euler line meets the circumcircle,
(10) the perspectors of the 27 Morley triangles and $ABC$.\(^{43}\)

**Proposition 15.** The quartic $Q_2$ contains the 61 following points:

(1) the vertices $A, B, C$,\(^{44}\)
(2) the circular points at infinity,\(^{45}\)
(3) the three points where the Thomson cubic meets the circumcircle again,
(4) the in/excenters $I, I_a, I_b, I_c$, with tangents all passing through $O$, and the intersections of these tangents $OI_x$ with the trilinear polars of the corresponding in/excenters,
(5) $O$ and $K$,\(^{46}\)
(6) the six points where a symmedian intersects a circle centered at the corresponding vertex of the tangential triangle passing through the remaining two vertices of $ABC$,\(^{47}\)
(7) the six feet of bisectors,
(8) the isodynamic points $X_{15}$ and $X_{16}$, with tangents passing through $X_{23}$,
(9) the two infinite points of the Jerabek hyperbola,\(^{48}\)
(10) the isogonal conjugates of the perspectors of the 27 Morley triangles and $ABC$.\(^{49}\)

We give a proof of (10). Let $k_1, k_2, k_3 = 0, \pm 1$, and consider
\[
\varphi_1 = \frac{A + 2k_1 \pi}{3}, \quad \varphi_2 = \frac{B + 2k_2 \pi}{3}, \quad \varphi_3 = \frac{C + 2k_3 \pi}{3}.
\]
Denote by $M$ one of the 27 points with barycentric coordinates
\[
(a \cos \varphi_1 : b \cos \varphi_2 : c \cos \varphi_3).
\]

\(^{42}\)The two points on the median $AG$ have coordinates
\[
(2a : -a \pm \sqrt{2b^2 + 2c^2 - a^2} : -a \pm \sqrt{2b^2 + 2c^2 - a^2}).
\]

\(^{43}\)The existence of these points was brought to my attention by Edward Brisse. In particular, $X_{357}$, the perspector of $ABC$ and first Morley triangle.

\(^{44}\)These are inflection points, with tangents passing through $O$.

\(^{45}\)The singular focus is the inverse $X_{23}$ of $G$ in the circumcircle. This point is not on the curve $Q_2$.

\(^{46}\)Both tangents at $O$ and $K$ pass through the point $Z = (a^2S_A(b^2 + c^2 - a^2) : \cdots : \cdots)$, the intersection of the trilinear polar of $O$ with the orthotransversal of $X_{110}$. The tangent at $O$ is also tangent to the Jerabek hyperbola and the orthocubic.

\(^{47}\)The two points on the symmedian $AK$ have coordinates $(-a^2 \pm a\sqrt{2b^2 + 2c^2 - a^2} : 2b^2 : 2c^2)$.

\(^{48}\)The two real asymptotes of $Q_2$ are parallel to those of the Jerabek hyperbola and meet at $Z$ in footnote 46 above.

\(^{49}\)In particular, the Morley-Yff center $X_{358}$.
The isogonal conjugate of $M$ is the perspector of $ABC$ and one of the 27 Morley triangles. $^50$ We show that $M$ lies on the quartic $Q_2$. $^51$ Since $\cos A = \cos 3\varphi_1 = 4\cos^3\varphi_1 - 3\cos \varphi_1$, we have $\cos^3 \varphi_1 = \frac{1}{4} (\cos A + 3 \cos \varphi_1)$ and similar identities for $\cos^3 \varphi_2$ and $\cos^3 \varphi_3$. From this and the equation of $Q_2$, we obtain

$$\sum_{cyclic} a^4 S_{A b} \cos \varphi_2 c \cos \varphi_3 (c^2 b^2 \cos^2 \varphi_2 - b^2 c^2 \cos^2 \varphi_3)$$

$^50$For example, with $k_1 = k_2 = k_3 = 0$, $M^* = X_{357}$ and $M = X_{358}$.

$^51$Consequently, $M^*$ lies on the quintic $Q_1$. See Proposition 14(10).
Orthocorrespondence and orthopivotal cubics

\[ \sum_{\text{cyclic}} a^4 b^3 c^3 S_A (\cos \phi_3 \cos \phi_2 - \cos \phi_2 \cos \phi_3) \]

\[ \sum_{\text{cyclic}} \frac{1}{4} a^4 b^3 c^3 S_A (\cos \phi_3 \cos B - \cos \phi_2 \cos C) \]

\[ \sum_{\text{cyclic}} \frac{1}{4} a^4 b^3 c^3 S_A \left( \frac{S_B}{ac} \cos \phi_3 - \frac{S_C}{ab} \cos \phi_2 \right) \]

\[ \frac{1}{4} a^3 b^3 c^3 S_B S_C \sum_{\text{cyclic}} \left( \frac{\cos \phi_3}{c S_C} - \frac{\cos \phi_2}{b S_B} \right) \]

\[ = 0. \]

This completes the proof of (10).

**Remark.** \( Q_1 \) and \( Q_2 \) are strong curves in the sense that they are invariant under extraversions: any point lying on one of them has its three extraversions also on the curve. \(^{52}\)

**References**


Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

_E-mail address:_ b.gibert@free.fr

---

\(^{52}\) The extraversions of a point are obtained by replacing one of \( a, b, c \) by its opposite. For example, the extraversions of the incenter \( I \) are the three excenters and \( I \) is said to be a weak point. On the contrary, \( K \) is said to be a "strong" point.
On the Procircumcenter and Related Points

Alexei Myakishev

Abstract. Given a triangle $ABC$, we solve the construction problem of a point $P$, together with points $B_c, C_b$ on $BC$, $C_a, A_c$ on $CA$, and $A_b, B_a$ on $AB$ such that $P B_a C_a$, $A_b P C_b$, and $A_c B_c P$ are congruent triangles similar to $ABC$. There are altogether seven such triads. If these three congruent triangles are all oppositely similar to $ABC$, then $P$ must be the procircumcenter, with trilinear coordinates $(a^2 \cos A : b^2 \cos B : c^2 \cos C)$. If at least one of the triangles in the triad is directly similar to $ABC$, then $P$ is either a vertex or the midpoint of a side of the tangential triangle. We also determine the ratio of similarity in each case.

1. Introduction

Given a triangle $ABC$, we consider the construction of a point $P$, together with points $B_c, C_b$ on $BC$, $C_a, A_c$ on $CA$, and $A_b, B_a$ on $AB$ such that $P B_a C_a$, $A_b P C_b$, and $A_c B_c P$ are congruent triangles similar to $ABC$. We first consider in §§2,3 the case when these triangles are all oppositely similar to $ABC$. See Figure 1. In §4, the possibilities when at least one of these congruent triangles is directly similar to $ABC$ are considered. See, for example, Figure 2.

---


The author thanks Jean-Pierre Ehrmann and Paul Yiu for their helps in the preparation of this paper.
2. The case of opposite similarity: construction of $P$

With reference to Figure 1, we try to find the trilinear coordinates of $P$. As usual, we denote the lengths of the sides opposite to angles $A$, $B$, $C$ by $a$, $b$, $c$. Denote the oriented angles $C_bPB_c$ by $\varphi_a$, $A_cPC_a$ by $\varphi_b$, and $B_aPA_b$ by $\varphi_c$. Since $PC_b = PB_c$, $\angle PB_cB = \frac{1}{2}(\pi - \varphi_a)$. Since also $\angle PB_cA_c = B$, we have $\angle A_cB_cC = \frac{1}{2}(\pi + \varphi_a) - B$. For the same reason, $\angle B_cA_cC = \frac{1}{2}(\pi + \varphi_b) - A$. Considering the sum of the angles in triangle $A\angle B_C$, we have $\frac{1}{2}(\varphi_a + \varphi_b) = \pi - 2C$. Since $\varphi_a + \varphi_b + \varphi_c = \pi$, we have $\varphi_c = 4C - \pi$. Similarly, $\varphi_a = 4A - \pi$ and $\varphi_b = 4B - \pi$.

Let $k$ be the ratio of similarity of the triangles $PB_bC_a$, $A_bPC_b$, and $A_cB_cP$ with $ABC$, i.e., $B_aC_a = PC_b = B_cP = k \cdot BC = ka$. The perpendicular distance from $P$ to the line $BC$ is

$$h_a = ka \cos \frac{\varphi_a}{2} = ka \cos \left(2A - \frac{\pi}{2}\right) = ka \sin 2A.$$ 

Similarly, the perpendicular distances from $P$ to $CA$ and $AB$ are $h_b = kb \sin 2B$ and $h_c = kc \sin 2C$. It follows that $P$ has trilinear coordinates,

$$(a \sin 2A : b \sin 2B : c \sin 2C) \sim (a^2 \cos A : b^2 \cos B : c^2 \cos C). \quad (1)$$

Note that we have found not only the trilinears of $P$, but also the angles of isosceles triangles $PC_bB_c$, $PA_aC_a$, $PB_aB_c$. It is therefore easy to construct the triangles by ruler and compass from $P$. Now, we easily identify $P$ as the isogonal conjugate of the isotomic conjugate of the circumcenter $O$, which has trilinear coordinates $(\cos A : \cos B : \cos C)$. We denote this point by $\overline{O}$ and follow John H. Conway in calling it the procircumcenter of triangle $ABC$. We summarize the results in the following proposition.

**Proposition 1.** Given a triangle $ABC$ not satisfying (2), the point $P$ for which there are congruent triangles $PB_bC_a$, $A_bPC_b$, $A_cB_cP$ oppositely similar to $ABC$ (with $B_a$, $C_b$ on $BC$, $C_a$, $A_b$, $A_c$ on $CA$, and $A_b$, $B_a$ on $AB$) is the procircumcenter $\overline{O}$. This is a finite point unless the given triangle satisfies

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = 0. \quad (2)$$

The procircumcenter $\overline{O}$ appears as $X_{184}$ in [3], and is identified as the inverse of the Jerabek center $X_{125}$ in the Brocard circle. A simple construction of $\overline{O}$ is made possible by the following property discovered by Fred Lang.

**Proposition 2** (Lang [4]). Let the perpendicular bisectors of $BC$, $CA$, $AB$ intersect the other pairs of sides at $B_1$, $C_1$, $C_2$, $A_2$, $A_3$, $B_3$ respectively. The perpendicular bisectors of $B_1C_1$, $C_2A_2$ and $A_3B_3$ bound a triangle homothetic to $ABC$ at the procircumcenter $\overline{O}$. 

---

1 We regard the orientation of triangle $ABC$ as positive. The oriented angles are defined modulo $2\pi$. 
3. The case of opposite similarity: ratio of similarity

We proceed to determine the ratio of similarity $k$. We shall make use of the following lemmas.

**Lemma 3.** Let $\triangle$ denote the area of triangle $ABC$, and $R$ its circumradius.

(1) $\triangle = 2R^2 \sin A \sin B \sin C$;
(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$;
(3) $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C$;
(4) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.

**Proof.** (1) By the law of sines,

$$\triangle = \frac{1}{2} bc \sin A = \frac{1}{2} (2R \sin B)(2R \sin C) \sin A = 2R^2 \sin A \sin B \sin C.$$

For (2),

$$\sin 2A + \sin 2B + \sin 2C = 2 \sin A \cos A + 2 \sin(B + C) \cos(B - C) = 2 \sin A(- \cos(B + C) + \cos(B - C)) = 4 \sin A \sin B \sin C.$$

The proof of (3) is similar. For (4),

$$\sin^2 A + \sin^2 B + \sin^2 C = \sin^2 A + 1 - \frac{1}{2}(\cos 2B + \cos 2C) = \sin^2 A + 1 - \cos(B + C) \cos(B - C) = 2 - \cos^2 A + \cos A \cos(B - C) = 2 + \cos A(\cos(B + C) + \cos(B - C)) = 2 + 2 \cos A \cos B \cos C.$$

□

**Lemma 4.** $a^2 + b^2 + c^2 = 9R^2 - OH^2$, where $R$ is the circumradius, and $O$, $H$ are respectively the circumcenter and orthocenter of triangle $ABC$.

This was originally due to Euler. An equivalent statement

$$a^2 + b^2 + c^2 = 9(R^2 - OG^2),$$

where $G$ is the centroid of triangle $ABC$, can be found in [2, p.175].

**Proposition 5** (Dergiades [1]). The ratio of similarity of $\overrightarrow{OB_a C_a}$, $A_b \overrightarrow{OC_b}$, and $A_c B_c \overrightarrow{O}$ with $ABC$ is

$$k = \sqrt{\frac{R^2}{3R^2 - OH^2}}.$$
Proof. Since $2\triangle = a \cdot h_a + b \cdot h_b + c \cdot h_c$, and $h_a = ka \sin 2A$, $h_b = kb \sin 2B$, and $h_c = kc \sin 2C$, the ratio of similarity is the absolute value of

\[
\frac{2\triangle}{a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C} = \frac{4R^2(\sin^2 A \sin 2A + \sin^2 B \sin 2B + \sin^2 C \sin 2C)}{4 \sin A \sin B \sin C} \quad \text{(Lemma 3(1))}
\]

\[
= \frac{2(\sin 2A + \sin 2B + \sin 2C) - (\sin 4A + \sin 4B + \sin 4C)}{4 \sin A \sin B \sin C} \quad \text{(Lemma 3(2, 3))}
\]

\[
= \frac{2 + 8 \cos A \cos B \cos C}{4(\sin^2 A + \sin^2 B + \sin^2 C)} - 6 \quad \text{(Lemma 3(4))}
\]

\[
= \frac{a^2 + b^2 + c^2 - 6R^2}{R^2} = \frac{3R^2 - OH^2}{R^2}
\]

by Lemma 4. \hfill \square

From Proposition 5, we also infer that $O$ is an infinite point if and only if $OH = \sqrt{3}R$. More interesting is that for triangles satisfying $OH = 2R$ or $\sqrt{2}R$, the congruent triangles in the triad are also congruent to the reference triangle $ABC$. See Figures 3 and 4. These are triangles satisfying

\[
a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = \pm a^2b^2c^2.
\]
4. Cases allowing direct similarity with $ABC$

As Jean-Pierre Ehrmann has pointed out, by considering all possible orientations of the triangles $PB_aC_a$, $A_bPC_b$, $A_cB_cP$, there are other points, apart from the procircumcenter $O$, that yield triads of congruent triangles similar to $ABC$.

4.1. Exactly one of the triangles oppositely similar to $ABC$. Suppose, for example, that among the three congruent triangles, only $PB_aC_a$ be oppositely similar to $ABC$, the other two, $A_bPC_b$ and $A_cB_cP$ being directly similar. We denote by $P_a^+$ the point $P$ satisfying these conditions. Modifying the calculations in §2, we have
\[
\varphi_a = \pi + 2A, \quad \varphi_b = \pi - 2A, \quad \varphi_c = \pi - 2A.
\]
From these, we obtain the trilinears of $P_a^+$ as
\[
(-a \sin A : b \sin A : c \sin A) = (-a : b : c).
\]
It follows that $P_a^+$ is the $A$-vertex of the tangential triangle of $ABC$. See Figure 5.

![Figure 5](image)

The ratio of similarity, by a calculation similar to that performed in §3, is $k = \left| \frac{1}{2 \cos A} \right|$. This is equal to 1 only when $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. In these cases, the three triangles are congruent to $ABC$.

Clearly, there are two other triads of congruent triangles corresponding to the other two vertices of the tangential triangle.

4.2. Exactly one of triangles directly similar to $ABC$. Suppose, for example, that among the three congruent triangles, only $PB_aC_a$ be directly similar to $ABC$, the other two, $A_bPC_b$ and $A_cB_cP$ being oppositely similar. We denote by $P_a^-$ the point $P$ satisfying these conditions. See Figure 6. In this case, we have
\[
\varphi_a = 2A - \pi, \quad \varphi_b = \pi + 2B - 2C, \quad \varphi_c = \pi + 2C - 2B.
\]
From these, we obtain the trilinears of $P_a^-$ as

$$(-a \sin A : b \sin(B - C) : c \sin(C - B)) = (-a^3 : b(b^2 - c^2) : c(c^2 - b^2)).$$

It is easy to check that this is the midpoint of the side $B_1C_1$ of the tangential triangle of $ABC$. In this case, the ratio of similarity is $k = \frac{1}{4 \cos B \cos C}$.

Clearly, there are two other triads of congruent triangles corresponding to the midpoints of the remaining two sides of the tangential triangle.

We conclude with the remark that it is not possible for all three of the congruent triangles to be directly similar to $ABC$, since this would require $\varphi_a = \varphi_b = \varphi_c = \pi$.

References


Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445

E-mail address: alex.geom@mtu-net.ru
Bicentric Pairs of Points and Related Triangle Centers

Clark Kimberling

Abstract. Bicentric pairs of points in the plane of triangle $ABC$ occur in connection with three configurations: (1) cevian traces of a triangle center; (2) points of intersection of a central line and central circumconic; and (3) vertex-products of bicentric triangles. These bicentric pairs are formulated using trilinear coordinates. Various binary operations, when applied to bicentric pairs, yield triangle centers.

1. Introduction

Much of modern triangle geometry is carried out in one or the other of two homogeneous coordinate systems: barycentric and trilinear. Definitions of triangle center, central line, and bicentric pair, given in [2] in terms of trilinears, carry over readily to barycentric definitions and representations. In this paper, we choose to work in trilinears, except as otherwise noted.

Definitions of triangle center (or simply center) and bicentric pair will now be briefly summarized. A triangle center is a point (as defined in [2] as a function of variables $a, b, c$ that are sidelengths of a triangle) of the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b),$$

where $f$ is homogeneous in $a, b, c$, and

$$|f(a, c, b)| = |f(a, b, c)|. \tag{1}$$

If a point satisfies the other defining conditions but (1) fails, then the points

$$F_{ab} := f(a, b, c) : f(b, c, a) : f(c, a, b),$$
$$F_{ac} := f(a, c, b) : f(b, a, c) : f(c, b, a) \tag{2}$$

are a bicentric pair. An example is the pair of Brocard points,

$$c/b : a/c : b/a \quad \text{and} \quad b/c : b/a : c/b.$$
bicentric pairs associated with bicentric triangles. §5 supports §6, and §12 revisits two operations discussed in §3.

2. Products: trilinear and barycentric

Suppose \( U = u : v : w \) and \( X = x : y : z \) are points expressed in general homogeneous coordinates. Their product is defined by

\[
U \cdot X = ux : vy : wz.
\]

Thus, when coordinates are specified as trilinear or barycentric, we have here two distinct product operations, corresponding to constructions of barycentric product [8] and trilinear product [6]. Because we have chosen trilinears as the primary means of representation in this paper, it is desirable to write, for future reference, a formula for barycentric product in terms of trilinear coordinates. To that end, suppose \( u : v : w \) and \( x : y : z \) are trilinear representations, so that in barycentrics,

\[
U = au : bv : cw \quad \text{and} \quad X = ax : by : cz.
\]

Then the barycentric product is \( a^2ux : b^2vy : c^2wz \), and we conclude as follows: the trilinear representation for the barycentric product of \( U = u : v : w \) and \( X = x : y : z \), these being trilinear representations, is given by

\[
U \cdot X = aux : bvy : cwz.
\]

3. Other centralizing operations

Given a bicentric pair, aside from their trilinear and barycentric products, various other binary operations applied to the pair yield a center. Consider the bicentric pair (2). In [2, p. 48], the points

\[
F_{ab} \oplus F_{ac} := f_{ab} + f_{ac} : f_{bc} + f_{ba} : f_{ca} + f_{cb}
\]

and

\[
F_{ab} \ominus F_{ac} := f_{ab} - f_{ac} : f_{bc} - f_{ba} : f_{ca} - f_{cb}
\]

are observed to be triangle centers. See §8 for a geometric discussion.

Next, suppose that the points \( F_{ab} \) and \( F_{ac} \) do not lie on the line at infinity, \( L^\infty \), and consider normalized trilinears, represented thus:

\[
F'_{ab} = (k_{ab}f_{ab}, k_{ab}f_{bc}, k_{ab}f_{ca}), \quad F'_{ac} = (k_{ac}f_{ac}, k_{ac}f_{ba}, k_{ac}f_{cb}),
\]

where

\[
k_{ab} := \frac{2\sigma}{af_{ab} + bf_{bc} + cf_{ca}}, \quad k_{ac} := \frac{2\sigma}{af_{ac} + bf_{ba} + cf_{cb}}, \quad \sigma := \text{area}(\triangle ABC).
\]

These representations give

\[
F'_{ab} \oplus F'_{ac} = k_{ab}f_{ab} + k_{ac}f_{ac} : k_{ab}f_{bc} + k_{ac}f_{ba} : k_{ab}f_{ca} + k_{ac}f_{cb},
\]

which for many choices of \( f(a, b, c) \) differs from (3). In any case, (6) gives the the midpoint of the bicentric pair (2), and the harmonic conjugate of this midpoint with respect to \( F_{ab} \) and \( F_{ac} \) is the point in which the line \( F_{ab}F_{ac} \) meets \( L^\infty \).
We turn now to another centralizing operation on the pair \((2)\). Their line is given by the equation

\[
\begin{vmatrix}
\alpha & \beta & \gamma \\
 f_{ab} & f_{bc} & f_{ca} \\
 f_{ac} & f_{ba} & f_{cb}
\end{vmatrix} = 0
\]

and is a central line. Its trilinear pole, \(P\), and the isogonal conjugate of \(P\), given by

\[
f_{bc}f_{cb} - f_{ca}f_{ba} : f_{cb}f_{bc} - f_{ca}f_{ac} = f_{bc}f_{ac} - f_{ca}f_{ab} - f_{bc}f_{ac},
\]

are triangle centers.

If \(X := x : y : z = f(a, b, c) : f(b, c, a) : f(c, a, b)\) is a triangle center other than \(X_1\), then the points \(Y := y : z : x\) and \(Z := z : x : y\) are clearly bicentric. The operations discussed in §§2,3, applied to \(\{Y, Z\}\), yield the following centers:

- trilinear product \(= X_1/X\) (the indexing of centers as \(X_i\) follows [3]);
- barycentric product \(= X_0/X\) (here, “/” signifies trilinear division);
- \(Y \oplus Z = y + z : z + x : x + y\);
- \(Y \ominus Z = y - z : z - x : x - y\);
- midpoint \(= m(a, b, c) : m(b, c, a) : m(c, a, b)\), where
  \[
  m(a, b, c) = cy^2 + bz^2 + 2ayz + x(by + cz); 
  \]
- \(YZ \cap L^\infty = n(a, b, c) : n(b, c, a) : n(c, a, b)\), where
  \[
  n(a, b, c) = cy^2 - bz^2 + x(by - cz); 
  \]
- (isogonal conjugate of trilinear pole of \(YZ\))
  \[
  = x^2 - yz : y^2 - zx : z^2 - xy 
  \]
  \[
  = (X_1\text{-Hirst inverse of } X). 
  \]

The points \(Z/Y\) and \(Y/Z\) are bicentric and readily yield the centers with first coordinates \(x(y^2 + z^2), x(y^2 - z^2), \) and \(x^3 - y^2z^2/x\). One more way to make bicentric pairs from triangle centers will be mentioned: if \(U = r : s : t\) and \(X := x : y : z\) are centers, then ([2, p.49])

\[
U \otimes X := sz : tx : ry, \quad X \otimes U := ty : rz : sx
\]

are a bicentric pair. For example, \((U \otimes X) \ominus (X \otimes U)\) has for trilinears the coefficients for line \(UX\).
4. Bicentric pairs associated with cevian traces

Suppose \( P \) is a point in the plane of \( \triangle ABC \) but not on one of the sidelines \( BC, CA, AB \) and not on \( C^\infty \). Let \( A', B', C' \) be the points in which the lines \( AP, BP, CP \) meet the sidelines \( BC, CA, AB \), respectively. The points \( A', B', C' \) are the cevian traces of \( P \). Letting \( |XY| \) denote the directed length of a segment from a point \( X \) to a point \( Y \), we recall a fundamental theorem of triangle geometry (often called Ceva’s Theorem, but Hogendijk [1] concludes that it was stated and proved by an ancient king) as follows:

\[
|BA'| \cdot |CB'| \cdot |AC'| = |A'C| \cdot |B'A| \cdot |C'B|.
\]

(The theorem will not be invoked in the sequel.) We shall soon see that if \( P \) is a center, then the points

\[
P_{BC} := |BA'| : |CB'| : |AC'| \quad \text{and} \quad P_{CB} := |A'C| : |B'A| : |C'B|
\]

comprise a bicentric pair, except for \( P = \text{centroid} \), in which case both points are the incenter. Let \( \sigma \) denote the area of \( \triangle ABC \), and write \( P = x : y : z \). Then the actual trilinear distances are given by

\[
B = \left( 0, \frac{2\sigma}{b}, 0 \right) \quad \text{and} \quad A' = \left( \frac{2\sigma y}{by + cz}, \frac{2\sigma z}{by + cz} \right).
\]

Substituting these into a distance formula (e.g. [2, p. 31]) and simplifying give

\[
|BA'| = \frac{z}{b(by + cz)}; \\
P_{BC} = \frac{z}{b(by + cz)} : \frac{x}{c(cz + ax)} : \frac{y}{a(ax + by)}; \\
P_{CB} = \frac{y}{c(by + cz)} : \frac{z}{a(cz + ax)} : \frac{x}{b(ax + by)}.
\]

(7)

So represented, it is clear that \( P_{BC} \) and \( P_{CB} \) comprise a bicentric pair if \( P \) is a center other than the centroid. Next, let

\[
P'_{BC} = \frac{|BA'|}{|CA'|} : \frac{|CB'|}{|AB'|} : \frac{|AC'|}{|BC'|} \quad \text{and} \quad P'_{CB} = \frac{|CA'|}{|BA'|} : \frac{|AB'|}{|CB'|} : \frac{|BC'|}{|AC'|}.
\]

Equation (7) implies

\[
P'_{BC} = \frac{cz}{by} : \frac{ax}{cz} : \frac{by}{ax} \quad \text{and} \quad P'_{CB} = \frac{by}{cz} : \frac{ax}{by}.
\]

(9)

Thus, using "\( \cdot \)" for trilinear quotient, or for barycentric quotient in case the coordinates in (7) and (8) are barycentrics, we have \( P'_{BC} = P_{BC}/P_{CB} \) and \( P'_{CB} = P_{CB}/P_{BC} \). The pair of isogonal conjugates in (9) generalize the previously mentioned Brocard points, represented by (9) when \( P = X_1 \).

As has been noted elsewhere, the trilinear (and hence barycentric) product of a bicentric pair is a triangle center. For both kinds of product, the representation is given by

\[
P_{BC} \cdot P_{CB} = \frac{a}{x(by + cz)^2} : \frac{b}{y(cz + ax)^2} : \frac{c}{z(ax + by)^2}.
\]
The line of a bicentric pair is clearly a central line. In particular, the line $P_{BC}'P_{CB}'$ is given by the equation

$$
\left(\frac{a^2x^2}{bcyz} - \frac{bcyz}{a^2x^2}\right)\alpha + \left(\frac{b^2y^2}{cazx} - \frac{cazx}{b^2y^2}\right)\beta + \left(\frac{c^2z^2}{abxy} - \frac{abxy}{c^2z^2}\right)\gamma = 0.
$$

This is the trilinear polar of the isogonal conjugate of the $E$-Hirst inverse of $F$, where $E = ax : by : cz$, and $F$ is the isogonal conjugate of $E$. In other words, the point whose trilinears are the coefficients for the line $P_{BC}'P_{CB}'$ is the $E$-Hirst inverse of $F$.

The line $P_{BC}P_{CB}$ is given by $x'\alpha + y'\beta + z'\gamma = 0$, where

$$
x' = bc(by + cz)(a^2x^2 - bcyz),
$$

so that $P_{BC}P_{CB}$ is seen to be a certain product of centers if $P$ is a center.

Regarding a euclidean construction for $P_{BC}$, it is easy to transfer distances for this purpose. Informally, we may describe $P_{BC}$ and $P_{BC}'$ as points constructed “by rotating through 90° the corresponding relative distances of the cevian traces from the vertices $A, B, C$”.

### 5. The square of a line

Although this section does not involve bicentric pairs directly, the main result will make an appearance in §7, and it may also be of interest per se.

Suppose that $U_1 = u_1 : v_1 : w_1$ and $U_2 = u_2 : v_2 : w_2$ are distinct points on an arbitrary line $L$, represented in general homogeneous coordinates relative to $\triangle ABC$. For each point $X := u_1 + u_2t : v_1 + v_2t : w_1 + w_2t$,

let

$$
X^2 := (u_1 + u_2t)^2 : (v_1 + v_2t)^2 : (w_1 + w_2t)^2.
$$

The locus of $X^2$ as $t$ traverses the real number line is a conic section. Following the method in [4], we find an equation for this locus:

$$
l^4\alpha^2 + m^4\beta^2 + n^4\gamma^2 - 2m^2n^2\beta\gamma - 2n^2l^2\gamma\alpha - 2l^2m^2\alpha\beta = 0, \tag{10}
$$

where $l, m, n$ are coefficients for the line $U_1U_2$; that is,

$$
l : m : n = v_1w_2 - w_1v_2 : u_1w_2 - w_1u_2 : u_1v_2 - v_1u_2.
$$

Equation (10) represents an inscribed ellipse, which we denote by $I^2$. If the coordinates are trilinears, then the center of the ellipse is the point $bn^2 + cm^2 : cl^2 + an^2 : am^2 + bl^2$. 

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_2$</th>
<th>$X_4$</th>
<th>$X_{75}$</th>
<th>$X_{59}$</th>
<th>$X_{92}$</th>
<th>$X_{57}$</th>
<th>$X_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{BC} \cdot P_{CB}$</td>
<td>$X_{31}$</td>
<td>$X_{593}$</td>
<td>$X_{593}$</td>
<td>$X_{92}$</td>
<td>$X_{92}$</td>
<td>$X_{57}$</td>
<td>$X_{57}$</td>
</tr>
<tr>
<td>$P_{BC} \circ P_{CB}$</td>
<td>$X_{32}$</td>
<td>$X_{849}$</td>
<td>$X_{849}$</td>
<td>$X_{4}$</td>
<td>$X_{4}$</td>
<td>$X_{56}$</td>
<td>$X_{56}$</td>
</tr>
</tbody>
</table>

**Table 1. Examples of trilinear and barycentric products**
6. (Line \(L\)) \(\cap\) (Circumconic \(\Gamma\)), two methods

Returning to general homogeneous coordinates, suppose that line \(L\), given by 
\[ l\alpha + m\beta + n\gamma = 0, \]
meets circumconic \(\Gamma\), given by 
\[ u/\alpha + v/\beta + w/\gamma = 0. \]
Let \(R\) and \(S\) denote the points of intersection, where \(R = S\) if \(L\) is tangent to \(\Gamma\).
Substituting \(-\alpha\) for \(\alpha\) yields
\[ mw\beta^2 + (mv + nw - lu)\beta\gamma + n\gamma^2 = 0, \]
(11)
with discriminant
\[ D := l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmw, \]
(12)
so that solutions of (11) are given by
\[ \frac{\beta}{\gamma} = \frac{lu - mv - nw \pm \sqrt{D}}{2mw}. \]
(13)
Putting \(\beta\) and \(\gamma\) equal to the numerator and denominator, respectively, of the right-hand side (13), putting \(\alpha = -(\beta + n\gamma)/l\), and simplifying give for \(R\) and \(S\) the representation
\[ x_1 : y_1 : z_1 = m(mv - lu - nw \mp \sqrt{D}) : l(lu - mv - nw \pm \sqrt{D}) : 2lmw. \]
(14)
Cyclically, we obtain two more representations for \(R\) and \(S\):
\[ x_2 : y_2 : z_2 = 2mn : n(nw - mv - lu \mp \sqrt{D}) : m(mv - nw - lu \pm \sqrt{D}) \]
(15)
and
\[ x_3 : y_3 : z_3 = n(nw - lu - mv \mp \sqrt{D}) : 2nlv : l(lu - nw - mv \mp \sqrt{D}). \]
(16)
Multiplying the equal points in (14)-(16) gives \(R^3\) and \(S^3\) as
\[ x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3 \]
in cyclic form. The first coordinates in this form are
\[ 2m^2n^2u(mv - lu - nw \mp \sqrt{D})(nw - lu - mv \pm \sqrt{D}), \]
and these yield
\[ (1\text{st coordinate of } R^3) = m^2n^2u[l^2u^2 - (mv - nw - \sqrt{D})^2] \]
(17)
\[ (1\text{st coordinate of } S^3) = m^2n^2u[l^2u^2 - (mv - nw + \sqrt{D})^2]. \]
(18)
The 2nd and 3rd coordinates are determined cyclically.
In general, products (as in §2) of points on \(\Gamma\) intercepted by a line are notable:
multiplying the first coordinates shown in (17) and (18) gives
\[ (1\text{st coordinate of } R^3 \cdot S^3) = l^2m^5n^5u^4vw, \]
so that
\[ R \cdot S = mnu : nvl : lmw. \]
Thus, on writing \(L = l : m : n\) and \(U = u : v : w\), we have \(R \cdot S = U/L\).
The above method for finding coordinates of $R$ and $S$ in symmetric form could be called the multiplicative method. There is also an additive method. Adding the coordinates of the points in (14) gives

$$m(mv - lu - nw) : l(lu - mv - nw) : 2lmw.$$  

Do the same using (15) and (16), then add coordinates of all three resulting points, obtaining the point $U = u_1 : u_2 : u_3$, where

$$u_1 = (lm + ln - 2mn)u + (m - n)(nw - mv)$$
$$u_2 = (mn + ml - 2nl)v + (n - l)(lu - nw)$$
$$u_3 = (nl + nm - 2lm)w + (l - m)(mv - lu).$$

Obviously, the point $V = v_1 : v_2 : v_3 = m - n : n - l : l - m$ also lies on $L$, so that $L$ is given parametrically by

$$u_1 + tv_1 : u_2 + tv_2 : u_3 + tv_3.$$  

(19)

Substituting into the equation for $\Gamma$ gives

$$u(u_2 + tv_2)(u_3 + tv_3) + v(u_3 + tv_3)(u_1 + tv_1) + w(u_1 + tv_1)(u_2 + tv_2) = 0.$$  

The expression of the left side factors as

$$(t^2 - D)F = 0,$$  

(20)

where

$$F = u(n - l)(l - m) + v(l - m)(m - n) + w(m - n)(n - l).$$

Equation (20) indicates two cases:

**Case 1:** $F = 0$. Here, $V$ lies on both $L$ and $\Gamma$, and it is then easy to check that the point

$$W = mnu(n - l)(l - m) : nlv(l - m)(m - n) : lmw(m - n)(n - l)$$

also lies on both.

**Case 2:** $F \neq 0$. By (20), the points of intersection are

$$u_1 \pm v_1\sqrt{D} : u_2 \pm v_2\sqrt{D} : u_3 \pm v_3\sqrt{D}.$$  

(21)

As an example to illustrate Case 1, take $u(a, b, c) = (b - c)^2$ and $l(a, b, c) = a$. Then $D = (b - c)^2(a - c)^2(a - b)^2$, and the points of intersection are $b - c : c - a : a - b$ and $(b - c)/a : (c - a)/b : (a - b)/c$.

---

1I thank the Jean-Pierre Ehrmann for describing this method and its application.
7. $L \cap \Gamma$ when $D = 0$

The points $R$ and $S$ are identical if and only if $D = 0$. In this case, if in equation (12) we regard either $l : m : n$ or $u : v : w$ as a variable $\alpha : \beta : \gamma$, then the resulting equation is that of a conic inscribed in $\triangle ABC$. In view of equation (10), we may also describe this locus in terms of squares of lines; to wit, if $u : v : w$ is the variable $\alpha : \beta : \gamma$, then the locus is the set of squares of points on the four lines indicated by the equations

$$\sqrt{|l|} \alpha \pm \sqrt{|m|} \beta \pm \sqrt{|n|} \gamma = 0.$$ 

Taking the coordinates to be trilinears, examples of centers $X_i = l : m : n$ and $X_j = u : v : w$ for which $D = 0$ are given in Table 2. It suffices to show results for $i \leq j$, since $L$ and $U$ are interchangeable in (12).

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>244, 678</td>
</tr>
<tr>
<td>2</td>
<td>1015, 1017</td>
</tr>
<tr>
<td>3</td>
<td>125</td>
</tr>
<tr>
<td>6</td>
<td>115</td>
</tr>
<tr>
<td>11</td>
<td>55, 56, 181, 202, 203, 215</td>
</tr>
<tr>
<td>31</td>
<td>244, 1099, 1109, 1111</td>
</tr>
<tr>
<td>44</td>
<td>44</td>
</tr>
</tbody>
</table>

Table 2. Examples for $D = 0$

8. $L \cap \Gamma$ when $D \neq 0$ and $l : m : n = u : v : w$

Returning to general homogeneous coordinates, suppose now that $l : m : n$ and $u : v : w$ are triangle centers for which $D \neq 0$. Then, sometimes, $R$ and $S$ are centers, and sometimes, a bicentric pair. We begin with the case $l : m : n = u : v : w$, for which (12) gives

$$D := (u + v + w)(u - v + w)(u + v - w)(u - v - w).$$

This factorization shows that if $u + v + w = 0$, then $D = 0$. We shall prove that converse also holds. Suppose $D = 0$ but $u + v + w \neq 0$. Then one of the other three factors must be 0, and by symmetry, they must all be 0, so that $u = v + w$, so that

$$u(a, b, c) = v(a, b, c) + w(a, b, c)$$
$$u(a, b, c) = u(b, c, a) + u(c, a, b)$$
$$u(b, c, a) = u(c, a, b) + u(a, b, c).$$

Applying the third equation to the second gives $u(a, b, c) = u(c, a, b) + u(a, b, c) + u(a, b, c)$, so that $u(a, b, c) = 0$, contrary to the hypothesis that $U$ is a triangle center.

Writing the roots of (11) as $r_2/r_3$ and $s_2/s_3$, we find

$$\frac{r_2 s_2}{r_3 s_3} = \frac{(u^2 - v^2 - w^2 + \sqrt{D})(u^2 - v^2 - w^2 - \sqrt{D})}{4v^2 w^2} = 1,$$
which proves that $R$ and $S$ are a conjugate pair (isogonal conjugates in case the coordinates are trilinears). Of particular interest are cases for which these points are polynomial centers, as listed in Table 3, where, for convenience, we put
\[ E := (b^2 - c^2)(c^2 - a^2)(a^2 - b^2). \]

\[ \begin{array}{|c|c|c|c|}
\hline
u & \sqrt{D} & r_1 & s_1 \\
\hline
a(b^2 - c^2) & E & a & bc \\
\hline
a(b^2 - c^2)(b^2 + c^2 - a^2) & 16\sigma^4E & \sec A & \cos A \\
\hline
a(b - c)(b + c - a) & 4abc(b - c)(c - a)(a - b) & \cot(A/2) & \tan(A/2) \\
\hline
a^2(b^2 - c^2)(b^2 + c^2 - a^2) & 4a^2b^2c^2E & \tan A & \cot A \\
\hline
bc(a^4 - b^2c^2) & (a^4 - b^2c^2)(b^4 - c^2a^2)(c^4 - a^2b^2) & b/c & c/b \\
\hline
\end{array} \\
\]

Table 3. Points $R = r_1 : r_2 : r_3$ and $S = s_1 : s_2 : s_3$ of intersection

In Table 3, the penultimate row indicates that for $u : v : w = X_{647}$, the Euler line meets the circumconic $u/a + v/b + w/c = 0$ in the points $X_4$ and $X_3$. The final row shows that $R$ and $S$ can be a bicentric pair.

9. $L \cap \Gamma$: Starting with Intersection Points

It is easy to check that a point $R$ lies on $\Gamma$ if and only if there exists a point $x : y : z$ for which
\[ R = \frac{u}{by - cz} : \frac{v}{cz - ax} : \frac{w}{ax - by}. \]

From this representation, it follows that every line that meets $\Gamma$ in distinct points
\[ \frac{u}{by_i - cz_i} : \frac{v}{cz_i - ax_i} : \frac{w}{ax_i - by_i}, \quad i = 1, 2, \]
has the form
\[ \frac{(by_1 - cz_1)(by_2 - cz_2)a}{u} + \frac{(cz_1 - ax_1)(cz_2 - ax_2)b}{v} + \frac{(ax_1 - by_1)(ax_2 - by_2)c}{w} = 0, \]
and conversely. In this case,
\[ D = u^2v^2w^2 \left| \begin{array}{ccc}
bc & ca & ab \\
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2
\end{array} \right|^2, \]
indicating that $D = 0$ if and only if the points $x_i : y_i : z_i$ are collinear with the $bc : ca : ab$, which, in case the coordinates are trilinears, is the centroid of $\triangle ABC$.

Example 1. Let
\[ x_1 : y_1 : z_1 = c/b : a/c : b/a \quad \text{and} \quad x_2 : y_2 : z_2 = b/c : c/a : a/b. \]
These are the 1st and 2nd Brocard points in case the coordinates are trilinears, but in any case, (22) represents the central line
\[ \frac{\alpha}{ua^2(b^2 - c^2)} + \frac{\beta}{vb^2(c^2 - a^2)} + \frac{\gamma}{wc^2(a^2 - b^2)} = 0, \]
meeting $\Gamma$ in the bicentric pair

\[
\begin{align*}
\frac{u}{b^2(a^2 - c^2)} : \frac{v}{c^2(b^2 - a^2)} & : \frac{w}{a^2(c^2 - b^2)} \\
\frac{u}{a^2(b^2 - c^2)} : \frac{v}{c^2(b^2 - a^2)} & : \frac{w}{b^2(c^2 - a^2)}.
\end{align*}
\]

**Example 2.** Let $X = x : y : z$ be a triangle center other than $X_1$, so that $y : z : x$ and $z : x : y$ are a bicentric pair. The points

\[
\frac{u}{b z - c x} : \frac{v}{c x - a y} : \frac{w}{a y - b z}, \quad \text{and} \quad \frac{u}{c y - b x} : \frac{v}{a z - c y} : \frac{w}{b x - a z}
\]

are the bicentric pair in which the central line

\[
v w (b x - c y) (c x - b z) + w u (c y - a z) (a y - c x) + v u (a z - b x) (b z - a y) \gamma = 0
\]

meets $\Gamma$.

10. $L \cap \Gamma :$ **Euler Line and Circumcircle**

**Example 3.** Using trilinears, the circumcircle is given by $u(a, b, c) = a$ and the Euler line by

\[
l(a, b, c) = a(b^2 - c^2)(b^2 + c^2 - a^2).
\]

The discriminant $D = 4 a^2 b^2 c^2 d^2$, where

\[
d = \sqrt{a^6 + b^6 + c^6 + 3a^2 b^2 c^2 - b^2 c^2 (b^2 + c^2) - c^2 a^2 (c^2 + a^2) - a^2 b^2 (a^2 + b^2)}.
\]

Substitutions into (17) and (18) and simplification give the points of intersection, centers $R$ and $S$, represented by 1st coordinates

\[
\left\{ \frac{[ca(a^2 - c^2) \pm bd][ba(a^2 - b^2) \pm cd]}{(b^2 - c^2)^2 (b^2 + c^2 - a^2)^2} \right\}^{1/3}.
\]

11. **Vertex-products of bicentric triangles**

Suppose that $f(a, b, c) : g(b, c, a) : h(c, a, b)$ is a point, as defined in [2] We abbreviate this point as $f_{ab} : g_{bc} : h_{ca}$ and recall from [5, 7] that bicentric triangles are defined by the forms

\[
\begin{pmatrix}
    f_{ab} & g_{bc} & h_{ca} \\
    h_{ab} & f_{bc} & g_{ca} \\
    g_{ab} & h_{bc} & f_{ca}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    f_{ac} & h_{ba} & g_{cb} \\
    g_{ac} & f_{ba} & h_{cb} \\
    h_{ac} & g_{ba} & f_{cb}
\end{pmatrix}.
\]

The vertices of the first of these two triangles are the rows of the first matrix, etc. We assume that $f_{ab} g_{ab} h_{ab} \neq 0$. Then the product of the three vertices, namely

\[
f_{ab} g_{ab} h_{ab} : f_{bc} g_{bc} h_{bc} : f_{ca} g_{ca} h_{ca}
\]

and the product of the vertices of the second triangle, namely

\[
f_{ac} g_{ac} h_{ac} : f_{ba} g_{ba} h_{ba} : f_{cb} g_{cb} h_{cb}
\]

clearly comprise a bicentric pair if they are distinct, and a triangle center otherwise.

Examples of bicentric pairs thus obtained will now be presented. An inductive method [6] of generating the non-circle-dependent objects of triangle geometry enumerates such objects in sets formally of size six. When the actual size is six, which means that no two of the six objects are identical, the objects form a pair
of bicentric triangles. The least such pair for which \( f_{ab}g_{ab}/h_{ab} \neq 0 \) are given by Objects 31-36:

\[
\begin{pmatrix}
  b & c \cos B & -b \cos B \\
  -c \cos C & c & a \cos C \\
  b \cos A & -a \cos A & a
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  c & -c \cos C & b \cos C \\
  c \cos A & a & -a \cos A \\
  -b \cos B & a \cos B & -b
\end{pmatrix}.
\]

In this example, the bicentric pair of points (23) and (24) are

\[
\frac{b}{a \cos B} : \frac{c}{b \cos C} : \frac{a}{c \cos A} \quad \text{and} \quad \frac{c}{a \cos C} : \frac{a}{b \cos A} : \frac{b}{c \cos B},
\]

and the product of these is the center \( \cos A \csc^3 A : \cos B \csc^3 B : \cos C \csc^3 C \).

This example and others obtained successively from Generation 2 of the aforementioned enumeration are presented in Table 4. Column 1 tells the Object numbers in [5]; column 2, the \( A \)-vertex of the least Object; column 3, the first coordinate of point (23) after canceling a symmetric function of \((a, b, c)\); and column 4, the first coordinate of the product of points (23) and (24) after canceling a symmetric function of \((a, b, c)\). In Table 4, \( \cos A, \cos B, \cos C \) are abbreviated as \( a_1, b_1, c_1 \), respectively.

<table>
<thead>
<tr>
<th>Objects</th>
<th>( f_{ab} : g_{ab} : h_{ab} )</th>
<th>( f_{ab}g_{ab}/h_{ab} )</th>
<th>( f_{ab}f_{ac}g_{ac}h_{ac} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>31-36</td>
<td>( b : cb_1 : -bb_1 )</td>
<td>( b/ab_1 )</td>
<td>( a_1/a )</td>
</tr>
<tr>
<td>37-42</td>
<td>( bc_1 : -ca_1 : ba_1 )</td>
<td>( bc_1/aa_1 )</td>
<td>( (aa_1)^{-3} )</td>
</tr>
<tr>
<td>43-48</td>
<td>( bb_1 : c : -b )</td>
<td>( bb_1/a )</td>
<td>( (a_1 a^2)^{-1} )</td>
</tr>
<tr>
<td>49-54</td>
<td>( ab : -c^2 : bc )</td>
<td>( b/c )</td>
<td>1</td>
</tr>
<tr>
<td>58-63</td>
<td>( c + ba_1 : cc_1 : -bc_1 )</td>
<td>( (ba_1 + c)/ac_1 )</td>
<td>( a_1 (ba_1 + c)(ca_1 + b)a^{-2} )</td>
</tr>
<tr>
<td>71-76</td>
<td>( -b_1^2 : c_1 : b_1 )</td>
<td>( b_1^2/a_1 )</td>
<td>( a_1^{-4} )</td>
</tr>
<tr>
<td>86-91</td>
<td>( c_1 - a_1 b_1 : c_1^2 : b_1 c_1 )</td>
<td>( b_1 (c_1 - a_1 b_1) )</td>
<td>( (a_1 (a_1 - b_1 c_1))^{-1} )</td>
</tr>
<tr>
<td>92-97</td>
<td>( a_1 b_1 : 1 : -a_1 )</td>
<td>( b_1/c_1 )</td>
<td>1</td>
</tr>
<tr>
<td>98-103</td>
<td>( 1 : -c_1 : c_1 a_1 )</td>
<td>( b_1/c_1 )</td>
<td>1</td>
</tr>
<tr>
<td>104-109</td>
<td>( aa_1 : -c : ca_1 )</td>
<td>( a/cc_1 )</td>
<td>( a^2 a_1 )</td>
</tr>
<tr>
<td>110-115</td>
<td>( a : b : -ba_1 )</td>
<td>( ab_1/b )</td>
<td>( a^3/a_1 )</td>
</tr>
<tr>
<td>116-121</td>
<td>( c_1 - a_1 b_1 : 1 : -a_1 )</td>
<td>( b_1 (c_1 - a_1 b_1) )</td>
<td>( (a_1 (a_1 - b_1 c_1))^{-1} )</td>
</tr>
<tr>
<td>122-127</td>
<td>( 1 + a_1^2 : c_1 : -c_1 a_1 )</td>
<td>( b_1 (1 + a_1^2)/c_1 )</td>
<td>( (1 + a_1^2) )</td>
</tr>
<tr>
<td>128-133</td>
<td>( 2a_1 : -b_1 : a_1 b_1 )</td>
<td>( a_1 )</td>
<td>( a_1^2 )</td>
</tr>
</tbody>
</table>

Table 4. Bicentric triangles, bicentric points, and central vertex-products

Table 4 includes examples of interest: (i) bicentric triangles for which (23) and (24) are identical and therefore represent a center; (ii) distinct pairs of bicentric triangles that yield the identical bicentric pairs of points; and (iii) cases in which the pair (23) and (24) are isogonal conjugates. Note that Objects 49-54 yield for (23) and (24) the 2nd Brocard point, \( \Omega_2 = b/c : c/a : a/b \) and the 1st Brocard point, \( \Omega_1 = c/b : a/c : b/a \).
12. Geometric discussion: ⊕ and ⊖

Equations (3) and (4) define operations ⊕ and ⊖ on pairs of bicentric points. Here, we shall consider the geometric meaning of these operations. First, note that one of the points in (2) lies on $\mathcal{L}^\infty$ if and only if the other lies on $\mathcal{L}^\infty$, since the transformation $(a, b, c) \to (a, c, b)$ carries each of the equations

$$af_{ab} + bf_{bc} + cf_{ca} = 0, \quad af_{ac} + bf_{ba} + cf_{cb} = 0$$

to the other. Accordingly, the discussion breaks into two cases.

Case 1: $F_{ab}$ not on $\mathcal{L}^\infty$. Let $k_{ab}$ and $k_{ac}$ be the normalization factors given in §3. Then the actual directed trilinear distances of $F_{ab}$ and $F_{ac}$ (to the sidelines $BC, CA, AB$) are given by (5). The point $F$ that separates the segment $F_{ab}F_{ac}$ into segments satisfying

$$\frac{|F_{ab}F|}{|F_{ac}|} = \frac{k_{ab}}{k_{ac}},$$

where $||$ denotes directed length, is then

$$\frac{k_{ac}}{k_{ab} + k_{ac}} F'_{ab} + \frac{k_{ab}}{k_{ab} + k_{ac}} F'_{ac} = \frac{k_{ac}k_{ab}}{k_{ab} + k_{ac}} F_{ab} + \frac{k_{ab}k_{ac}}{k_{ab} + k_{ac}} F_{ac},$$

which, by homogeneity, equals $F_{ab} \oplus F_{ac}$. Similarly, the point “constructed” as

$$\frac{k_{ab}}{k_{ab} + k_{ac}} F'_{ab} - \frac{k_{ab}}{k_{ab} + k_{ac}} F'_{ac}$$

equals $F_{ab} \ominus F_{ac}$. These representations show that $F_{ab} \oplus F_{ac}$ and $F_{ab} \ominus F_{ac}$ are a harmonic conjugate pair with respect to $F_{ab}$ and $F_{ac}$.

Case 2: $F_{ab}$ on $\mathcal{L}^\infty$. In this case, the isogonal conjugates $F_{ab}^{-1}$ and $F_{ac}^{-1}$ lie on the circumcircle, so that Case 1 applies:

$$F_{ab}^{-1} \oplus F_{ac}^{-1} = \frac{f_{ab} + f_{ac}}{f_{ab}f_{ac}} : \frac{f_{bc} + f_{ba}}{f_{bc}f_{ba}} : \frac{f_{ca} + f_{cb}}{f_{ca}f_{cb}}.$$ Trilinear multiplication [6] by the center $F_{ab} \cdot F_{ac}$ gives

$$F_{ab} \oplus F_{ac} = (F_{ab}^{-1} \oplus F_{ac}^{-1}) \cdot F_{ab} \cdot F_{ac}.$$ In like manner, $F_{ab} \ominus F_{ac}$ is “constructed”.

It is easy to prove that a pair $P_{ab}$ and $P_{ac}$ of bicentric points on $\mathcal{L}^\infty$ are necessarily given by

$$P_{ab} = bf_{ca} - cf_{bc} : cf_{ab} - af_{ca} : af_{bc} - bf_{ab}$$

for some bicentric pair as in (2). Consequently,

$$P_{ab} \oplus P_{ac} = g(a, b, c) : g(b, c, a) : g(c, a, b),$$

$$P_{ab} \ominus P_{ac} = h(a, b, c) : h(b, c, a) : h(c, a, b),$$

where

$$g(a, b, c) = b(f_{ca} + f_{cb}) - c(f_{bc} + f_{ba}),$$

$$h(a, b, c) = b(f_{ca} - f_{cb}) + c(f_{ba} - f_{bc}).$$
Example 4. We start with \( f_{ab} = c/b \), so that \( F_{ab} \) and \( F_{ac} \) are the Brocard points, and \( P_{ab} \) and \( P_{ac} \) are given by 1st coordinates \( a - c^2/a \) and \( a - b^2/a \), respectively, yielding 1st coordinates \((2a^2 - b^2 - c^2)/a\) and \((b^2 - c^2)/a\) for \( P_{ab} \oplus P_{ac} \) and \( P_{ab} \ominus P_{ac} \). These points are the isogonal conjugates of \( X_{111} \) (the Parry point) and \( X_{110} \) (focus of the Kiepert parabola), respectively.

References


Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu
Some Configurations of Triangle Centers

Lawrence S. Evans

Abstract. Many collections of triangle centers and symmetrically defined triangle points are vertices of configurations. This illustrates a high level of organization among the points and their collinearities. Some of the configurations illustrated are inscriptible in Neuberg’s cubic curve and others arise from Monge’s theorem.

1. Introduction

By a configuration $\mathcal{K}$ we shall mean a collection of $p$ points and $g$ lines with $r$ points on each line and $q$ lines meeting at each point. This implies the relationship $pq = gr$. We then say that $\mathcal{K}$ is a $(p,g)$ configuration. The simplest configuration is a point with a line through it. Another example is the triangle configuration, $(3_2, 3_2)$ with $p = g = 3$ and $q = r = 2$. When $p = g$, $\mathcal{K}$ is called self-dual, and then we must also have $q = r$. The symbol for the configuration is now simplified to read $(p,q)$. The smallest $(n_3)$ self-dual configurations exist combinatorially, when the “lines” are considered as suitable triples of points (vertices), but they cannot be realized with lines in the Euclidean plane. Usually when configurations are presented graphically, the lines appear as segments to make the figure compact and easy to interpret. Only one $(7_3)$ configuration exists, the Fano plane of projective geometry, and only one $(8_3)$ configuration exists, the Möbius-Kantor configuration. Neither of these can be realized with straight line segments. For larger $n$, the symbol may not determine a configuration uniquely. The smallest $(n_3)$ configurations consisting of line segments in the Euclidean plane are $(9_3)$, and there are three of them, one of which is the familiar Pappus configuration [4, pp.94–170]. The number of distinct $(n_3)$ configurations grows rapidly with $n$. For example, there are 228 different $(12_3)$ configurations [11, p.40]. In the discussion here, we shall only be concerned with configurations lying in a plane.

While configurations have long been studied as combinatorial objects, it does not appear that in any examples the vertices have been identified with triangle-derived points. In recent years there has been a resurgence of interest in triangle geometry along with the recognition of many new special points defined in different very ways. Since each point is defined from original principles, it is somewhat surprising that so many of them are collinear in small sets. An even higher level of relationship among special points is seen when they can be incorporated into...
certain configurations of moderate size. Then the collinearities and their incidences are summarized in a tidy, symmetrical, and graphic way. Here we exhibit several configurations whose vertices are naturally defined by triangles and whose lines are collinearities among them. It happens that the general theory for the first three examples was worked out long ago, but then the configurations were not identified as consisting of familiar triangle points and their collinearities.

2. Some configurations inscriptable in a cubic

First let us set the notation for several triangles. Given a triangle $T$ with vertices $A$, $B$, and $C$, let $A^*$ be the reflection of vertex $A$ in side $BC$, $A_+$ the apex of an equilateral triangle erected outward on $BC$, and $A_-$ the apex of an equilateral triangle erected inward on $BC$. Similarly define the corresponding points for $B$ and $C$. Denote the triangle with vertices $A^*$, $B^*$, $C^*$ as $T^*$ and similarly define the triangles $T_+$ and $T_-$. Using trilinear coordinates it is straightforward to verify that the four triangles above are pairwise in perspective to one another. The points of perspective are as follows.

\[
\begin{array}{c|cccc}
 & T & T^* & T_+ & T_- \\
T & H & F_+ & F_- \\
T^* & H & J_+ & J_- \\
T_+ & F_+ & J_- & O \\
T_- & F_- & J_+ & O \\
\end{array}
\]

Here, $O$ and $H$ are respectively the circumcenter and orthocenter, $F_\pm$ the isogonic (Fermat) points, and $J_\pm$ the isodynamic points. They are triangle centers as defined by Kimberling [5, 6, 7, 8], who gives their trilinear coordinates and discusses their geometric significance. See also the in §5. For a simple simultaneous construction of all these points, see Evans [2].

To assemble the configurations, we first need to identify certain sets of collinear points. Now it is advantageous to introduce a notation for collinearity. Write $\mathcal{L}(X, Y, Z, \ldots)$ to denote the line containing $X, Y, Z, \ldots$. The key to identifying configurations among all the previously mentioned points depends on the observation that $A^*$, $B_+$, and $C_-$ are always collinear, so we may write $\mathcal{L}(A^*, B_+, C_-)$. One can easily verify this using trilinear coordinates. This is also true for any permutation of $A$, $B$, and $C$, so we have

(I): the 6 lines $\mathcal{L}(A^*, B_+, C_-)$, $\mathcal{L}(A^*, B_-, C_+)$, $\mathcal{L}(B^*, C_+, A_-)$, $\mathcal{L}(B^*, C_-, A_+)$, $\mathcal{L}(C_+, A_+, B_-)$, $\mathcal{L}(C_-, A_-, B_+)$.

They all occur in Figures 1, 2, and 3. In fact the nine points $A_+, A_-, A^*$, $\ldots$ themselves form the vertices of a $(9_2, 6_3)$ configuration.

It is easy to see other collinearities, namely 3 from each pair of triangles in perspective. For example, triangles $T_+$ and $T_-$ are in perspective from $O$, so we have

(II): the 3 lines $\mathcal{L}(A_+, O, A_-)$, $\mathcal{L}(B_+, O, B_-)$ and $\mathcal{L}(C_+, O, C_-)$.

See Figure 2.
2.1. The Cremona-Richmond configuration \((15_3)\). Consider the following sets of collinearities of three points:

(III): the 3 lines \(\mathcal{L}(A, F_+, A_+)\), \(\mathcal{L}(B, F_+, B_+)\) and \(\mathcal{L}(C, F_+, C_+)\);

(IV): the 3 lines \(\mathcal{L}(A, F_-, A_-)\), \(\mathcal{L}(B, F_-, B_-)\) and \(\mathcal{L}(C, F_-, C_-)\);

(V): the 3 lines \(\mathcal{L}(A, H, A^*)\), \(\mathcal{L}(B, H, B^*)\) and \(\mathcal{L}(C, H, C^*)\).

The 15 points \((A, B, C, A^*, B^*, C^*, A_\pm, B_\pm, C_\pm, H, F_\pm)\) and 15 lines in (I), (III), (IV), and (V) form a figure which is called the Cremona-Richmond configuration \([7]\). See Figure 1. It has 3 lines meeting at each point with 3 points on each line, so it is self-dual with symbol \((15_3)\). Inspection reveals that this configuration itself contains no triangles.

The reader may have noticed that the fifteen points in the configuration all lie on Neuberg’s cubic curve, which is known to contain many triangle centers \([7]\). Recently a few papers, such as Pinkernell’s \([10]\) discussing Neuberg’s cubic have
Figure 2

appeared, so we shall not elaborate on the curve itself. It has been known for a long time that many configurations are inscriptable in cubic curves, possibly first noticed by Schoenflies circa 1888 according to Feld [3]. However, it does not appear to be well-known that Neuberg’s cubic in particular supports such configurations of familiar points. We shall exhibit two more configurations inscriptable in Neuberg’s cubic.

2.2. A (183) associated with the excentral triangle. For another configuration, this one of the type (183), we employ the excentral triangle, that is, the triangle whose vertices are the excenters of $T$. Denote the excenter opposite vertex $A$ by $I_a$, etc., and denote the extriangle as $T_x$. Triangles $T$ and $T_x$ are in perspective from the incenter, $I$. This introduces two more sets of collinearities involving the excenters:

- **(VI):** the 3 lines $\mathcal{L}(A, I, I_a)$, $\mathcal{L}(B, I, I_b)$ and $\mathcal{L}(C, I, I_c)$;
- **(VII):** the 3 lines $\mathcal{L}(I_b, A, I_c)$, $\mathcal{L}(I_c, B, I_a)$ and $\mathcal{L}(I_a, C, I_b)$.
The 18 lines of (I), (II), (V), (VI), (VII) and the 18 points $A$, $B$, $C$, $I_a$, $I_b$, $I_c$, $A^*$, $B^*$, $C^*$, $A_{\pm}$, $B_{\pm}$, $C_{\pm}$, $O$, $H$, and $I$ form an (18_3) configuration. See Figure 2. There are enough points to suggest the outline of Neuberg’s cubic, which is bipartite. The 10 points in the lower right portion of the figure lie on the ovoid portion of the curve. The 8 other points lie on the serpentine portion, which has an asymptote parallel to Euler’s line (dashed). For other shapes of the basic triangle $T$, these points will not necessarily lie on the same components of the curve.

2.3. A configuration $(12_4, 16_3)$. Now we define two more sets of collinearities involving the isodynamic points:

- (VIII): the 3 lines $L(A^*, J_-, A_{+})$, $L(B^*, J_-, B_{+})$ and $L(C^*, J_-, C_{+})$;
- (IX): the 3 lines $L(A^*, J_+, A_{-})$, $L(B^*, J_+, B_{-})$ and $L(C^*, J_+, C_{-})$.

Among the centers of perspective we have defined so far, there is an additional collinearity, $L(J_+, O, J_-)$, which is the Brocard axis. See Figure 3.

Using Weierstrass elliptic functions, Feld proved that within any bipartite cubic, a real configuration can be inscribed which has 12 points and 16 lines, with 4 lines meeting at each point and 3 points on each line [11], so that is, its symbol is $(12_4, 16_3)$. Now the Neuberg cubic of a non-equilateral triangle is bipartite, consisting of an ovoid portion and a serpentine portion whose asymptote is parallel to the Euler line of the triangle. Here one such inscriptable configuration consists of the following sets of lines: (I), (II), (VIII), (IX), and the line, $L(J_+, O, J_-)$. See Figure 3. The three triangles $T_+$, $T_-$, and $T^*$ are pair-wise in perspective.
with collinear perspectors \(J_+\), \(J_-\), and \(O\). The vertices of the basic triangle \(T\) are not in this configuration.

3. A Desargues configuration with triangle centers as vertices

There are so many collinearities involving triangle centers that we can also exhibit a Desargues \((10_3)\) configuration with vertices consisting entirely of basic centers. Let \(K\) denote the symmedian (Lemoine’s) point, \(N_p\) the center of the nine-point circle, \(G\) the centroid, \(N_+\) the first Napoleon point, and \(N_-\) the second Napoleon point. Then the ten points \(F_+, F_-, J_+, J_-, N_+, N_-, K, G, H\) and \(N_p\) form the vertices of such a configuration. This is seen on noting that the triangles \(F_-J_+N_+\) and \(F_+J_-N_-\) are in perspective from \(K\) with the line of perspective \(L(G, N_p, H)\), which is Euler’s line. See Figure 4. In a Desargues configuration any vertex may be chosen as the center of perspective of two suitable triangles. For simplicity we have chosen \(K\) in this example. Unlike the previous examples, Desargues configurations are not inscriptable in cubic curves [9].

4. Configurations from Monge’s theorem

Another way triangle centers form vertices of configurations arises from Monge’s theorem [4, 11]. This theorem states that if we have three circles, then the 3 external centers of similitude (ecs) are collinear and that each external center of similitude is collinear with two of the internal centers of similitude (ics). These 4 collinearities form a \((4_3, 6_2)\) configuration, i.e., a complete quadrilateral with the centers of similitude as vertices. This is best illustrated by an example. Suppose we have the circumcircle, the nine-point circle, and the incircle of a triangle. The ics of the circumcircle and the incircle is the centroid, \(G\), and their ecs is the orthocenter, \(H\). The ics of the nine-point circle and the incircle is \(X_{12}\) in Kimberling’s list and the ecs is Feuerbach’s point, \(X_{11}\). The ics of the circumcircle and the incircle is \(X_{55}\), and the ecs is \(X_{56}\). The lines of the configuration
Some configurations of triangle centers

are then $\mathcal{L}(H, X_{56}, X_{11}), \mathcal{L}(G, X_{55}, X_{11}), \mathcal{L}(G, X_{56}, X_{12}),$ and $\mathcal{L}(H, X_{55}, X_{12})$. This construction, of course, applies to any group of three circles related to the triangle. In the example given, the circles can be nested, so it may not be easy to see the centers of similitude. In such a case, the radii of the circles can be reduced in the same proportion to make the circles small enough that they do not overlap. The $\text{ecs}$’s and $\text{ics}$’s remain the same. The $\text{ecs}$ of two such circles is the point where the two common external tangents meet, and the $\text{ics}$ is the point where the two common internal tangents meet. When two of the circles have the same radii, their $\text{ics}$ is the midpoint of the line joining their centers and their $\text{ecs}$ is the point at infinity in the direction of the line joining their centers.

One may ask what happens when a fourth circle whose center is not collinear with any other two is also considered. Monge’s theorem applies to each group of three circles. First it happens that the four lines containing only $\text{ecs}$’s themselves form a $(6_2, 4_3)$ configuration. Second, when the twelve lines containing an $\text{ecs}$ and two $\text{ics}$’s are annexed, the result is a $(12_4, 16_3)$ configuration. This is a projection onto the plane of Reye’s three-dimensional configuration, which arises from a three-dimensional analog of Monge’s theorem for four spheres [4]. This is illustrated in Figure 5 with the vertices labelled with the points of Figure 3, which shows that these two $(12_4, 16_3)$ configurations are actually the same even though the representation in Figure 5 may not be inscriptable in a bipartite cubic. Evidently larger configurations arise by the same process when yet more circles are considered.

5. Final remarks

We have see that certain collections of collinear triangle points can be knitted together into highly symmetrical structures called configurations. Furthermore some relatively large configuratons such as the $(18_3)$ shown above are inscriptable in low degree algebraic curves, in this case a cubic.

General information about configurations can be found in Hilbert and Cohn-Vossen [4]. Also we recommend Coxeter [1], which contains an extensive bibliography of related material pre-dating 1950.

The centers here appear in Kimberling [5, 6, 7, 8] as $X_n$ for $n$ below.

<table>
<thead>
<tr>
<th>center</th>
<th>$I$</th>
<th>$G$</th>
<th>$O$</th>
<th>$H$</th>
<th>$N_p$</th>
<th>$K$</th>
<th>$F_+$</th>
<th>$F_-$</th>
<th>$J_+$</th>
<th>$J_-$</th>
<th>$N_+$</th>
<th>$N_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
</tbody>
</table>

While not known by eponyms, $X_{12}$, $X_{55}$, and $X_{56}$ are also geometrically significant in elementary ways [7, 8].

References


Figure 5


Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: 75342.3052@compuserve.com
On the Circumcenters of Cevasix Configurations

Alexei Myakishev and Peter Y. Woo

Abstract. We strengthen Floor van Lamoen’s theorem that the 6 circumcenters of the cevasix configuration of the centroid of a triangle are concyclic by giving a proof which at the same time shows that the converse is also true with a minor qualification, i.e., the circumcenters of the cevasix configuration of a point $P$ are concyclic if and only if $P$ is the centroid or the orthocenter of the triangle.

1. Introduction

Let $P$ be a point in the plane of triangle $ABC$, with traces $A', B', C'$ on the sidelines $BC, CA, AB$ respectively. We assume that $P$ does not lie on any of the sidelines. Triangle $ABC$ is then divided by its cevians $AA', BB', CC'$ into six triangles, giving rise to what Clark Kimberling [2, pp.257–260] called the cevasix configuration of $P$. See Figure 1. Floor van Lamoen has discovered that when $P$ is the centroid of triangle $ABC$, the 6 circumcenters of the cevasix configuration are concyclic. See Figure 2. This was posed as a problem in the American Mathematical Monthly [3]. Solutions can be found in [3, 4]. In this note we study the converse.

![Figure 1](image1.png)

![Figure 2](image2.png)

Theorem 1. The circumcenters of the cevasix configuration of $P$ are concyclic if and only if $P$ is the centroid or the orthocenter of triangle $ABC$. 

2. Preliminary results

We adopt the following notations.

<table>
<thead>
<tr>
<th>Triangle</th>
<th>$PCB'$</th>
<th>$PC'B$</th>
<th>$PAC'$</th>
<th>$PA'C$</th>
<th>$PBA'$</th>
<th>$PB'A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation</td>
<td>$\triangle(A_+)$</td>
<td>$\triangle(A_-)$</td>
<td>$\triangle(B_+)$</td>
<td>$\triangle(B_-)$</td>
<td>$\triangle(C_+)$</td>
<td>$\triangle(C_-)$</td>
</tr>
<tr>
<td>Circumcenter</td>
<td>$A_+$</td>
<td>$A_-$</td>
<td>$B_+$</td>
<td>$B_-$</td>
<td>$C_+$</td>
<td>$C_-$</td>
</tr>
</tbody>
</table>

It is easy to see that two of these triangle may possibly share a common circumcenter only when they share a common vertex of triangle $ABC$.

**Lemma 2.** The circumcenters of triangles $APB'$ and $APC'$ coincide if and only if $P$ lies on the reflection of the circumcircle in the line $BC$.

*Proof.* Triangles $APB'$ and $APC'$ have the same circumcenter if and only if the four points $A, B', P, C'$ are concyclic. In terms of directed angles, $\angle BPC = \angle B'PC' = \angle B'AC' = \angle CAB = -\angle BAC$. See, for example, [1, §§16–20]. It follows that the reflection of $A$ in the line $BC$ lies on the circumcircle of triangle $PBC$, and $P$ lies on the reflection of the circumcircle in $BC$. The converse is clear.

Thus, if $B_+ = C_-$ and $C_+ = A_-$, then necessarily $P$ is the orthocenter $H$, and also $A_+ = B_-$. In this case, there are only three distinct circumcenters. They clearly lie on the nine-point circle of triangle $ABC$. We shall therefore assume $P \neq H$, so that there are at least five distinct points in the set $\{A_\pm, B_\pm, C_\pm\}$.

The next proposition appears in [2, p.259].

**Proposition 3.** The 6 circumcenters of the cevasix configuration of $P$ lie on a conic.

*Proof.* We need only consider the case when these 6 circumcenters are all distinct. The circumcenters $B_+$ and $C_-$ lie on the perpendicular bisector of the segment $AP$; similarly, $B_-$ and $C_+$ lie on the perpendicular bisector of $PA'$. These two perpendicular bisectors are clearly parallel. This means that $B_+C_-$ and $B_-C_+$ are parallel. Similarly, $C_+A_+//C_-A_-$ and $A_+B_-//A_-B_+$. The hexagon $A_+C_-B_+A_-C_+B_-$ has three pairs of parallel opposite sides. By the converse of Pascal’s theorem, there is a conic passing through the six vertices of the hexagon.

**Proposition 4.** The vertices of a hexagon $A_+C_-B_+A_-C_+B_-$ with parallel opposite sides $B_+C_-//C_+B_-, C_+A_-//A_+C_-, A_+B_-//B_+A_-$ lie on a circle if and only if the main diagonals $A_+A_-, B_+B_-$ and $C_+C_-$ have equal lengths.

*Proof.* If the vertices are concyclic, then $A_+C_-A_-C_+$ is an isosceles trapezoid, and $A_+A_-=C_+C_-$. Similarly, $C_+B_-B_+$ is also an isosceles trapezoid, and $C_+C_- = B_+B_-$. Conversely, consider the triangle $XYZ$ bounded by the three diagonals $A_+, A_-, B_+, B_-$ and $C_+, C_-$. If these diagonals are equal in length, then the trapezoids $A_+C_-A_-C_+, C_+B_-B_+ and B_+A_-A_+$ are isosceles. From these we immediately conclude that the common perpendicular bisector of $A_+C_-$ and $A_-C_+$.
is the biser of angle \(XYZ\). Similarly, the common perpendicular bisector of \(B_+C_-\) and \(B_-C_+\) is the biser of angle \(X\), and that of \(A_+B_-\) and \(A_-B_+\) the biser of angle \(Z\). These three perpendicular bisectors clearly intersect at a point, the incenter of triangle \(XYZ\), which is equidistant from the six vertices of the hexagon. \(\square\)

**Proposition 5.** The vector sum \(AA' + BB' + CC' = 0\) if and only if \(P\) is the centroid.

**Proof.** Suppose with reference to triangle \(ABC\), the point \(P\) has absolute barycentric coordinates \(uA + vB + wC\), where \(u + v + w = 1\). Then,
\[
A' = \frac{1}{v + w}(vB + wC), \quad B' = \frac{1}{w + u}(wC + uA), \quad C' = \frac{1}{u + v}(uA + vB).
\]
From these,
\[
AA' + BB' + CC' = (A' + B' + C') - (A + B + C)
\]
\[
= \frac{u^2 - vw}{(w + u)(u + v)} \cdot A + \frac{v^2 - uw}{(u + v)(v + w)} \cdot B + \frac{w^2 - uv}{(v + w)(w + u)} \cdot C.
\]
This is zero if and only if
\[
u^2 - vw = v^2 - uw = w^2 - uv = 0,
\]
and \(u = v = w = \frac{1}{3}\) since they are all real, and \(u + v + w = 1\). \(\square\)

We denote by \(\pi_a\), \(\pi_b\), \(\pi_c\) the orthogonal projections on the lines \(AA', BB', CC'\) respectively.

**Proposition 6.**
\[
\begin{align*}
\pi_b(A_+A_-) &= -\frac{1}{2}BB', & \pi_c(A_+A_-) &= \frac{1}{2}CC', \\
\pi_c(B_+B_-) &= -\frac{1}{2}CC', & \pi_a(B_+B_-) &= \frac{1}{2}AA', \\
\pi_a(C_+C_-) &= -\frac{1}{2}AA', & \pi_b(C_+C_-) &= \frac{1}{2}BB'.
\end{align*}
\]

**Proof.** The orthogonal projections of \(A_i\) and \(A_-\) on the cevian \(BB'\) are respectively the midpoints of the segments \(PB'\) and \(BP\). Therefore,
\[
\pi_b(A_+A_-) = \frac{B + P}{2} - \frac{P + B'}{2} = \frac{B' - B}{2} = -\frac{1}{2}BB'.
\]
The others follow similarly. \(\square\)

**3. Proof of Theorem 1**

**Sufficiency part.** Let \(P\) be the centroid \(G\) of triangle \(ABC\). By Proposition 4, it is enough to prove that the diagonals \(A_+A_-\), \(B_+B_-\) and \(C_+C_-\) have equal lengths. By Proposition 5, we can construct a triangle \(A^*B^*C^*\) whose sides as vectors \(B^*C^*, C^*A^*\) and \(A^*B^*\) are equal to the medians \(AA', BB', CC'\) respectively.
Consider the vector $A^*Q$ equal to $A_+A_-$. By Proposition 6, the orthogonal projections of $A_+A_-$ on the two sides $C^*A^*$ and $A^*B^*$ are the midpoints of the sides. This means that $Q$ is the circumcenter of triangle $A'B'C^*$, and the length of $A_+A_-$ is equal to the circumradius of triangle $A'B'C^*$. The same is true for the lengths of $B_+B_-$ and $C_+C_-$. The case $P = H$ is trivial.

**Necessity part.** Suppose the 6 circumcenters $A_\pm, B_\pm, C_\pm$ lie on a circle. By Proposition 3, the diagonals $A_+A_-, B_+B_-, C_+C_-$ have equal lengths. We show that $AA' + BB' + CC' = 0$, so that $P$ is the centroid of triangle $ABC$ by Proposition 5. In terms of scalar products, we rewrite equation (1) as

$$
A_+A_- \cdot BB' = -\frac{1}{2} BB' \cdot BB', \quad A_+A_- \cdot CC' = \frac{1}{2} CC' \cdot CC',
$$

$$
B_+B_- \cdot CC' = -\frac{1}{2} CC' \cdot CC', \quad B_+B_- \cdot AA' = \frac{1}{2} AA' \cdot AA',
$$

$$
C_+C_- \cdot AA' = -\frac{1}{2} AA' \cdot AA', \quad C_+C_- \cdot BB' = \frac{1}{2} BB' \cdot BB'.
$$

From these, $(B_+B_- + C_+C_-) \cdot AA' = 0$, and $AA'$ is orthogonal to $B_+B_- + C_+C_-$. Since $B_+B_-$, and $C_+C_-$ have equal lengths, $B_+B_- + C_+C_-$ and $B_+B_- - C_+C_-$ are orthogonal. We may therefore write $B_+B_- - C_+C_- = kAA'$ for a scalar $k$. From (2) above,

$$
kAA' \cdot AA' = (B_+B_- - C_+C_-) \cdot AA'
$$

$$
= \frac{1}{2} AA' \cdot AA' + \frac{1}{2} AA' \cdot AA'
$$

$$
= AA' \cdot AA'.
$$

From this, $k = 1$ and we have

$$
AA' = B_+B_- - C_+C_-.
$$

The same reasoning shows that

$$
BB' = C_+C_- - A_+A_-,
$$

$$
CC' = A_+A_- - B_+B_-.
$$

Combining the three equations, we have

$$
AA' + BB' + CC' = 0.
$$

It follows from Proposition 5 that $P$ must be the centroid of triangle $ABC$.

**4. An alternative proof of Theorem 1**

We present another proof of Theorem 1 by considering an auxiliary hexagon. Let $L_a$ and $L'_a$ be the lines perpendicular to $AA'$ at $A$ and $A'$ respectively; similarly, $L_b, L'_b,$ and $L_c$ and $L'_c$. Consider the points

$$
X_+ = L_a \cap L'_b, \quad X_- = L_b \cap L'_a,
$$

$$
Y_+ = L_a \cap L'_c, \quad Y_- = L_c \cap L'_a,
$$

$$
Z_+ = L_b \cap L'_a, \quad Z_- = L_a \cap L'_b.
$$
Note that the circumcenters $A_\pm$, $B_\pm$, $C_\pm$ are respectively the midpoints of $PX_\pm$, $PY_\pm$, $PZ_\pm$. Hence, the six circumcenters are concyclic if and only if $X_\pm$, $Y_\pm$, $Z_\pm$ are concyclic.

In Figure 3, let $\angle CPA' = \angle APC' = \alpha$. Since angles $PA'Y_-$ and $PCY_-$ are both right angles, the four points $P$, $A'$, $C$, $Y_-$ are concyclic and $\angle Z_+Y_-X_+ = \angle A'Y_-X_+ = \angle A'PC = \alpha$. Similarly, $\angle CPB' = \angle BPC' = \angle Y_-X_+Z_-$, and we denote the common measure by $\beta$.

**Lemma 7.** If the four points $X_+$, $Y_-$, $Z_+$, $Z_-$ are concyclic, then $P$ lies on the median through $C$.

**Proof.** Let $x = \frac{AP}{AA'}$ and $y = \frac{BP}{BB'}$. If the four points $X_+$, $Y_-$, $Z_+$, $Z_-$ are concyclic, then $\angle Z_+Z_-X_+ = \alpha$ and $\angle Y_-Z_+Z_- = \beta$. Now,

$$\frac{|BB'|}{|AA'|} = \frac{|Z_+Z_-| \cdot \sin \alpha'}{|Z_+Z_-| \cdot \sin \beta'} = \frac{\sin \alpha}{\sin \beta} = \frac{|AC'|}{|AP|} \cdot \frac{|AC'|}{|BP|}.$$ 

It follows that

$$\frac{|BP|}{|BB'| \cdot |BC'|} = \frac{|AA'| \cdot |AC'|}{|AP|},$$

and, as a ratio of signed lengths,

$$\frac{BC'}{AC'} = \frac{y}{x}. \quad (3)$$
Now applying Menelaus’ theorem to triangle \(APC'\) with transversal \(A'CB\), and triangle \(BGA'\) with transversal \(B'CA\), we have

\[
\frac{AA'}{AP} \cdot \frac{PC}{CC'} \cdot \frac{C'B}{BA} = -1 = \frac{BB'}{B'P} \cdot \frac{PC}{CC'} \cdot \frac{C'A}{AB}.
\]

From this, \(\frac{AA'}{AP} \cdot \frac{BC'}{1-x} = -\frac{AC'}{1-y}\), or

\[
\frac{BC'}{1-x} = -\frac{AC'}{1-y}.
\]

Comparing (3) and (4), we have \(\frac{1-x}{1-y} = \frac{x}{y}, (x - y)(x + y - 1) = 0\). Either \(x = y\) or \(x + y = 1\). It is easy to eliminate the possibility \(x + y = 1\). If \(P\) has homogeneous barycentric coordinates \((u : v : w)\) with reference to triangle \(ABC\), then \(x = \frac{u + w}{u + v + w}\) and \(y = \frac{v + u}{u + v + w}\). Thus, \(x + y = 1\) requires \(w = 0\) and \(P\) lies on the sideline \(AB\), contrary to the assumption. It follows that \(x = y\), and from (3), \(C'\) is the midpoint of \(AB\), and \(P\) lies on the median through \(C\). \(\square\)

The necessity part of Theorem 1 is now an immediate corollary of Lemma 7.

5. Concluding remark

We conclude with a remark on triangles for which two of the circumcenters of the cevasix configuration of the centroid coincide. Clearly, \(B_6 = C_1\) if and only if \(A, B', G, C'\) are concyclic. Equivalently, the image of \(G\) under the homothety \(h(A, 2)\) lies on the circumcircle of triangle \(ABC\). This point has homogeneous barycentric coordinates \((-1 : 2 : 2)\). Since the circumcircle has equation \(a^2yz + b^2zx + c^2xy = 0\), we have \(2a^2 = b^2 + c^2\). There are many interesting properties of such triangles. We simply mention that it is similar to its own triangle of medians. Specifically,

\[
m_a = \frac{\sqrt{3}}{2}a, \quad m_b = \frac{\sqrt{3}}{2}c, \quad m_c = \frac{\sqrt{3}}{2}b.
\]

Editor’s endnote. John H. Conway [5] has located the center of the Van Lamoen circle (of the circumcenters of the cevasix configuration of the centroid) as

\[
F = mN + \frac{\cot^2 \omega}{12} \cdot (G - K),
\]

where \(mN\) is the medial Ninecenter, \(G\) the centroid, \(K\) the symmedian point, and \(\omega\) the Brocard angle of triangle \(ABC\). In particular, the parallel through \(F\) to the symmedian line \(GK\) hits the Euler line in \(mN\). See Figure 4. The point \(F\) has homogeneous barycentric coordinates

\[(10a^4 - 13a^2(b^2 + c^2) + (4b^4 - 10b^2c^2 + 4c^4) : \ldots : \ldots).
\]

This appears as \(X_{1153}\) of [6].

\(^1\)This is the point which divides \(OH\) in the ratio \(1:3\).
References


Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex_geom@mtu-net.ru

Peter Y. Woo: Department of Mathematics, Biola University, 13800 Biola Avenue, La Mirada, California 90639, USA
E-mail address: woobiola@yahoo.com
Napoleon Triangles and Kiepert Perspectors

Two examples of the use of complex number coordinates

Floor van Lamoen

Abstract. In this paper we prove generalizations of the well known Napoleon Theorem and Kiepert Perspectors. We use complex numbers as coordinates to prove the generalizations, because this makes representation of isosceles triangles built on given segments very easy.

1. Introduction

In [1, XXVII] O. Bottema describes the famous (first) Fermat-Torricelli point of a triangle $ABC$. This point is found by attaching outwardly equilateral triangles to the sides of $ABC$. The new vertices form a triangle $A'B'C'$ that is perspective to $ABC$, that is, $AA', BB'$ and $CC'$ have a common point of concurrency, the perspector of $ABC$ and $A'B'C'$. A lot can be said about this point, but for this paper we only need to know that the lines $AA'$, $BB'$ and $CC'$ make angles of 60 degrees (see Figure 1), and that this is also the case when the equilateral triangles are pointed inwardly, which gives the second Fermat-Torricelli point.

![Figure 1]

It is well known that to yield a perspector, the triangles attached to the sides of $ABC$ do not need to be equilateral. For example they may be isosceles triangles with base angle $\phi$, like Bottema tells us in [1, XI]. It was Ludwig Kiepert who studied these triangles - the perspectors with varying $\phi$ lie on a rectangular hyperbola...
named after Kiepert. See [4] for some further study on this hyperbola, and some references. See Figure 2. However, it is already sufficient for the lines $AA'$, $BB'$, $CC'$ to concur when the attached triangles have oriented angles satisfying

$$\angle BAC' = \angle CBA', \quad \angle ABC' = \angle CBA', \quad \angle ACB = \angle BCA'.$$

When the attached triangles are equilateral, there is another nice geometric property: the centroids of the triangles $A'B'C$, $A'B'C$ and $A'B'C'$ form a triangle that is equilateral itself, a fact that is known as Napoleon's Theorem. The triangles are referred to as the first and second Napoleon triangles (for the cases of outwardly and inwardly pointed attached triangles). See Figures 3a and 3b. The perspectors of these two triangles with $ABC$ are called first and second Napoleon points. General informations on Napoleon triangles and Kiepert perspectors can be found in [2, 3, 5, 6].

![Figure 3a](image1)

![Figure 3b](image2)

2. The equation of a line in the complex plane

Complex coordinates are not that much different from the rectangular $(x, y)$ - the two directions of the axes are now hidden in one complex number, that we call the affix of a point. Of course such an affix just exists of a real ($x$) and imaginary ($y$) part - the complex number $z = p + qi$ in fact resembles the point $(p, q)$.

If $z = p + qi$, then the number $\bar{z} = p - qi$ is called complex conjugate of $z$. The combination of $z$ and $\bar{z}$ is used to make formulas, since we do not have $x$ and $y$ anymore! A parametric formula for the line through the points $a_1$ and $a_2$ is $z = a_1 + t(a_2 - a_1)$, where $t$ runs through the real numbers. The complex conjugate of this formula is $\bar{z} = \overline{a_1} + t(\overline{a_2} - \overline{a_1})$. Elimination of $t$ from these two formulas gives the formula for the line through the points with affixes $a_1$ and $a_2$:

$$z(\overline{a_1} - a_2) - \overline{z}(a_1 - a_2) + (a_1 \overline{a_2} - \overline{a_1}a_2) = 0.$$
3. Isosceles triangle on a segment

Let the points \( A \) and \( B \) have affixes \( a \) and \( b \). We shall find the affix of the point \( C \) for which \( ABC \) is an isosceles triangle with base angle \( \phi \) and apex \( C \). The midpoint of \( AB \) has affix \( \frac{1}{2}(a + b) \). The distance from this midpoint to \( C \) is equal to \( \frac{1}{2}|AB| \tan \phi \). With this we find the affix for \( C \) as

\[
c = \frac{a + b}{2} + i \tan \phi \left( \frac{b - a}{2} \right) = \frac{1 - i \tan \phi}{2}a + \frac{1 + i \tan \phi}{2}b \]

where \( \chi = \frac{1}{2} + i \frac{1}{2} \tan \phi \), so that \( \chi + \chi^{-1} = 1 \).

The special case that \( ABC \) is equilateral, yields for \( \chi \) the sixth root of unity \( \zeta = \frac{1}{2} + \frac{1}{2}\sqrt{3} = e^{i\pi/3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \). This number \( \zeta \) is a sixth root of unity, because it satisfies

\[
\zeta^6 = e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1.
\]

It also satisfies the identities \( \zeta^3 = -1 \) and \( \zeta \cdot \zeta^{-1} = \zeta + \zeta^{-1} = 1 \). Depending on orientation one can find two vertices \( C \) that together with \( AB \) form an equilateral triangle, for which we have respectively \( c = \zeta a + \zeta b \) (negative orientation) and \( c = \overline{\zeta} a + \zeta b \) (positive orientation). From this one easily derives

**Proposition 1.** The complex numbers \( a, b \) and \( c \) are affixes of an equilateral triangle if and only if

- for positive orientation or
  \[
a + \zeta^2 b + \zeta^4 c = 0
\]

- for negative orientation
  \[
a + \zeta b^2 + \zeta^2 c = 0
\]

4. Napoleon triangles

We shall generalize Napoleon’s Theorem, by extending the idea of the use of centroids. Napoleon triangles were indeed built in a triangle \( ABC \) by attaching to the sides of a triangle equilateral triangles, and taking the centroids of these. We now start with two triangles \( A_kB_kC_k \) for \( k = 1, 2 \), and attach equilateral triangles to the connecting segments between the \( A \)’s, the \( B \)’s and the \( C \)’s. This seems to be entirely different, but Napoleon’s Theorem will be a special case by starting with triangles \( BCA \) and \( CAB \).

So we start with two triangles \( A_kB_kC_k \) for \( k = 1, 2 \) with affixes \( a_k, b_k \) and \( c_k \) for the vertices. The centroids \( Z_k \) have affixes \( z_k = \frac{1}{3}(a_k + b_k + c_k) \). Now we attach positively orientated equilateral triangles to segments \( A_1A_2, B_1B_2 \) and \( C_1C_2 \) having \( A_{3+}, B_{3+}, C_{3+} \) as third vertices. In the same way we find \( A_{3-}, B_{3-}, C_{3-} \) from equilateral triangles with negative orientation. We find as affixes

\[
a_{3+} = \zeta a_2 + \overline{\zeta} a_1
\]

and

\[
a_{3-} = \zeta a_1 + \overline{\zeta} a_2,
\]

and similar expressions for \( b_{3+}, b_{3-}, c_{3+} \) and \( c_{3-} \). The centroids \( Z_{3+} \) and \( Z_{3-} \) now have affixes \( z_{3+} = \zeta z_2 + \overline{\zeta} z_1 \) and \( z_{3-} = \zeta z_1 + \overline{\zeta} z_2 \) respectively, from which we
see that \(Z_1Z_2Z_{3+}\) and \(Z_1Z_2Z_{3-}\) are equilateral triangles of positive and negative orientation respectively.

We now work with the following centroids: 

- \(D_+\), \(E_+\) and \(F_+\) of triangles \(B_1C_2A_{3+}\), \(C_1A_2B_{3+}\) and \(A_1B_2C_{3+}\) respectively; 
- \(D_-\), \(E_-\) and \(F_-\) of triangles \(C_1B_2A_{3-}\), \(A_1C_2B_{3-}\) and \(B_1A_2C_{3-}\) respectively.

For these we claim

**Theorem 2.** Given triangles \(A_kB_kC_k\) and points \(Z_k\) for \(k = 1, 2, 3+, 3-\) and \(D_\pm E_\pm F_\pm\) as described above, triangles \(D_+E_+F_+\) and \(D_-E_-F_-\) are equilateral triangles of negative orientation, congruent and parallel, and their centroids coincide with the centroids of \(Z_1Z_2Z_{3+}\) and \(Z_1Z_2Z_{3-}\) respectively. (See Figure 4).

**Proof.** To prove this we find the following affixes

\[
\begin{align*}
d_+ &= \frac{1}{3}(b_1 + c_2 + \zeta a_2 + \overline{\zeta} a_1), & d_- &= \frac{1}{3}(b_2 + c_1 + \zeta a_1 + \overline{\zeta} a_2), \\
e_+ &= \frac{1}{3}(c_1 + a_2 + \zeta b_2 + \overline{\zeta} b_1), & e_- &= \frac{1}{3}(c_2 + a_1 + \zeta b_1 + \overline{\zeta} b_2), \\
f_+ &= \frac{1}{3}(a_1 + b_2 + \zeta c_2 + \overline{\zeta} c_1), & f_- &= \frac{1}{3}(a_2 + b_1 + \zeta c_1 + \overline{\zeta} c_2).
\end{align*}
\]

Using Proposition 1 it is easy to show that \(D_+E_+F_+\) and \(D_-E_-F_-\) are equilateral triangles of negative orientation. For instance, the expression \(d_+ + \zeta^4 e_+ + \zeta^2 f_+\) has as ‘coefficient’ of \(b_1\) the number \(\frac{1}{3}(1 + \zeta^4 \overline{\zeta}) = 0\). We also find that

\[
d_+ - e_+ - d_- = \overline{\zeta}(a_1 - a_2) + \zeta(b_1 - b_2) + (c_2 - c_1),
\]

Figure 4
from which we see that $D_+E_+ + D_-E_- \equiv 0$. Finally it is easy to check that $rac{1}{3}(d_+ + e_+ + f_+) = \frac{1}{3}(z_1 + z_2 + z_3^+)$ and $rac{1}{3}(d_- + e_- + f_-) = \frac{1}{3}(z_1 + z_2 + z_3^-)$, and the theorem is proved.

We can make a variation of Theorem 2 if in the creation of $D_\pm E_\pm F_\pm$ we interchange the roles of $A_3^+ B_3^+ C_3^+$ and $A_3^- B_3^- C_3^-$. The roles of $Z_3^+$ and $Z_3^-$ change as well, and the equilateral triangles found have positive orientation.

We note that if the centroids $Z_1$ and $Z_2$ coincide, then they coincide with $Z_3^+$ and $Z_3^-$, so that $D_+ E_+ F_+ D_+ E_+ F_+$ is a regular hexagon, of which the center coincides with $Z_1$ and $Z_2$.

Napoleon’s Theorem is a special case. If we take $A_1 B_1 C_1 = BCA$ and $A_2 B_2 C_2 = CAB$, then $D_+ E_+ F_+$ is the second Napoleon Triangle, and indeed appears equilateral. We get as a bonus that $D_+ E_+ F_+ D_+ E_+ F_-$ is a regular hexagon. Now $D_-$ is the centroid of $AAA_3^-$, that is, $D_-$ is the point on $AA_3^-$ such that $AD_- : D_- A_3^- = 1 : 2$. In similar ways we find $E_-$ and $F_-$. Triangles $ABC$ and $A_3^- B_3^- C_3^-$ have the first point of Fermat-Torricelli $F_1$ as perspector, and the lines $AA_3^-, BB_3^-$ and $CC_3^-$ make angles of 60 degrees. From this it is easy to see (congruent inscribed angles) that $F_1$ must be on the circumcircle of $D_- E_- F_-$ and thus also on the circumcircle of $D_+ E_+ F_+$. See Figure 5. In the same way, now using the variation of Theorem 2, we see that the second Fermat-Torricelli point lies on the circumcircle of the first Napoleon Triangle.

![Figure 5](image-url)
5. Kiepert perspectors

To generalize the Kiepert perspectors we start with two triangles as well. We label these $ABC$ and $A'B'C'$ to distinguish from Theorem 2. These two triangles we take to be directly congruent (hence $A$ corresponds to $A'$, etc.) and of the same orientation. This means that the two triangles can be mapped to each other by a combination of a rotation and a translation (in fact one of both is sufficient). We now attach isosceles triangles to segments connecting $ABC$ and $A'B'C'$. While we usually find Kiepert perspectors on a line, for example, from $A$ to the apex of an isosceles triangle built on $BC$, now we start from the apex of an isosceles triangle on $AA'$ and go to the apex of an isosceles triangle on $BC$. This gives the following theorem:

![Diagram of isosceles triangles and Kiepert perspectors]

**Theorem 3.** Given two directly congruent triangles $ABC$ and $A'B'C'$ with the same orientation, attach to the segments $AA'$, $BB'$, $CC'$, $A'C'$ and $BA'$ similar isosceles triangles with the same orientation and apexes $A''$, $B''$, $C''$, $A'''$, $B'''$ and $C'''$. The lines $A''A'''$, $B''B'''$ and $C''C'''$ are concurrent, so triangles $A''B''C''$ and $A'''B'''C'''$ are perspective. (See Figure 6).

**Proof.** For the vertices $A$, $B$ and $C$ we take the affixes $a$, $b$ and $c$. Because triangles $ABC$ and $A'B'C'$ are directly congruent and of equal orientation, we can get $A'B'C'$ by applying on $ABC$ a rotation about the origin, followed by a translation. This rotation about the origin can be represented by multiplication by a number $\tau$ on the unit circle, so that $\tau \bar{\tau} = 1$. The translation is represented by addition with a number $\sigma$. So the affixes of $A'$, $B'$ and $C'$ are the numbers $\tau a + \sigma$, $\tau b + \sigma$ and $\tau c + \sigma$. 

Figure 6
We take for the base angles of the isosceles triangle $\phi$ again, and let $\chi = \frac{1}{2} + \frac{i}{2} \tan \phi$, so that the affix for $A''$ is $(\chi + \chi^\tau)a + \chi \sigma$. For $A'''$ we find $\chi c + \chi^\tau b + \chi \sigma$. The equation of the line $A''A'''$ we can find after some calculations as

$$
(\chi \bar{a} + \chi^\tau \bar{a} - \chi c - \chi^\tau \bar{a})z = (\chi a + \chi^\tau a - \chi c - \chi^\tau b)z
+ (\chi + \chi^\tau)\bar{a}(\chi c + \chi^\tau b) + \chi \sigma(\chi a + \chi^\tau b) + \chi \bar{a}(\chi c + \chi^\tau a) - (\chi + \chi^\tau)\bar{a}(\chi c + \chi^\tau b) - \chi \sigma(\chi a + \chi^\tau b) = 0.
$$

In a similar fashion we find for $B''B'''$, 

$$
(\chi \bar{b} + \chi^\tau \bar{b} - \chi a - \chi^\tau c)z = (\chi b + \chi^\tau b - \chi a - \chi^\tau c)z
+ (\chi + \chi^\tau)b(\chi a + \chi^\tau c) + \chi \sigma(\chi a + \chi^\tau c) + \chi \bar{b}(\chi a + \chi^\tau a) - (\chi + \chi^\tau)b(\chi a + \chi^\tau c) - \chi \sigma(\chi a + \chi^\tau a) = 0,
$$

and for $C''C'''$, 

$$
(\chi \bar{c} + \chi^\tau \bar{c} - \chi b - \chi^\tau a)z = (\chi c + \chi^\tau c - \chi b - \chi^\tau a)z
+ (\chi + \chi^\tau)c(\chi b + \chi^\tau a) + \chi \sigma(\chi b + \chi^\tau a) + \chi \bar{c}(\chi b + \chi^\tau b) - (\chi + \chi^\tau)c(\chi b + \chi^\tau a) - \chi \sigma(\chi b + \chi^\tau a) = 0.
$$

We must do some more effort to see what happens if we add the three equations. Our effort is rewarded by noticing that the sum gives $0 = 0$. The three equations are dependent, so the lines are concurrent. This proves the theorem. \(\square\)

We end with a question on the locus of the perspector for varying $\phi$. It would have been nice if the perspector would, like in Kiepert’s hyperbola, lie on an equilateral hyperbola. This, however, does not seem to be generally the case.

References


Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl
On the Fermat Lines

Paul Yiu

Abstract. We study the triangle formed by three points each on a Fermat line of a given triangle, and at equal distances from the vertices. For two specific values of the common distance, the triangle degenerates into a line. The two resulting lines are the axes of the Steiner ellipse of the triangle.

1. The Fermat lines

This paper is on a variation of the theme of Bottema [2]. Bottema studied the triangles formed by three points each on an altitude of a given triangle, at equal distances from the respective vertices. See Figure 1. He obtained many interesting properties of this configuration. For example, these three points are collinear when the common distance is $R \pm d$, where $R$ is the circumradius and $d$ the distance between the circumcenter and the incenter of the reference triangle. The two lines containing the two sets of collinear points are perpendicular to each other at the incenter, and are parallel to the asymptotes of the Feuerbach hyperbola, the rectangular hyperbola through the vertices, the orthocenter, and the incenter. See Figure 2.

In this paper we consider the Fermat lines, which are the lines joining a vertex of the given triangle $ABC$ to the apex of an equilateral triangle constructed on its opposite side. We label these triangles $BCA_\epsilon$, $CAB_\epsilon$, and $ABC_\epsilon$, with $\epsilon = +1$.
for those erected externally, and $\epsilon = -1$ otherwise. There are 6 of such lines, $AA_+, BB_+, CC_+$, $AA_-, BB_-, CC_-$. See Figure 3. The reason for choosing these lines is that, for $\epsilon = \pm 1$, the three segments $AA_\epsilon$, $BB_\epsilon$, and $CC_\epsilon$ have equal lengths $\tau_\epsilon$ given by

$$
\tau_\epsilon^2 = \frac{1}{2}(a^2 + b^2 + c^2) + \epsilon \cdot 2\sqrt{3}\triangle,
$$

where $a$, $b$, $c$ are the side lengths, and $\triangle$ the area of triangle $ABC$. See, for example, [1, XXVII.3].

It is well known that the three Fermat lines $AA_\epsilon$, $BB_\epsilon$, and $CC_\epsilon$ intersect each other at the $\epsilon$-Fermat point $F_\epsilon$ at $60^\circ$ angles. The centers of the equilateral triangles $BCA_\epsilon$, $CAB_\epsilon$, and $ABC_\epsilon$ form the $\epsilon$-Napoleon equilateral triangle. The circumcircle of the $\epsilon$-Napoleon triangle has radius $\tau_\epsilon/3$ and passes through the $(-\epsilon)$-Fermat point. See, for example, [5].

2. The triangles $T_\epsilon(t)$

We shall label points on the Fermat lines by their distances from the corresponding vertices of $ABC$, positive in the direction from the vertex to the Fermat point, negative otherwise. Thus, $A_+(t)$ is the unique point $X$ on the positive Fermat line $AF_+$ such that $AX = t$. In particular,

$$
A_\epsilon(\tau_\epsilon) = A_\epsilon, \quad B_\epsilon(\tau_\epsilon) = B_\epsilon, \quad C_\epsilon(\tau_\epsilon) = C_\epsilon.
$$

We are mainly interested in the triangles $T_\epsilon(t)$ whose vertices are $A_\epsilon(t)$, $B_\epsilon(t)$, $C_\epsilon(t)$, for various values of $t$. Here are some simple observations.

(1) The centroid of $AA_+A_-$ is $G$. This is because the segments $A_+A_-$ and $BC$ have the same midpoint.
(2) The centers of the equilateral triangles $BCA_+$ and $BCA_-$ trisect the segment $A_+A_-$. Therefore, the segment joining $A_+(\frac{\epsilon}{3})$ to the center of $BCA_-$ is parallel to the Fermat line $AA_-$ and has midpoint $G$.

(3) This means that $A_+(\frac{\epsilon}{3})$ is the reflection of the $A$-vertex of the $(-\epsilon)$-Napoleon triangle in the centroid $G$. See Figure 4, in which we label $A_+(\frac{\epsilon}{3})$ by $X$ and $A_-(\frac{\epsilon}{3})$ by $X'$ respectively.

This is the same for the other two points $B_+(\frac{-\epsilon}{3})$ and $C_+(\frac{-\epsilon}{3})$.

(4) It follows that the triangle $T(\frac{\epsilon}{3})$ is the reflection of the $(-\epsilon)$-Napoleon triangle in $G$, and is therefore equilateral.

(5) The circle through the vertices of $T(\frac{\epsilon}{3})$ and the $(-\epsilon)$-Napoleon triangle has radius $\frac{\tau-\epsilon}{3}$ and also passes through the Fermat point $F$. See Figure 5.

Since $GA_+(\frac{\epsilon}{3}) = \frac{\tau+\epsilon}{3}$, (see Figure 4), the circle, center $X$, radius $\frac{\tau+\epsilon}{3}$, passes through $G$. See Figure 6A. Likewise, the circle, center $X'$, radius $\frac{\tau}{3}$ also passes through $G$. See Figure 6B. In these figures, we label

\[
Y = A_+ \left( \frac{\tau_+ - \tau_-}{3} \right), \quad Z = A_+ \left( \frac{\tau_+ + \tau_-}{3} \right), \\
Y' = A_- \left( \frac{\tau_+ - \tau_-}{3} \right), \quad Z' = A_- \left( \frac{\tau_+ + \tau_-}{3} \right).
\]

It follows that $GY$ and $GZ$ are perpendicular to each other; so are $GY'$ and $GZ'$.
(6) For \( \epsilon = \pm 1 \), the lines joining the centroid \( G \) to \( A_\epsilon(\frac{\tau_+ + \tau_3}{3}) \) and \( A_\epsilon(\frac{\tau_+ - \tau_3}{3}) \) are perpendicular to each other. Similarly, the lines joining \( G \) to \( B_\epsilon(\frac{\tau_+ + \tau_3}{3}) \) and \( B_\epsilon(\frac{\tau_+ - \tau_3}{3}) \) are perpendicular to each other; so are the lines joining \( G \) to \( C_\epsilon(\frac{\tau_+ + \tau_3}{3}) \) and \( C_\epsilon(\frac{\tau_+ - \tau_3}{3}) \).

In Figure 6A, since \( \angle XGY = \angle XYG \) and \( AXGX' \) is a parallelogram, the line \( GY \) is the bisector of angle \( XGX' \), and is parallel to the bisector of angle \( A_+AA_- \). If the internal bisector of angle \( A_+AA_- \) intersects \( A_+A_- \) at \( A' \), then it is easy to see that \( A' \) is the apex of the isosceles triangle constructed inwardly on \( BC \) with base angle \( \varphi \) satisfying

\[
\cot \varphi = \frac{\tau_+ + \tau_-}{\sqrt{3}(\tau_+ - \tau_-)}. \tag{†}
\]

Similarly, in Figure 6B, the line \( GZ' \) is parallel to the external bisector of the same angle. We summarize these as follows.

(7) The lines joining \( A_+\left(\frac{\tau_+ + \tau_3}{3}\right) \) to \( A_-\left(\frac{\tau_+ - \tau_3}{3}\right) \) and \( A_+\left(\frac{\tau_+ + \tau_3}{3}\right) \) to \( A_-\left(\frac{\tau_+ + \tau_3}{3}\right) \) are perpendicular at \( G \), and are respectively parallel to the internal and external bisectors of angle \( A_+AA_- \). Similarly, the two lines joining \( B_\epsilon\left(\frac{\tau_+ + \tau_3}{3}\right) \) to \( B_-\left(\frac{\tau_+ - \tau_3}{3}\right) \) and \( B_\epsilon\left(\frac{\tau_+ + \tau_3}{3}\right) \) to \( B_-\left(\frac{\tau_+ + \tau_3}{3}\right) \) are perpendicular at \( G \), being parallel to the internal and external bisectors of angle \( B_+BB_- \); so are the lines joining
On the Fermat lines

\[ C_+\left(\frac{T_+ - \tau}{3}\right) \text{ to } C_-\left(\frac{T_+ - \tau}{3}\right), \text{ and } C_+\left(\frac{T_+ + \tau}{3}\right) \text{ to } C_-\left(\frac{T_+ + \tau}{3}\right), \]
being parallel to the internal and external bisectors of angle \(C_+CC_-)\.

3. Collinearity

What is interesting is that these 3 pairs of perpendicular lines in (7) above form the same right angles at the centroid \(G\). Specifically, the six points

\[ A_+\left(\frac{T_+ + \tau}{3}\right), B_+\left(\frac{T_+ + \tau}{3}\right), C_+\left(\frac{T_+ + \tau}{3}\right), A_-\left(\frac{T_+ + \tau}{3}\right), B_-\left(\frac{T_+ + \tau}{3}\right), C_-\left(\frac{T_+ + \tau}{3}\right) \]

are collinear with the centroid \(G\) on a line \(L_+\); so are the 6 points

\[ A_+\left(\frac{T_+ - \tau}{3}\right), B_+\left(\frac{T_+ - \tau}{3}\right), C_+\left(\frac{T_+ - \tau}{3}\right), A_-\left(\frac{T_+ - \tau}{3}\right), B_-\left(\frac{T_+ - \tau}{3}\right), C_-\left(\frac{T_+ - \tau}{3}\right) \]

on a line \(L_-\) through \(G\). See Figure 7. To justify this, we consider the triangle \(T_\epsilon(t) := A_\epsilon(t)B_\epsilon(t)C_\epsilon(t)\) for varying \(t\).

(8) For \(\epsilon = \pm 1\), the triangle \(T_\epsilon(t)\) degenerates into a line containing the centroid \(G\) if and only if \(t = \frac{\tau_+ + \delta \tau - \epsilon}{3}, \delta = \pm 1\).

Figure 7
4. Barycentric coordinates

To prove (8) and to obtain further interesting geometric results, we make use of coordinates. Bottema has advocated the use of homogeneous barycentric coordinates. See [3, 6]. Let \( P \) be a point in the plane of triangle \( ABC \). With reference to \( ABC \), the homogeneous barycentric coordinates of \( P \) are the ratios of signed areas \((\triangle PBC : \triangle PCA : \triangle PAB)\).

The coordinates of the vertex \( A_+ \) of the equilateral triangle \( BCA_+ \), for example, are \((-\sqrt{3}a^2 : \sqrt{3}(a^2 + b^2 - c^2) + 4\Delta : \sqrt{3}(c^2 + a^2 - b^2) + 4\Delta)\). More generally, for \( \epsilon = \pm 1 \), the vertices of the equilateral triangles erected on the sides of triangle \( ABC \) are the points

\[
A_\epsilon = (-2\sqrt{3}a^2 : \sqrt{3}(a^2 + b^2 - c^2) + 4\epsilon\Delta : \sqrt{3}(c^2 + a^2 - b^2) + 4\epsilon\Delta),
\]
\[
B_\epsilon = (\sqrt{3}(a^2 + b^2 - c^2) + 4\epsilon\Delta : -2\sqrt{3}b^2 : \sqrt{3}(b^2 + c^2 - a^2) + 4\epsilon\Delta),
\]
\[
C_\epsilon = (\sqrt{3}(c^2 + a^2 - b^2) + 4\epsilon\Delta : \sqrt{3}(b^2 + c^2 - a^2) + 4\epsilon\Delta : -2\sqrt{3}c^2).
\]

Note that in each case, the coordinate sum is \( 8\epsilon\Delta \). From this we easily compute the coordinates of the centroid by simply adding the corresponding coordinates of the three vertices.

(9A) For \( \epsilon = \pm 1 \), triangles \( A_\epsilon B_\epsilon C_\epsilon \) and \( ABC \) have the same centroid.

Sometimes it is convenient to work with absolute barycentric coordinates. For a finite point \( P = (u : v : w) \), we obtain the absolute barycentric coordinates by normalizing its homogeneous barycentric coordinates, namely, by dividing by the coordinate sum. Thus,

\[
P = \frac{1}{u + v + w}(uA + vB + wC),
\]

provided \( u + v + w \) is nonzero.

The absolute barycentric coordinates of the point \( A_\epsilon(t) \) can be easily written down. For each value of \( t \),

\[
A_\epsilon(t) = \frac{1}{\tau_\epsilon}((\tau_\epsilon - t)A + t \cdot A_\epsilon),
\]

and similarly for \( B_\epsilon(t) \) and \( C_\epsilon(t) \).

This, together with (9A), leads easily to the more general result.

(9B) For arbitrary \( t \), the triangles \( T_\epsilon(t) \) and \( ABC \) have the same centroid.
5. Area of $T_\epsilon(t)$

Let $X = (x_1 : x_2 : x_3), Y = (y_1 : y_2 : y_3)$ and $Z = (z_1 : z_2 : z_3)$ be finite points with homogeneous coordinates with respect to triangle $ABC$. The signed area of the oriented triangle $XYZ$ is

$$\frac{|x_1 x_2 x_3|}{(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)(z_1 + z_2 + z_3)} \cdot \triangle.$$  

A proof of this elegant formula can be found in [1, VII] or [3]. A direct application of this formula yields the area of triangle $T_\epsilon(t)$.

(10) The area of triangle $T_\epsilon(t)$ is

$$\frac{3\sqrt{3}\epsilon}{4} \left( t - \frac{\tau_\epsilon + \tau - \epsilon}{3} \right) \left( t - \frac{\tau_\epsilon - \tau - \epsilon}{3} \right) \triangle.$$  

Statement (8) follows immediately from this formula and (9B).

(11) $T_\epsilon(t)$ has the same area as $ABC$ if and only if $t = 0$ or $\frac{2\tau_\epsilon}{3}$. In fact, the two triangles $T_+(\frac{2\tau_\epsilon}{3})$ and $T_-(\frac{2\tau_\epsilon}{3})$ are symmetric with respect to the centroid. See Figures 8A and 8B.

6. Kiepert hyperbola and Steiner ellipse

The existence of the line $L_-$ (see §3) shows that the internal bisectors of the angles $A_+ A_-, B_+ B_-,$ and $C_+ C_-$ are parallel. These bisectors contain the apexes $A', B', C'$ of isosceles triangles constructed inwardly on the sides with the same base angle given by (†). It is well known that $AB'C'$ and $ABC$ are perspective at a point on the Kiepert hyperbola, the rectangular circum-hyperbola
through the orthocenter and the centroid. This perspector is necessarily an infinite point (of an asymptote of the hyperbola). In other words, the line $L_-$ is parallel to an asymptote of this rectangular hyperbola.

(12) The lines $L_{\pm}$ are the parallels through $G$ to the asymptotes of the Kiepert hyperbola.

(13) It is also known that the asymptotes of the Kiepert hyperbola are parallel to the axes of the Steiner in-ellipse, (see [4]), the ellipse that touches the sides of triangle $ABC$ at their midpoints, with center at the centroid $G$. See Figure 9.

![Figure 9](image)

(14) The Steiner in-ellipse has major and minor axes of lengths $\frac{\tau_+ + \tau_-}{3}$. From this, we have the following construction of its foci. See Figure 9.

- Construct the concentric circles $C_{\pm}$ at $G$ through $A_i(\frac{\tau_i}{3})$.
- Construct a circle $C$ with center on $L_+$ tangent to the circles $C_+$ internally and $C_-$ externally. There are two such circles; any one of them will do.
- The intersections of the circle $C$ with the line $L_-$ are the foci of Steiner in-ellipse.

We conclude by recording the homogeneous barycentric coordinates of the two foci of the Steiner in-ellipse. Let

$$Q = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$
The line $\mathcal{L}_- \ldots$ containing the two foci has infinite point

$$I_\infty = \left((b - c)(a(a + b + c) - (b^2 + bc + c^2) - \sqrt{Q}),ight.$$

$$\left.(c - a)(b(a + b + c) - (c^2 + ca + a^2) - \sqrt{Q}),ight.$$

$$\left.(a - b)(c(a + b + c) - (a^2 + ab + b^2) - \sqrt{Q})\right).$$

As a vector, this has square length $2\sqrt{Q}(f + g\sqrt{Q})$, where

$$f = \sum_{\text{cyclic}} a^6 - bc(b^4 + c^4) + a^2bc(ab + ac - bc),$$

$$g = \sum_{\text{cyclic}} a^4 - bc(b^2 + c^2 - a^2).$$

Since the square distance from the centroid to each of the foci is $\frac{1}{3}\sqrt{Q}$, these two foci are the points

$$G \pm \frac{1}{3\sqrt{2}(f + g\sqrt{Q})}I_\infty.$$

References


Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA

*E-mail address*: yiu@fau.edu
Triangle Centers Associated with the Malfatti Circles

Milorad R. Stevanović

Abstract. Various formulae for the radii of the Malfatti circles of a triangle are presented. We also express the radii of the excircles in terms of the radii of the Malfatti circles, and give the coordinates of some interesting triangle centers associated with the Malfatti circles.

1. The radii of the Malfatti circles

The Malfatti circles of a triangle are the three circles inside the triangle, mutually tangent to each other, and each tangent to two sides of the triangle. See Figure 1. Given a triangle $ABC$, let $a$, $b$, $c$ denote the lengths of the sides $BC$, $CA$, $AB$, $s$ the semiperimeter, $I$ the incenter, and $r$ its inradius. The radii of the Malfatti circles of triangle $ABC$ are given by

$$r_1 = \frac{r}{2(s-a)} \left( s - r - (IB + IC - IA) \right),$$

$$r_2 = \frac{r}{2(s-b)} \left( s - r - (IC + IA - IB) \right),$$

$$r_3 = \frac{r}{2(s-c)} \left( s - r - (IA + IB - IC) \right).$$

The author is grateful to the referee and the editor for their valuable comments and helps.
According to F.G.-M. [1, p.729], these results were given by Malfatti himself, and were published in [7] after his death. See also [6]. Another set of formulae give the same radii in terms of \( a, b, c \) and \( r \):

\[
\begin{align*}
r_1 &= \frac{(IB + r - (s - b))(IC + r - (s - c))}{2(IA + r - (s - a))}, \\
r_2 &= \frac{(IC + r - (s - c))(IA + r - (s - a))}{2(IB + r - (s - b))}, \\
r_3 &= \frac{(IA + r - (s - a))(IB + r - (s - b))}{2(IC + r - (s - c))}.
\end{align*}
\] (2)

These easily follow from (1) and the following formulae that express the radii \( r_1, r_2, r_3 \) in terms of \( r \) and trigonometric functions:

\[
\begin{align*}
r_1 &= \frac{(1 + \tan \frac{B}{2})(1 + \tan \frac{C}{2})}{1 + \tan \frac{A}{2}} \cdot \frac{r}{2}, \\
r_2 &= \frac{(1 + \tan \frac{C}{2})(1 + \tan \frac{A}{2})}{1 + \tan \frac{B}{2}} \cdot \frac{r}{2}, \\
r_3 &= \frac{(1 + \tan \frac{A}{2})(1 + \tan \frac{B}{2})}{1 + \tan \frac{C}{2}} \cdot \frac{r}{2}.
\end{align*}
\] (3)

These can be found in [10]. They can be used to obtain the following formula which is given in [2, pp.103–106]. See also [12].

\[
\frac{2}{r} = \frac{1}{\sqrt{r_1 r_2}} + \frac{1}{\sqrt{r_2 r_3}} + \frac{1}{\sqrt{r_1 r_3}} - \sqrt{\frac{1}{r_1 r_2}} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}.
\] (4)

2. Exradii in terms of Malfatti radii

Antreas P. Hatzipolakis [3] asked for the exradii \( r_a, r_b, r_c \) of triangle \( ABC \) in terms of the Malfatti radii \( r_1, r_2, r_3 \) and the inradius \( r \).

Proposition 1.

\[
\begin{align*}
r_a - r_1 &= \frac{\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)}, \\
r_b - r_2 &= \frac{\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)}, \\
r_c - r_3 &= \frac{\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)},
\end{align*}
\] (5)
Proof. For convenience we write
\[ t_1 := \tan \frac{A}{4}, \quad t_2 := \tan \frac{B}{4}, \quad t_3 := \tan \frac{C}{4}. \]
Note that from \( \tan \left( \frac{A}{4} + \frac{B}{4} + \frac{C}{4} \right) = 1, \) we have
\[ 1 - t_1 - t_2 - t_3 - t_1 t_2 - t_2 t_3 - t_3 t_1 + t_1 t_2 t_3 = 0. \tag{6} \]
From (3) we obtain
\[
\begin{align*}
\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}} &= \frac{t_1}{1 + t_1} \cdot \frac{2}{r}, \\
\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}} &= \frac{t_2}{1 + t_2} \cdot \frac{2}{r}, \\
\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}} &= \frac{t_3}{1 + t_3} \cdot \frac{2}{r}. \tag{7}
\end{align*}
\]
For the exradius \( r_a, \) we have
\[ r_a = \frac{s}{s-a} \cdot r = \cot \frac{B}{2} \cot \frac{C}{2} \cdot r = \frac{(1 - t_2^2)(1 - t_3^2)}{4t_2 t_3} \cdot r. \]
It follows that
\[
\begin{align*}
\frac{r_a}{r} - 1 &= (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \left( \frac{(1 - t_2)(1 - t_3)}{2t_2 t_3} - \frac{1}{1 + t_1} \right) \\
&= (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \cdot \frac{(1 + t_1)(1 - t_2)(1 - t_3) - 2t_2 t_3}{2t_2 t_3(1 + t_1)} \\
&= (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \cdot \frac{2t_1}{2t_2 t_3(1 + t_1)} \quad \text{(from (6))} \\
&= \frac{t_1}{1 + t_1} \cdot \frac{1 + t_2}{t_2} \cdot \frac{1 + t_3}{t_3} \cdot \frac{r}{2}.
\end{align*}
\]
Now the result follows from (7). \qed

Note that with the help of (4), the exradii \( r_a, r_b, r_c \) can be explicitly written in terms of the Malfatti radii \( r_1, r_2, r_3. \) We present another formula useful in the next sections in the organization of coordinates of triangle centers.

**Proposition 2.**
\[ \frac{1}{r_1} - \frac{1}{r_a} = \frac{a}{rs} \cdot \frac{(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2})}{1 + \cos \frac{A}{2}}. \]

3. Triangle centers associated with the Malfatti circles

Let \( A' \) be the point of tangency of the Malfatti circles \( C_2 \) and \( C_3. \) Similarly define \( B' \) and \( C'. \) It is known ([4, p.97]) that triangle \( A'B'C' \) is perspective with \( ABC \) at the first Ajima-Malfatti point \( X_{179}. \) See Figure 3. We work out the details here and construct a few more triangle centers associated with the Malfatti circles. In particular, we find two new triangle centers \( P_+ \) and \( P_- \) which divide the incenter \( I \) and the first Ajima-Malfatti point harmonically.
3.1. The centers of the Malfatti circles. We begin with the coordinates of the centers of the Malfatti circles.

Since $O_1$ divides the segment $AI_a$ in the ratio $AO_1 : O_1I_a = r_1 : r_a - r_1$, we have \[
\frac{O_1}{r_1} = \left(1 - \frac{1}{r_a}\right) A + \frac{1}{r_a} : I_a.\]
With $r_a = \frac{r}{s-a}$ we rewrite the absolute barycentric coordinates of $O_1$, along with those of $O_2$ and $O_3$, as follows.

\[
\begin{align*}
O_1 &= \frac{1}{r_1} \left(\frac{1}{r_1} - \frac{1}{r_a}\right) A + \frac{s-a}{rs} \cdot I_a, \\
O_2 &= \frac{1}{r_2} \left(\frac{1}{r_2} - \frac{1}{r_b}\right) B + \frac{s-b}{rs} \cdot I_b, \\
O_3 &= \frac{1}{r_3} \left(\frac{1}{r_3} - \frac{1}{r_c}\right) C + \frac{s-c}{rs} \cdot I_c.
\end{align*}
\]  

From these expressions we have, in homogeneous barycentric coordinates,

\[
\begin{align*}
O_1 &= \left(2rs \left(\frac{1}{r_1} - \frac{1}{r_a}\right) - a : b : c\right), \\
O_2 &= \left(a : 2rs \left(\frac{1}{r_2} - \frac{1}{r_b}\right) - b : c\right), \\
O_3 &= \left(a : b : 2rs \left(\frac{1}{r_3} - \frac{1}{r_c}\right) - c\right).
\end{align*}
\]
3.2. The triangle center $P_-$. It is clear that $O_1O_2O_3$ is perspective with $ABC$ at the incenter ($a : b : c$). However, it also follows that if we consider

$$A'' = BO_3 \cap CO_2, \quad B'' = CO_1 \cap AO_3, \quad C'' = AO_2 \cap BO_1,$$

then triangle $A''B''C''$ is perspective with $ABC$ at

$$P_- = \left( \frac{2rs}{r_1} - \frac{1}{r_a} - a : \frac{2rs}{r_2} - \frac{1}{r_b} - b : \frac{2rs}{r_3} - \frac{1}{r_c} - c \right)$$

$$= \left( \frac{1}{r_1} - \frac{1}{r_a} - a : \frac{1}{r_2} - \frac{1}{r_b} - b : \frac{1}{r_3} - \frac{1}{r_c} - c \right)$$

$$= \left( a \left( \frac{1 + \cos \frac{B}{2}}{1 + \cos \frac{A}{2}} \right) \frac{1}{2} : \cdots : \cdots \right)$$

by Proposition 2. See Figure 2.

**Remark.** The point $P_-$ appears in [5] as the first Malfatti-Rabinowitz point $X_{1142}$.

3.3. The first Ajima-Malfatti point. For the points of tangency of the Malfatti circles, note that $A'$ divides $O_2O_3$ in the ratio $O_2A' : A'O_3 = r_2 : r_3$. We have, in absolute barycentric coordinates,

$$\left( \frac{1}{r_2} + \frac{1}{r_3} \right) A' = \frac{O_2}{r_2} + \frac{O_3}{r_3} = \frac{a}{rs} \cdot \frac{A}{r_2} + \left( \frac{1}{r_2} - \frac{1}{r_3} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C;$$

similarly for $B'$ and $C'$. In homogeneous coordinates,

$$A' = \left( \frac{a}{rs} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c} \right),$$

$$B' = \left( \frac{1}{r_1} - \frac{1}{r_a} : b \cdot \frac{1}{r_3} - \frac{1}{r_c} \right),$$

$$C' = \left( \frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : c \cdot \frac{1}{r_s} \right).$$

From these, it is clear that $A'B'C'$ is perspective with $ABC$ at

$$P = \left( \frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c} \right)$$

$$= \left( \frac{a(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2})}{1 + \cos \frac{A}{2}} : \frac{b(1 + \cos \frac{C}{2})(1 + \cos \frac{A}{2})}{1 + \cos \frac{B}{2}} \right)$$

$$= \left( \frac{c(1 + \cos \frac{A}{2})(1 + \cos \frac{B}{2})}{1 + \cos \frac{C}{2}} \right)$$

$$= \left( \frac{a}{(1 + \cos \frac{A}{2})^2} : \frac{b}{(1 + \cos \frac{B}{2})^2} : \frac{c}{(1 + \cos \frac{C}{2})^2} \right)$$

(11)
by Proposition 2. The point \( P \) appears as \( X_{179} \) in [4, p.97], with trilinear coordinates

\[
\left( \sec^4 \frac{A}{4} : \sec^4 \frac{B}{4} : \sec^4 \frac{C}{4} \right)
\]

computed by Peter Yff, and is named the first Ajima-Malfatti point. See Figure 3.

![Figure 3](image)

\[3.4. \text{The triangle center } P_+. \] Note that the circle through \( A', B', C' \) is orthogonal to the Malfatti circles. It is the radical circle of the Malfatti circles, and is the incircle of \( O_1O_2O_3 \). The lines \( O_1A', O_2B', O_3C' \) are concurrent at the Gergonne point of triangle \( O_1O_2O_3 \). See Figure 4. As such, this is the point \( P_+ \) given by

\[
\left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) P_+ = \frac{O_1}{r_1} + \frac{O_2}{r_2} + \frac{O_3}{r_3}
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{I_a}{r_a} + \frac{I_b}{r_b} + \frac{I_c}{r_c}
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \frac{I_a}{r_a} + \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{I}{r}
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \frac{1}{r_a} B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{1}{2rs} (aA + bB + cC)
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs} \right) A + \left( \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs} \right) C.
\]
It follows that in homogeneous coordinates,
\[
P_+ = \left( \frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs} \right)
\]
\[
= \left( a \left( \frac{1 + \cos \frac{B}{2} \left( 1 + \cos \frac{C}{2} \right)}{1 + \cos \frac{A}{2}} + \frac{1}{2} \right) : \cdots : \cdots \right) \quad (12)
\]
by Proposition 2.

**Proposition 3.** The points \( P_+ \) and \( P_- \) divide the segment \( IP \) harmonically.

**Proof.** This follows from their coordinates given in (12), (9), and (11). \( \square \)

From the coordinates of \( P, P_+ \) and \( P_- \), it is easy to see that \( P_+ \) and \( P_- \) divide the segment \( IP \) harmonically.

3.5. *The triangle center* \( Q \). Let the Malfatti circle \( C_1 \) touch the sides \( CA \) and \( AB \) at \( Y_1 \) and \( Z_1 \) respectively. Likewise, let \( C_2 \) touch \( AB \) and \( BC \) at \( Z_2 \) and \( X_2 \), \( C_3 \) touch \( BC \) and \( CA \) at \( X_3 \) and \( Y_3 \) respectively. Denote by \( X, Y, Z \) the midpoints of the segments \( X_2X_3, Y_3Y_1, Z_1Z_2 \) respectively. Stanley Rabinowitz [9] asked if the lines \( AX, BY, CZ \) are concurrent. We answer this in the affirmative.

**Proposition 4.** The lines \( AX, BY, CZ \) are concurrent at a point \( Q \) with homogeneous barycentric coordinates
\[
\left( \tan \frac{A}{4} : \tan \frac{B}{4} : \tan \frac{C}{4} \right). \quad (13)
\]
Proof. In Figure 5, we have

\[
BX = \frac{1}{2}(a + BX_2 - X_3C) \\
= \frac{1}{2}\left(a + \frac{r_2(s - b)}{r} - \frac{r_3(s - c)}{r}\right) \\
= \frac{1}{2}\left(a + IB - IC\right) \quad \text{(from (1))} \\
= \frac{1}{2}\left(2R\sin A + \frac{r}{\sin B} - \frac{r}{\sin C}\right) \\
= 4R\sin A \frac{B}{4} \sin C \frac{C}{4} \cos B + C \frac{B + C}{4}
\]

by making use of the formula

\[
 r = 4R\sin A \frac{B}{2} \sin C \frac{C}{2}.
\]

Similarly,

\[
XC = \frac{1}{2}(a - BX_2 + X_3C) = 4R\sin A \frac{B}{2} \sin C \frac{C}{4} \cos B + C \frac{B + C}{4}.
\]

It follows that

\[
\frac{BX}{XC} = \frac{\cos B \sin C}{\sin B \cos C} = \tan \frac{C}{4}.
\]

Likewise,

\[
\frac{CY}{YA} = \tan \frac{B}{4} \quad \text{and} \quad \frac{AZ}{ZB} = \tan \frac{A}{4}.
\]
and it follows from Ceva’s theorem that $AX, BY, CZ$ are concurrent since
\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.
\]
In fact, we can easily identify the homogeneous barycentric coordinates of the intersection $Q$ as given in (13) above since those of $X, Y, Z$ are
\[
X = \left(0 : \tan \frac{B}{4} : \tan \frac{C}{4}\right),
\]
\[
Y = \left(\tan \frac{A}{4} : 0 : \tan \frac{C}{4}\right),
\]
\[
Z = \left(\tan \frac{A}{4} : \tan \frac{B}{4} : 0\right).
\]
\[\square\]

**Remark.** The coordinates of $Q$ can also be written as
\[
\left(\frac{\sin \frac{A}{2}}{1 + \cos \frac{A}{2}} : \frac{\sin \frac{B}{2}}{1 + \cos \frac{B}{2}} : \frac{\sin \frac{C}{2}}{1 + \cos \frac{C}{2}}\right)
\]
or
\[
\left(\frac{a}{(1 + \cos \frac{A}{2})(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2})} : \frac{b}{(1 + \cos \frac{B}{2})^2(1 + \cos \frac{C}{2})^2} : \frac{c}{(1 + \cos \frac{C}{2})^2(1 + \cos \frac{A}{2})^2}\right).
\]

### 3.6. The radical center of the Malfatti circles.

Note that the common tangent of $C_2$ and $C_3$ at $A'$ passes through $X$. This means that $A'X$ is perpendicular to $O_2O_3$ at $A'$. This line therefore passes through the incenter $I'$ of $O_1O_2O_3$. Now, the homogeneous coordinates of $A'$ and $X$ can be rewritten as
\[
A' = \left(\frac{a}{(1 + \cos \frac{A}{2})} : \frac{b}{(1 + \cos \frac{B}{2})} : \frac{c}{(1 + \cos \frac{C}{2})}\right),
\]
\[
X = \left(0 : \frac{b}{(1 + \cos \frac{B}{2})} : \frac{c}{(1 + \cos \frac{C}{2})}\right).
\]
It is easy to verify that these two points lie on the line
\[
\frac{(1 + \cos \frac{A}{2})(\cos \frac{B}{2} - \cos \frac{C}{2})}{a} x - \frac{(1 + \cos \frac{A}{2})\cos \frac{B}{2}}{b} y + \frac{(1 + \cos \frac{A}{2})\cos \frac{C}{2}}{c} z = 0,
\]
which also contains the point
\[
\left(\frac{a}{1 + \cos \frac{A}{2}} : \frac{b}{1 + \cos \frac{B}{2}} : \frac{c}{1 + \cos \frac{C}{2}}\right).
\]
Similar calculations show that the latter point also lies on the lines $BY$ and $C'Z$. It is therefore the incenter $I'$ of triangle $O_1O_2O_3$. See Figure 6. This point appears in [5] as $X_{183}$, the radical center of the Malfatti circles.
Remarks. (1) The line joining $Q$ and $I'$ has equation

$$\frac{(1 + \cos \frac{A}{2})(\cos \frac{B}{2} - \cos \frac{C}{2})}{\sin \frac{A}{2}} x + \frac{(1 + \cos \frac{B}{2})(\cos \frac{C}{2} - \cos \frac{A}{2})}{\sin \frac{B}{2}} y + \frac{(1 + \cos \frac{C}{2})(\cos \frac{A}{2} - \cos \frac{B}{2})}{\sin \frac{C}{2}} z = 0.$$ 

This line clearly contains the point $(\sin \frac{A}{2} : \sin \frac{B}{2} : \sin \frac{C}{2})$, which is the point $X_{174}$, the Yff center of congruence in [4, pp.94–95].

(2) According to [4], the triangle $A'B'C'$ in §3.3 is also perspective with the excentral triangle. This is because cevian triangles and anticevian triangles are always perspective. The perspector

$$\left( a \left( \frac{2 + \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}}{1 + \cos \frac{A}{2}} \right)^2 + \cos \frac{A}{2} \left( \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - (2 + \cos \frac{A}{2})^2 \right) \right) : \cdots : \cdots$$

is named the second Ajima-Malfatti point $X_{180}$. For the same reason, the triangle $XYZ$ in §3.5 is also perspective with the excentral triangle. The perspector is the point

$$\left( a \left( -\cos \frac{A}{2} \left( 1 + \cos \frac{A}{2} \right) + \cos \frac{B}{2} \left( 1 + \cos \frac{B}{2} \right) + \cos \frac{C}{2} \left( 1 + \cos \frac{C}{2} \right) \right) : \cdots : \cdots \right).$$

This point and the triangle center $P_+$ apparently do not appear in the current edition of [5].

Editor’s endnote. The triangle center $Q$ in §3.5 appears in [5] as the second Malfatti-Rabinowitz point $X_{1143}$. Its coordinates given by the present editor [13] were not correct owing to a mistake in a sign in the calculations. In the notations of [13], if
Triangle centers associated with the Malfatti circles

$\alpha, \beta, \gamma$ are such that

$$\sin^2 \alpha = \frac{a}{s}, \quad \sin^2 \beta = \frac{b}{s}, \quad \sin^2 \gamma = \frac{c}{s},$$

and $\lambda = \frac{1}{2}(\alpha + \beta + \gamma)$, then the homogeneous barycentric coordinates of $Q$ are

$$(\cot(\lambda - \alpha) : \cot(\lambda - \beta) : \cot(\lambda - \gamma)).$$

These are equivalent to those given in (13) in simpler form.

References


Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia, Yugoslavia

E-mail address: milmath@ptt.yu
The Lucas Circles and the Descartes Formula

Wilfred Reyes

Abstract. We determine the radii of the three circles each tangent to the circumcircle of a given triangle at a vertex, and mutually tangent to each other externally. The calculations are then reversed to give the radii of the two Soddy circles associated with three circles tangent to each other externally.

1. The Lucas circles

Consider a triangle $ABC$ with circumcircle $C$. We set up a coordinate system with the circumcenter $O$ at the origin and $A, B, C$ represented by complex numbers of moduli $R$, the circumradius. If the lengths of the sides $BC, CA, AB$ are $a, b, c$ respectively, then

$$||A - B|| = c \quad \text{and} \quad \langle A, B \rangle = R^2 - \frac{c^2}{2}. \quad (1)$$

Analogous relations hold for the pairs $B, C$ and $C, A$. Let $0 \leq \alpha < R$, and consider the circle $C_A(\alpha)$ with center $\frac{R - \alpha}{R} \cdot A$ and radius $\alpha$. This is internally tangent to the circumcircle at $A$, and is the image of $C$ under the homothety $h(A, \frac{\alpha}{R})$. See Figure 1. For real numbers $\beta, \gamma$ satisfying $0 \leq \beta, \gamma < R$, we consider the circles $C_B(\beta)$ and $C_C(\gamma)$ analogously defined. Now, the circles $C_A(\alpha)$ and $C_B(\beta)$ are tangent externally if and only if

$$\left| \left| \frac{R - \alpha}{R} A - \frac{R - \beta}{R} B \right| \right| = \alpha + \beta.$$

This is equivalent, by a simple application of (1), to

$$c^2 = \frac{4\alpha \beta}{(R - \alpha)(R - \beta)}.$$ 

Therefore, the three circles $C_A(\alpha), C_B(\beta)$ and $C_C(\gamma)$ are tangent externally to each other if and only if

$$a^2 = \frac{4R^2\beta\gamma}{(R - \beta)(R - \gamma)}, \quad b^2 = \frac{4R^2\gamma\alpha}{(R - \gamma)(R - \alpha)}, \quad c^2 = \frac{4R^2\alpha\beta}{(R - \alpha)(R - \beta)}. \quad (2)$$
These equations can be solved for the radii $\alpha$, $\beta$, and $\gamma$ in terms of $a$, $b$, $c$, and $R$. In fact, multiplying the equations in (2), we obtain

$$abc = \frac{8R^3 \alpha \beta \gamma}{(R - \alpha)(R - \beta)(R - \gamma)}.$$ 

Consequently,

$$\frac{\alpha}{R - \alpha} = \frac{bc}{2Ra}, \quad \frac{\beta}{R - \beta} = \frac{ca}{2Rb}, \quad \frac{\gamma}{R - \gamma} = \frac{ab}{2Rc}.$$ 

From these, we obtain

$$\alpha = \frac{bc}{2Ra + bc} \cdot R, \quad \beta = \frac{ca}{2Rb + ca} \cdot R, \quad \gamma = \frac{ab}{2Rc + ab} \cdot R.$$ 

(3)

Denote by $\triangle$ the area of triangle $ABC$, and $h_a$, $h_b$, $h_c$ its three altitudes. We have $2\triangle = a \cdot h_a = b \cdot h_b = c \cdot h_c$. Since $abc = 4R\triangle$, the expression for $\alpha$ in (3) can be rewritten as

$$\alpha = \frac{abc}{2Ra^2 + abc} = \frac{4R\triangle}{2Ra^2 + 4R\triangle} = \frac{2\triangle}{a^2 + 2\triangle} = \frac{a \cdot h_a}{a^2 + a \cdot h_a} = \frac{h_a}{a + h_a}.$$ 

Therefore, the homothety $h(A, \frac{\alpha}{R})$ is the one that contracts the square on the side $BC$ (externally) into the inscribed square on this side. See Figure 1. The same is true for the other two circles. The three circles $C_A(\alpha)$, $C_B(\beta)$, $C_C(\gamma)$ are therefore the Lucas circles considered in [3]. See Figure 2.
2. Another triad of circles

A simple modification of the above calculations shows that for positive numbers \( \alpha', \beta', \gamma' \), the images of the circumcircle \( C \) under the homotheties \( h(A, -\frac{\alpha'}{R}) \), \( h(B, -\frac{\beta'}{R}) \) and \( h(C, -\frac{\gamma'}{R}) \) (each tangent to \( C \) at a vertex) are tangent to each other if and only if

\[
\alpha' = \frac{bc}{2Ra - bc} \cdot R, \quad \beta' = \frac{ca}{2Rb - ca} \cdot R, \quad \gamma' = \frac{ab}{2Rc - ab} \cdot R. \quad (4)
\]

The tangencies are all external provided \( 2Ra - bc, 2Rb - ca \) and \( 2Rc - ab \) are all positive. These quantities are essentially the excesses of the sides over the corresponding altitudes:

\[
2Ra - bc = \frac{bc}{a}(a - h_a), \quad 2Rb - ca = \frac{ca}{b}(b - h_b), \quad 2Rc - ab = \frac{ab}{c}(c - h_c).
\]

It may occur that one of them is negative. In that case, the tangencies with the corresponding circle are all internal. See Figure 4.
3. Inscribed squares

Consider the triad of circles in §2. The homothety $h(A, -\alpha' R)$ transforms the square erected on $BC$ on the same side of $A$ into an inscribed square since $-\alpha' R = -\frac{h_a}{a - h_a}$. See Figure 5.

Denote by $a_1$ and $a_2$ the lengths of sides of the two inscribed squares on $BC$, under the homotheties $h(A, \frac{\alpha}{R})$ and $h(A, -\frac{\alpha'}{R})$ respectively, i.e., $a_1 = \frac{\alpha}{R} \cdot a$ and
\[ a_2 = \frac{a'}{R} \cdot a. \] Making use of (3) and (4), we have
\[
\frac{1}{a_1} + \frac{1}{a_2} = \left( \frac{1}{\alpha} + \frac{1}{\alpha'} \right) \frac{R}{a} = \frac{4a}{bc} \cdot \frac{R}{a} = \frac{a}{\Delta} = \frac{2}{\sqrt{\frac{a}{\alpha}}}. \]

This means that the altitude \( h_a \) is the harmonic mean of the lengths of the sides of the two inscribed squares on the side \( BC \). See Figure 6.

4. The Descartes formula

We reverse the calculations in §§1,2 to give a proof of the Descartes formula. See, [2, pp.90–92]. Given three circles of radii \( \alpha, \beta, \gamma \) tangent to each other externally, we determine the radii of the two Soddy circles tangent to each of them. See, for example, [1, pp.13–16]. We first seek the radius \( R \) of the circle tangent internally to each of them, the outer Soddy circle. Regard, in equation (3), \( R, a, b, c \) as unknowns, and write \( \Delta \) for the area of the unknown triangle \( ABC \) whose vertices are the points of tangency. Thus, by Heron’s formula,
\[
16 \Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \tag{5}
\]

In terms of \( \Delta \), (3) can be rewritten as
\[
\alpha = \frac{2\Delta}{a^2 + 2\Delta} \cdot R, \quad \beta = \frac{2\Delta}{b^2 + 2\Delta} \cdot R, \quad \gamma = \frac{2\Delta}{c^2 + 2\Delta} \cdot R,
\]
or
\[
a^2 = \frac{2(R - \alpha)\Delta}{\alpha}, \quad b^2 = \frac{2(R - \beta)\Delta}{\beta}, \quad c^2 = \frac{2(R - \gamma)\Delta}{\gamma}. \tag{6}
\]

Substituting these into (5) and simplifying, we obtain
\[
\alpha^2 \beta^2 \gamma^2 - 2\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)R
+ (\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2)R^2 = 0.
\]

Dividing throughout by \( \alpha^2 \beta^2 \gamma^2 \cdot R^2 \), we have
\[
\frac{1}{R^2} + 2 \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \frac{1}{R} + \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha} \right) = 0.
\]

Since \( R > \alpha, \beta, \gamma \), we have
\[
\frac{1}{R} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2 \sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}} + \frac{1}{\alpha\beta}.
\]
This is positive if and only if
\[
\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha} > 0. \tag{7}
\]

This is the condition necessary and sufficient for the existence of a circle tangent internally to each of the given circles.

By reversing the calculations in §§2, the radius of the circle tangent to the three given circles externally, the inner Soddy circle, is given by
\[
\frac{1}{R'} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 2 \sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}} + \frac{1}{\alpha\beta}\].
If condition (7) is not satisfied, both Soddy circles are tangent to each of the given circles externally.

References


Wilfred Reyes: Departamento de Ciencias Básicas, Universidad del Bío-Bío, Chillán, Chile

E-mail address: wreyes@ubiobio.cl
Similar Pedal and Cevian Triangles

Jean-Pierre Ehrmann

Abstract. The only point with similar pedal and cevian triangles, other than the orthocenter, is the isogonal conjugate of the Parry reflection point.

1. Introduction

We begin with notation. Let \(ABC\) be a triangle with sidelengths \(a\), \(b\), \(c\), orthocenter \(H\), and circumcenter \(O\). Let \(K_A, K_B, K_C\) denote the vertices of the tangential triangle, \(O_A, O_B, O_C\) the reflections of \(O\) in \(A, B, C\), and \(A_S, B_S, C_S\) the reflections of the vertices of \(A\) in \(BC\), of \(B\) in \(CA\), and of \(C\) in \(AB\). Let \(M^*\) be isogonal conjugate of a point \(M\);
\(\overline{MM'}\) is inverse of \(M\) in the circumcircle;
\(\angle LL'\) is the measure, modulo \(\pi\), of the directed angle of the lines \(L, L'\);
\(S_A = bc \cos A = \frac{1}{2}(b^2 + c^2 - a^2)\), with \(S_B\) and \(S_C\) defined cyclically;
\(x : y : z\) is barycentric coordinates relative to triangle \(ABC\);
\(\Gamma_A\) is circle with diameter \(K_A\), with circles \(\Gamma_B\) and \(\Gamma_C\) defined cyclically.
The circle \(\Gamma_A\) passes through the points \(B_S, C_S\) and is the locus of \(M\) such that \(\angle B_SM_C = -2\angle BAC\). An equation for \(\Gamma_A\), in barycentrics, is
\[ 2S_A(a^2yz + b^2zx + c^2xy) + (b^2c^2x + 2c^2S_Cy + 2b^2S_Bz)(x + y + z) = 0. \]

Consider a triangle \(A'B'C'\), where \(A', B', C'\) lie respectively on the sidelines \(BC, CA, AB\). The three circles \(A'B'C', BC'A', CA'B'\) meet in a point \(S\) called the Miquel point of \(A'B'C'\). See [2, pp.131–135]. The point \(S\) (or \(\overline{S}\)) is the only point whose pedal triangle is directly (or indirectly) similar to \(AB'C'\).
The circles \(\Gamma_A, \Gamma_B, \Gamma_C\) have a common point \(T\): the Parry reflection point. \(X_{399}\) in [3]; the three radical axes \(TA_S, TB_S, TC_S\) are the reflections with respect to a sideline of \(ABC\) of the parallel to the Euler line going through the opposite vertex. See [3, 4], and Figure 1. \(T\) lies on the circle \((O, 2R)\), on the Neuberg cubic, and is the antipode of \(O\) on the Stammler hyperbola. See [1].

2. Similar triangles

Let \(A'B'C'\) be the cevian triangle of a point \(P = p : q : r\).

Lemma 1. The pedal and cevian triangles of \(P\) are directly (or indirectly) similar if and only if \(P\) (or \(\overline{P}\)) lies on the three circles \(ABC', BC'A', CA'B'\).

Proof. This is an immediate consequence of the properties of the Miquel point above. \(\square\)

Lemma 2. \(A, B', C', P\) are concyclic if and only if \(P\) lies on the circle \(BCH\).
Proof. A, B', C' and P are concyclic ⇔ \( \angle B'PC' = \angle B'AC' \) ⇔ \( \angle BPC = \angle BHC \) ⇔ P lies on the circle BCH. \( \square \)

**Proposition 3.** The pedal and cevian triangles of P are directly similar only in the trivial case of P = H.

**Proof.** By Lemma 1, the pedal and cevian triangles of P are directly similar if and only if P lies on the three circles ABC', BC'A', CA'B'. By Lemma 2, P lies on the three circles BCH, CAH, ABH. Hence, P = H. \( \square \)

**Lemma 4.** A, B', C', \( \overline{P} \) are concyclic if and only if \( P^* \) lies on the circle \( \Gamma_A \).
Proof. If \( P = p : q : r \), the circle \( \Phi_A \) passing through \( A, B', C' \) is given by
\[
a^2 yz + b^2 zx + c^2 xy - p(x + y + z) \left( \frac{c^2}{p+q} y + \frac{b^2}{p+r} z \right) = 0,
\]
and its inverse in the circumcircle is the circle \( \Phi_A \) given by
\[
(a^2(p^2 - qr) + (b^2 - c^2)p(q - r))(a^2 yz + b^2 zx + c^2 xy)
- pa^2(x + y + z)(c^2(p + r)y + b^2(p + q)z) = 0.
\]
Since \( \Phi_A \) contains \( P \), its inverse \( \Phi_A \) contains \( P \). Changing \((p, q, r)\) to \((x, y, z)\) gives the locus of \( P \) satisfying \( P \in \Phi_A \). Then changing \((x, y, z)\) to \((a^2 x, b^2 y, c^2 z)\) gives the locus \( \Phi_A \) of the point \( P^* \) such that \( P^* \in \Phi_A \). By examination, \( \Phi_A = \Gamma_A \).

**Proposition 5.** The pedal and cevian triangles of \( P \) are indirectly similar if and only if \( P \) is the isogonal conjugate of the Parry reflection point.

**Proof.** By Lemma 1, the pedal and cevian triangles of \( P \) are indirectly similar if and only if \( P \) lies on the three circles \( AB'C', BC'A', CA'B' \). By Lemma 4, \( P^* \) lies on each of the circles \( \Gamma_A, \Gamma_B, \Gamma_C \). Hence, \( P^* = T \), and \( P = T^* \).
Remarks. (1) The isogonal conjugate of $X_{399}$ is $X_{1138}$ in [3]: this point lies on the Neuberg cubic.

(2) We can deduce Lemma 4 from the relation $\angle B PC' - \angle B_s P^* C_s = \angle BAC$, which is true for every point $P$ in the plane of $ABC$ except the vertices $A, B, C$.

(3) As two indirectly similar triangles are orthologic and as the pedal and cevian triangles of $P$ are orthologic if and only if $P^*$ lies on the Stammler hyperbola, a point with indirectly similar cevian and pedal triangles must be the isogonal conjugate of a point of the Stammler hyperbola.

References


Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr
On the Kosnita Point and the Reflection Triangle

Darij Grinberg

Abstract. The Kosnita point of a triangle is the isogonal conjugate of the nine-point center. We prove a few results relating the reflections of the vertices of a triangle in their opposite sides to triangle centers associated with the Kosnita point.

1. Introduction

By the Kosnita point of a triangle we mean the isogonal conjugate of its nine-point center. The name Kosnita point originated from J. Rigby [5].

Theorem 1 (Kosnita). Let \(ABC\) be a triangle with the circumcenter \(O\), and \(X, Y, Z\) be the circumcenters of triangles \(BOC, COA, AOB\). The lines \(AX, BY, CZ\) concur at the isogonal conjugate of the nine-point center.

We denote the nine-point center by \(N\) and the Kosnita point by \(N^*\). Note that \(N^*\) is an infinite point if and only if the nine-point center is on the circumcircle. We study this special case in §5 below. The points \(N\) and \(N^*\) appear in [3] as \(X_5\) and \(X_{54}\) respectively. An old theorem of J. R. Musselman [4] relates the Kosnita

---

Published Date: April 18, 2003. Communicating Editor: Paul Yiu.

The author thanks the communicating editor for simplifications, corrections and numerous helpful comments, particularly in §§4-5.
point to the reflections \( A', B', C' \) of \( A, B, C \) in their opposite sides \( BC, CA, AB \) respectively.

**Theorem 2** (Musselman). The circles \( AOA', BOB', COC' \) pass through the inversive image of the Kosnita point in the circumcircle of triangle \( ABC \).

This common point of the three circles is the triangle center \( X_{157} \) in [3], which we denote by \( Q \) in Figure 1. The following theorem gives another triad of circles containing this point. It was obtained by Paul Yiu [7] by computations with barycentric coordinates. We give a synthetic proof in §2.

**Theorem 3** (Yiu). The circles \( AB'C', BC'A', CA'B' \) pass through the inversive image of the Kosnita point in the circumcircle.

On the other hand, it is clear that the circles \( A'BC, B'CA, \) and \( C'AB \) pass through the orthocenter of triangle \( ABC \). It is natural to inquire about the circumcenter of the *reflection triangle* \( A'B'C' \). A very simple answer is provided by the following characterization of \( A'B'C' \) by G. Boutte [1].

![Figure 2](image-url)
Theorem 4 (Boutte). Let $G$ be the centroid of $ABC$. The reflection triangle $A'B'C'$ is the image of the pedal triangle of the nine-point center $N$ under the homothety $h(G, 4)$.

Corollary 5. The circumcenter of the reflection triangle $A'B'C'$ is the reflection of the circumcenter in the Kosnita point.

2. Proof of Theorem 3

Denote by $Q$ the inverse of the Kosnita point $N^*$ in the circumcircle. By Theorem 2, $Q$ lies on the circles $BOB'$ and $COC'$. So $\angle B'QO = \angle B'BO$ and $\angle C'QO = \angle C'CO$. Since $\angle B'QC' = \angle B'QO + \angle C'QO$, we get

$$\angle B'QC' = \angle B'BO + \angle C'CO$$

$$= (\angle CBB' - \angle CBO) + (\angle BCC' - \angle BCO)$$

$$= \angle CBB' + \angle BCC' - (\angle CBO + \angle BCO)$$

$$= \angle CBB' + \angle BCC' - (\pi - \angle BOC)$$

$$= \angle CBB' + \angle BCC' - \pi + \angle BOC.$$

But we have $\angle CBB' = \frac{\pi}{2} - C$ and $\angle BCC' = \frac{\pi}{2} - B$. Moreover, from the central angle theorem we get $\angle BOC = 2A$. Thus,

$$\angle B'QC' = \left(\frac{\pi}{2} - C\right) + \left(\frac{\pi}{2} - B\right) - \pi + 2A$$

$$= \pi - B - C - \pi + 2A = 2A - B - C$$

$$= 3A - (A + B + C) = 3A - \pi,$$

and consequently

$$\pi - \angle B'QC' = \pi - (3A - \pi) = 2\pi - 3A.$$
But on the other hand, \( \angle BAC' = \angle BAC = A \) and \( \angle CAB' = A \), so
\[
\angle B'C = 2\pi - (\angle BAC' + \angle BAC + \angle CAB') = 2\pi - (A + A + A) = 2\pi - 3A.
\]
Consequently, \( \angle BAC' = \pi - \angle B'CQ' \). Thus, \( Q \) lies on the circle \( AB'C' \). Similar reasoning shows that \( Q \) also lies on the circles \( BCA' \) and \( CA'B' \).

This completes the proof of Theorem 3.

**Remark.** In general, if a triangle \( ABC \) and three points \( A', B', C' \) are given, and the circles \( A'BC, B'CA, \) and \( C'AB \) have a common point, then the circles \( AB'C', BC'A', \) and \( CA'B' \) also have a common point. This can be proved with some elementary angle calculations. In our case, the common point of the circles \( ABC, B'CA, \) and \( C'AB \) is the orthocenter of \( ABC \), and the common point of the circles \( AB'C', BC'A', \) and \( CA'B' \) is \( Q \).

3. Proof of Theorem 4

Let \( A_1, B_1, C_1 \) be the midpoints of \( BC, CA, AB \), and \( A_2, B_2, C_2 \) the midpoints of \( B_1C_1, C_1A_1, A_1B_1 \). It is clear that \( A_2B_2C_2 \) is the image of \( ABC \) under the homothety \( h(G, \frac{1}{4}) \). Denote by \( X \) the image of \( A' \) under this homothety. We show that this is the pedal of the nine-point center \( N \) on \( BC \).

![Figure 5](image-url)

First, note that \( X \), being the reflection of \( A_2 \) in \( B_2C_2 \), lies on \( BC \). This is because \( A_2X \) is perpendicular to \( B_2C_2 \) and therefore to \( BC \). The distance from
On the Kosnita point and the reflection triangle

$X$ to $A_2$ is twice of that from $A_2$ to $B_2C_2$. This is equal to the distance between the parallel lines $B_2C_2$ and $BC$.

The segment $A_2X$ is clearly the perpendicular bisector of $B_1C_1$. It passes through the circumcenter of triangle $A_1B_1C_1$, which is the nine-point $N$ of triangle $ABC$. It follows that $X$ is the pedal of $N$ on $BC$. For the same reasons, the images of $B', C'$ under the same homothety $h(G, \frac{1}{4})$ are the pedals of $N$ on $CA$ and $AB$ respectively.

This completes the proof of Theorem 4.

4. Proof of Corollary 5

It is well known that the circumcenter of the pedal triangle of a point $P$ is the midpoint of the segment $PP^*$, $P^*$ being the isogonal conjugate of $P$. See, for example, [2, pp.155–156]. Applying this to the nine-point center $N$, we obtain the circumcenter of the reflection triangle $A'B'C'$ as the image of the midpoint of $NN^*$ under the homothety $h(G, 4)$. This is the point

$$G + 4 \left( \frac{N + N^*}{2} - G \right) = 2(N + N^*) - 3G = 2N^* + 2N - 3G = 2N^* + (O + H) - (2 \cdot O + H) = 2N^* - O,$$

the reflection of $O$ in the Kosnita point $N^*$. Here, $H$ is orthocenter, and we have made use of the well known facts that $N$ is the midpoint of $OH$ and $G$ divides $OH$ in the ratio $HG : GO = 2 : 1$.

This completes the proof of Corollary 5.

This point is the point $X_{195}$ of [3]. Barry Wolk [6] has verified this theorem by computer calculations with barycentric coordinates.

5. Triangles with nine-point center on the circumcircle

Given a circle $O(R)$ and a point $N$ on its circumference, let $H$ be the reflection of $O$ in $N$. For an arbitrary point $P$ on the minor arc of the circle $N(R)$ inside $O(R)$, let (i) $A$ be the intersection of the segment $HP$ with $O(R)$, (ii) the perpendicular to $HP$ at $P$ intersect $O(R)$ at $B$ and $C$. Then triangle $ABC$ has nine-point center $N$ on its circumcircle $O(R)$. See Figure 6. This can be shown as follows. It is clear that $O(R)$ is the circumcircle of triangle $ABC$. Let $M$ be the midpoint of $BC$ so that $OM$ is orthogonal to $BC$ and parallel to $PH$. Thus, $OMPH$ is a (self-intersecting) trapezoid, and the line joining the midpoints of $PM$ and $OH$ is parallel to $PH$. Since the midpoint of $OH$ is $N$ and $PH$ is orthogonal to $BC$, we conclude that $N$ lies on the perpendicular bisector of $PM$. Consequently, $NM = NP = \frac{R}{2}$, and $M$ lies on the circle $N(R/2)$. This circle is the nine-point circle of triangle $ABC$, since it passes through the pedal $P$ of $A$ on $BC$ and through the midpoint $M$ of $BC$ and has radius $\frac{R}{2}$. 
Remark. As $P$ traverses the minor arc which the intersection of $N(R)$ with the interior of $O(R)$, the line $L$ passes through a fixed point, which is the reflection of $O$ in $H$.

**Theorem 6.** Suppose the nine-point center $N$ of triangle $ABC$ lies on the circum-circle.

1. The reflection triangle $A'B'C'$ degenerates into a line $L$.
2. If $X, Y, Z$ are the centers of the circles $BOC, COA, AOB$, the lines $AX$, $BY$, $CZ$ are all perpendicular to $L$.
3. The circles $AOA'$, $BOB'$, $COC'$ are mutually tangent at $O$. The line joining their centers is the parallel to $L$ through $O$.
4. The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through $O$.

**References**

Added in proof: Bernard Gibert has kindly communicated the following results. Let $A_1$ be the intersection of the lines $OA'$ and $B'C'$, and similarly define $B_1$ and $C_1$. Denote, as in §1, by $Q$ be the inverse of the Kosnita point in the circumcircle.

**Theorem 7** (Gibert). The lines $AA_1$, $BB_1$, $CC_1$ concur at the isogonal conjugate of $Q$.

This is the point $X_{1263}$ in [3]. The points $A$, $B$, $C$, $A'$, $B'$, $C'$, $O$, $Q$, $A_1$, $B_1$, $C_1$ all lie on the Neuberg cubic of triangle $ABC$, which is the isogonal cubic with pivot the infinite point of the Euler line. This cubic is also the locus of all points whose reflections in the sides of triangle $ABC$ form a triangle perspective to $ABC$. The point $Q$ is the unique point whose triangle of reflections has perspector on the circumcircle. This perspector, called the Gibert point $X_{1141}$ in [3], lies on the line joining the nine-point center to the Kosnita point.

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany

*E-mail address:* darij_grinberg@web.de
A Note on the Schiffler Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. We prove two interesting properties of the Schiffler point.

1. Main results

The Schiffler point is the intersection of four Euler lines. Let $I$ be the incenter of triangle $ABC$. The Schiffler point $S$ is the point common to the Euler lines of triangles $IBC$, $ICA$, $IAB$, and $ABC$. See [1, p.70]. Not much is known about $S$. In this note, we prove two interesting properties of this point.

Theorem 1. Let $A$ and $I_1$ be the circumcenter and $A$-excenter of triangle $ABC$, and $A_1$ the intersection of $OI_1$ and $BC$. Similarly define $B_1$ and $C_1$. The lines $AA_1$, $BB_1$ and $CC_1$ concur at the Schiffler point $S$.

Theorem 2. Let $A'$, $B'$, $C'$ be the touch points of the $A$-excircle and $BC$, $CA$, $AB$ respectively, and $A''$ the reflection of $A'$ in $B'C'$. Similarly define $B''$ and $C''$. The lines $AA''$, $BB''$ and $CC''$ concur at the Schiffler point $S$.

Figure 1
We make use of trilinear coordinates with respect to triangle $ABC$. According to [1, p.70], the Schiffler point has coordinates

$$
\left( \frac{1}{\cos B + \cos C} : \frac{1}{\cos C + \cos A} : \frac{1}{\cos A + \cos B} \right).
$$

2. Proof of Theorem 1

We show that $AA_1$ passes through the Schiffler point $S$. Because $O = (\cos A : \cos B : \cos C)$ and $I_1 = (-1 : 1 : 1)$, the line $OI_1$ is given by

$$(\cos B - \cos C)\alpha - (\cos C + \cos A)\beta + (\cos A + \cos B)\gamma = 0.$$ 

The line $BC$ is given by $\alpha = 0$. Hence the intersection of $OI_1$ and $BC$ is $A_1 = (0 : \cos A + \cos B : \cos A + \cos C)$. The collinearity of $A_1$, $S$ and $A$ follows from

$$
\begin{vmatrix}
1 & 0 & 0 \\
0 & \cos A + \cos B & \cos A + \cos C \\
\frac{1}{\cos B + \cos C} & \frac{1}{\cos C + \cos A} & \frac{1}{\cos A + \cos B}
\end{vmatrix} = \begin{vmatrix}
\cos A + \cos B & \cos A + \cos C \\
\frac{1}{\cos B + \cos C} & \frac{1}{\cos C + \cos A} & \frac{1}{\cos A + \cos B}
\end{vmatrix} = 0.
$$

This completes the proof of Theorem 1.

Remark. It is clear from the proof above that more generally, if $P$ is a point with trilinear coordinates $(p : q : r)$, and $A_1$, $B_1$, $C_1$ the intersections of $PI_a$ with $BC$, $PI_2$ with $CA$, $PI_3$ with $AB$, then the lines $AA_1$, $BB_1$, $CC_1$ intersect at a point with trilinear coordinates $\left( \frac{1}{q+r} : \frac{1}{r+p} : \frac{1}{p+q} \right)$. If $P$ is the symmedian point, for example, this intersection is the point $X_{81} = \left( \frac{1}{b+c} : \frac{1}{c+a} : \frac{1}{a+b} \right)$.

3. Proof of Theorem 2

We deduce Theorem 2 as a consequence of the following two lemmas.

Lemma 3. The line $OI_1$ is the Euler line of triangle $A'B'C'$.

Proof. Triangle $ABC$ is the tangential triangle of $A'B'C'$. It is known that the circumcenter of the tangential triangle lies on the Euler line. See, for example, [1, p.71]. It follows that $OI_1$ is the Euler line of triangle $A'B'C'$.

Lemma 4. Let $A^*$ be the reflection of vertex $A$ of triangle $ABC$ with respect to $BC$, $A_1B_1C_1$ be the tangential triangle of $ABC$. Then the Euler line of $ABC$ and line $A_1A^*$ intersect line $B_1C_1$ in the same point.
Proof. As is well known, the vertices of the tangential triangle are given by
\[ A_1 = (-a : b : c), \quad B_1 = (a : -b : c), \quad C_1 = (a : b : -c). \]
The line \( B_1C_1 \) is given by \( c\beta + b\gamma = 0 \). According to [1, p.42], the Euler line of triangle \( ABC \) is given by
\[ a(b^2-c^2)(b^2+c^2-a^2)\alpha + b(c^2-a^2)(c^2+a^2-b^2)\beta + c(a^2-b^2)(a^2+b^2-c^2)\gamma = 0. \]
Now, it is not difficult to see that
\[ A^* = (-1 : 2 \cos C : 2 \cos B) \]
\[ = (-abc : c(a^2 + b^2 - c^2) : b(c^2 + a^2 - b^2)). \]
The equation of the line \( A^*A_1 \) is then
\[ \begin{vmatrix} -abc & 2c(a^2 + b^2 - c^2) & 2b(c^2 + a^2 - b^2) \\ -a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = 0. \]
After simplification, this is
\[ -(b^2 - c^2)(b^2 + c^2 - a^2)\alpha + ab(a^2 - b^2)\beta - ac(a^2 - c^2)\gamma = 0. \]
Now, the lines \( B_1C_1, A^*A_1 \), and the Euler line are concurrent if the determinant
\[ \begin{vmatrix} 0 & c & b \\ -(b^2 - c^2)(b^2 + c^2 - a^2) & ab(a^2 - b^2) & -ac(a^2 - c^2) \\ a(b^2 - c^2)(b^2 + c^2 - a^2) & b(c^2 - a^2)(c^2 + a^2 - b^2) & c(a^2 - b^2)(a^2 + b^2 - c^2) \end{vmatrix} \]
is zero. Factoring out \((b^2 - c^2)(b^2 + c^2 - a^2)\), we have
\[ \begin{vmatrix} 0 & c & b \\ -1 & ab(a^2 - b^2) & -ac(a^2 - c^2) \\ a & b(c^2 - a^2)(c^2 + a^2 - b^2) & c(a^2 - b^2)(a^2 + b^2 - c^2) \end{vmatrix} \]
\[ = -c \begin{vmatrix} 0 & ab(a^2 - b^2) & -ac(a^2 - c^2) \\ -1 & -ac(a^2 - c^2) & b \\ a & c(a^2 - b^2)(a^2 + b^2 - c^2) & b(c^2 - a^2)(c^2 + a^2 - b^2) \end{vmatrix} \]
\[ = c^2((b^2 - c^2)(b^2 + c^2 - a^2) - a^2(a^2 - c^2)) 
- b^2((c^2 - a^2)(c^2 + a^2 - b^2) + a^2(a^2 - b^2)) 
= c^2 \cdot b^2(c^2 - b^2) - b^2 \cdot c^2(c^2 - b^2) 
= 0. \]
This confirms that the three lines are concurrent. \(\Box\)

To prove Theorem 2, it is enough to show that the line \( AA'' \) in Figure 1 contains \( S \). Now, triangle \( A'B'C' \) has tangential triangle \( ABC \) and Euler line \( OI_1 \) by Lemma 3. By Lemma 4, the lines \( OI_1, AA'' \) and \( BC \) are concurrent. This means that the line \( AA'' \) contains \( A_1 \). By Theorem 1, this line contains \( S \).
Reference


Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
*E-mail address*: emelyanov@kaluga.ru

Tatiana Emelyanova: 18-31 Proyezjaia Street, Kaluga, Russia 248009
*E-mail address*: emelyanov@kaluga.ru
Harcourt’s Theorem

Nikolaos Dergiades and Juan Carlos Salazar

Abstract. We give a proof of Harcourt’s theorem that if the signed distances from the vertices of a triangle of sides $a, b, c$ to a tangent of the incircle are $a_1, b_1, c_1$, then $aa_1 + bb_1 + cc_1$ is twice of the area of the triangle. We also show that there is a point on the circumconic with center $I$ whose distances to the sidelines of $ABC$ are precisely $a_1, b_1, c_1$. An application is given to the extangents triangle formed by the external common tangents of the excircles.

1. Harcourt’s Theorem

The following interesting theorem appears in F. G.-M.[1, p.750] as Harcourt’s theorem.

Theorem 1 (Harcourt). If the distances from the vertices $A, B, C$ to a tangent to the incircle of triangle $ABC$ are $a_1, b_1, c_1$ respectively, then the algebraic sum $aa_1 + bb_1 + cc_1$ is twice of the area of triangle $ABC$.

The distances are signed. Distances to a line from points on opposite sides are opposite in sign, while those from points on the same side have the same sign. For the tangent lines to the incircle, we stipulate that the distance from the incenter is positive. For example, in Figure 1, when the tangent line $\ell$ separates the vertex $A$ from $B$ and $C$, $a_1$ is negative while $b_1$ and $c_1$ are positive. With this sign convention, Harcourt’s theorem states that

$$aa_1 + bb_1 + cc_1 = 2\Delta,$$

(1)

---

Publication Date: June 2, 2003. Communicating Editor: Paul Yiu.

The authors thank the editor for his valuable comments, helps and strategic improvements. JCS also thanks Francisco Bellot Rosado and Darij Grinberg for their helpful remarks in an early stage of the preparation of this paper.
where $\triangle$ is the area of triangle $ABC$.

We give a simple proof of Harcourt’s theorem by making use of homogeneous barycentric coordinates with reference to triangle $ABC$. First, we establish a fundamental formula.

**Proposition 2.** Let $\ell$ be a line passing through a point $P$ with homogeneous barycentric coordinates $(x : y : z)$. If the signed distances from the vertices $A, B, C$ to a line $\ell$ are $d_1, d_2, d_3$ respectively, then

$$d_1x + d_2y + d_3z = 0. \quad (2)$$

**Proof.** It is enough to consider the case when $\ell$ separates $A$ from $B$ and $C$. We take $d_1$ as negative, and $d_2, d_3$ positive. See Figure 2. If $A'$ is the trace of $P$ on the side line $BC$, it is well known that

$$\frac{AP}{PA'} = \frac{x}{y+z}.$$  

Since $\frac{BA'}{A'C} = \frac{z}{y}$, the distance from $A'$ to $\ell$ is

$$d'_1 = \frac{yd_2 + zd_3}{y+z}.$$  

Since $\frac{-d_1}{d'_1} = \frac{AP}{PA} = \frac{y+z}{x}$, the equation $(2)$ follows. \qed

**Proof of Harcourt’s theorem.** We apply Proposition 2 to the line $\ell$ through the incenter $I = (a : b : c)$ parallel to the tangent. The signed distances from $A, B, C$ to $\ell$ are $d_1 = a_1 - r$, $d_2 = a_2 - r$, and $d_3 = a_3 - r$. From these,

$$aa_1 + bb_1 + cc_1 = a(d_1 + r) + b(d_2 + r) + c(d_3 + r)$$

$$= (ad_1 + bd_2 + cd_3) + (a + b + c)r$$

$$= 2\triangle,$$

since $ad_1 + bd_2 + cd_3 = 0$ by Proposition 2.
2. Harcourt’s theorem for the excircles

Harcourt’s theorem for the incircle and its proof above can be easily adapted to the excircles.

**Theorem 3.** If the distances from the vertices $A$, $B$, $C$ to a tangent to the $A$-excircle of triangle $ABC$ are $a_1$, $b_1$, $c_1$ respectively, then $-aa_1 + bb_1 + cc_1 = 2\Delta$. Analogous statements hold for the $B$- and $C$-excircles.

*Figure 3*

**Proof.** Apply Proposition 2 to the line $\ell$ through the excenter $I_a = (-a : b : c)$ parallel to the tangent. If the distances from $A$, $B$, $C$ to $\ell$ are $d_1$, $d_2$, $d_3$ respectively, then

$$-ad_1 + bd_2 + cd_3 = 0.$$  

Since $a_1 = d_1 + r_1$, $b_1 = d_2 + r_1$, $c_1 = d_3 + r_1$, where $r_1$ is the radius of the excircle, it easily follows that

$$-aa_1 + bb_1 + cc_1 = -a(d_1 + r_1) + b(d_2 + r_1) + c(d_3 + r_1) = (-ad_1 + bd_2 + cd_3) + r_1(-a + b + c) = r_1(-a + b + c) = 2\Delta.$$  

Consider the external common tangents of the excircles of triangle $ABC$. Let $\ell_a$ be the external common tangent of the $B$- and $C$-excircles. Denote by $d_{a1}$, $d_{a2}$, $d_{a3}$ the distances from the $A$, $B$, $C$ to this line. Clearly, $d_{a1} = h_a$, the altitude on $BC$. Similarly define $\ell_b$, $\ell_c$ and the associated distances.
**Theorem 4.** \( d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}. \)

**Proof.** Applying Theorem 3 to the tangent \( \ell_a \) of the \( B \)-excircle (respectively the \( C \)-excircle), we have

\[
\begin{align*}
ad_{a1} - bd_{a2} + cd_{a3} &= 2\Delta, \\
ad_{a1} + bd_{a2} - cd_{a3} &= 2\Delta.
\end{align*}
\]

From these it is clear that \( bd_{a2} = cd_{a3} \), and

\[
\frac{d_{a2}}{d_{a3}} = \frac{c}{b}.
\]

Similarly,

\[
\frac{d_{b3}}{d_{b1}} = \frac{a}{c} \quad \text{and} \quad \frac{d_{c1}}{d_{c2}} = \frac{b}{a}.
\]

Combining these three equations we have \( d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}. \) \( \square \)

It is clear that the perpendiculars from \( A \) to \( \ell_a \), being the reflection of the \( A \)-altitude, passes through the circumcenter; similarly for the perpendiculars from \( B \) to \( \ell_b \) and from \( C \) to \( \ell_c \).

Let \( X \) be the intersection of the perpendiculars from \( B \) to \( \ell_c \) and from \( C \) to \( \ell_b \). Note that \( OB \) and \( CX \) are parallel, so are \( OC \) and \( BX \). Since \( OB = OC \), it follows that \( OBXC \) is a rhombus, and \( BX = CX = R \), the circumradius.
Harcourt’s theorem 121

of triangle ABC. It also follows that X is the reflection of O in the side BC. Similarly, if Y is the intersection of the perpendiculars from C to ℓa and from A to ℓc, and Z that of the perpendiculars from A to ℓb and from B to ℓa, then XYZ is the triangle of reflections of the circumcenter O. As such, it is oppositely congruent to ABC, and the center of homothety is the nine-point center of triangle ABC.

3. The circum-ellipse with center I

Consider a tangent L to the incircle at a point P. If the signed distances from the vertices A, B, C to L are a1, b1, c1, then by Harcourt’s theorem, there is a point P# whose signed distances to the sides BC, CA, AB are precisely a1, b1, c1. What is the locus of the point P# as P traverses the incircle? By Proposition 2, the barycentric equation of L is

\[ a_1 x + b_1 y + c_1 z = 0. \]

This means that the point with homogeneous barycentric coordinates \((a : b : c)\) is a point on the dual conic of the incircle, which is the circumconic with equation

\[ (s - a)yz + (s - b)zx + (s - c)xy = 0. \]  

(3)

The point P# in question has barycentric coordinates \((a a_1 : b b_1 : c c_1)\). Since \((a_1, b_1, c_1)\) satisfies (3), if we put \((x, y, z) = (a a_1, b b_1, c c_1)\), then

\[ a(s - a)yz + b(s - b)zx + c(s - c)xy = 0. \]

Thus, the locus of P# is the circumconic with perspector \((a(s - a) : b(s - b) : c(s - c))\). 1 It is an ellipse, and its center is, surprisingly, the incenter I. 2 We denote this circum-ellipse by CI. See Figure 5.

---

1 This is the Mittenpunkt, the point X9 in [4]. It can be constructed as the intersection of the lines joining the excenters to the midpoints of the corresponding sides of triangle ABC.

2 In general, the center of the circumconic \(pyz + qzx + rxy = 0\) is the point with homogeneous barycentric coordinates \((p(q + r - p) : q(r + p - q) : r(p + q - r))\).
Let \( A_1, B_1, C_1 \) be the antipodes of the points of tangency of the incircle with the sidelines. It is quite easy to see that \( A_1^\# \), \( B_1^\# \), \( C_1^\# \) are the antipodes of \( A, B, C \) in the circum-ellipse \( C_I \). Note that \( A_1^\# B_1^\# C_1^\# \) and \( ABC \) are oppositely congruent at \( I \). It follows from Steiner’s porism that if we denote the intersections of \( L \) and this ellipse by \( Q^\# \) and \( R^\# \), then the lines \( P^\# Q^\# \) and \( P^\# R^\# \) are tangent to the incircle at \( Q \) and \( R \). This leads to the following construction of \( P^\# \).

**Construction.** If the tangent to the incircle at \( P \) intersects the ellipse \( G \) at two points, the second tangents from these points to the incircle intersect at \( P^\# \) on \( C_I \).

If the point of tangency \( P \) has coordinates \( \left( \frac{u^2}{s-a} : \frac{v^2}{s-b} : \frac{w^2}{s-c} \right) \), with \( u + v + w = 0 \), then \( P^\# \) is the point \( \left( \frac{a(s-a)}{u} : \frac{b(s-b)}{v} : \frac{c(s-c)}{w} \right) \). In particular, if \( L \) is the common tangent of the incircle and the nine-point circle at the Feuerbach point, which has coordinates \( (s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2 \), then \( P^\# \) is the point \( \left( \frac{u}{u-c} : \frac{v}{v-c} : \frac{w}{w-c} \right) \). This is \( X_{100} \) of \([3, 4]\). It is a point on the circumcircle, lying on the half line joining the Feuerbach point to the centroid of triangle \( ABC \). See [3, Figure 3.12, p.82].

4. The extangents triangle

Consider the external common tangent \( \ell_a \) of the excircles \((I_b) \) and \((I_c) \). Let \( d_{a1}, d_{a2}, d_{a3} \) be the distances from \( A, B, C \) to this line. We have shown that \( \frac{d_{a1}}{d_{a3}} = \frac{a}{b} \). On the other hand, it is clear that \( \frac{d_{a1}}{d_{a2}} = \frac{b}{b+c} \). See Figure 6. It follows that

\[ d_{a1} : d_{a2} : d_{a3} = bc : c(b+c) : b(b+c). \]

By Proposition 2, the barycentric equation of \( \ell_a \) is

\[ bcx + c(b+c)y + b(b+c)z = 0. \]

Similarly, the equations of \( \ell_b \) and \( \ell_c \) are

\[ c(c+a)x + cay + a(c+a)z = 0, \]
\[ b(a+b)x + a(a+b)y + abz = 0. \]

These three external common tangents bound a triangle called the **extangents triangle** in [3]. The vertices are the points\(^3\)

\[ A' = (-a^2s : b(c+a)(s-c) : c(a+b)(s-b)), \]
\[ B' = (a(b+c)(s-c) : -b^2s : c(a+b)(s-a)), \]
\[ C' = (a(b+c)(s-b) : b(c+a)(s-a) : -c^2s). \]

Let \( I'_a \) be the incenter of the reflection of triangle \( ABC \) in \( A \). It is clear that the distances from \( A \) and \( I'_a \) to \( \ell_a \) are respectively \( h_a \) and \( r \). Since \( A \) is the midpoint of \( II'_a \), the distance from \( I \) to \( \ell_a \) is \( 2h_a - r \).

---

\(^3\)The trilinear coordinates of these vertices given in [3, p.162, §6.17] are not correct. The diagonal entries of the matrices should read \( 1 + \cos A \) etc. and \( \frac{-a(\alpha \beta + \gamma)}{(\alpha + \beta + \gamma)(a+b+c)} \) etc. respectively.
Now consider the reflection of $I$ in $O$. We denote this point by $I'$. Since the distances from $I$ and $O$ to $\ell_a$ are respectively $2h_a - r$ and $R + h_a$, it follows that the distance from $I'$ to $\ell_a$ is $2(R + h_a) - (2h_a - r) = 2R + r$. For the same reason, the distances from $I'$ to $\ell_b$ and $\ell_c$ are also $2R + r$. From this we deduce the following interesting facts about the extangents triangle.

**Theorem 5.** The extangent triangle bounded by $\ell_a$, $\ell_b$, $\ell_c$

(1) has incenter $I'$ and inradius $2R + r$;
(2) is perspective with the excentral triangle at $I'$;
(3) is homothetic to the tangential triangle at the internal center of similitude of the circumcircle and the incircle of triangle $ABC$, the ratio of the homothety being $\frac{2R + r}{R}$.

**Proof.** It is enough to locate the homothetic center in (3). This is the point which divides $I'O$ in the ratio $2R + r : -R$, i.e.,

$$\frac{(2R + r)O - R(2O - I)}{R + r} = \frac{r \cdot O + R \cdot I}{R + r},$$

the internal center of similitude of the circumcircle and incircle of triangle $ABC$.\[5\]

**Remarks.** (1) The statement that the extangents triangle has inradius $2R + r$ can also be found in [2, Problem 2.5.4].

(2) Since the excentral triangle has circumcenter $I'$ and circumradius $2R$, it follows that the excenters and the incenters of the reflections of triangle $ABC$ in $A, B, C$ are concyclic. It is well known that since $ABC$ is the orthic triangle of the

---

\[4\]This point appears as $X_{30}$ in [4].

\[5\]This point appears as $X_{55}$ in [4].
excentral triangle, the circumcircle of $ABC$ is the nine-point circle of the excentral triangle.

(3) If the incircle of the extangents triangle touches its sides at $X$, $Y$, $Z$ respectively, then triangle $XYZ$ is homothetic to $ABC$, again at the internal center of similitude of the circumcircle and the incircle.

(4) More generally, the reflections of the traces of a point $P$ in the respective sides of the excentral triangle are points on the sidelines of the extangents triangle. They form a triangle perspective with $ABC$ at the isogonal conjugate of $P$. For example, the reflections of the points of tangency of the excircles (traces of the Nagel point $(s-a : s-b : s-c)$) form a triangle with perspector $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, the external center of similitude of the circumcircle and the incircle.\footnote{These are the reflections of the traces of the Gergonne point in the respective sides of the excentral triangle.}\footnote{This point appears as $X_{56}$ in \cite{Kimberling2003}.}

References


Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Juan Carlos Salazar: Calle Maturín N° C 19, Urb., Mendoza, Puerto Ordaz 8015, Estado Bolívar, Venezuela
E-mail address: caisersal@yahoo.com
Isotomic Inscribed Triangles and Their Residuals

Mario Dalcín

Abstract. We prove some interesting results on inscribed triangles which are isotomic. For examples, we show that the triangles formed by the centroids (respectively orthocenters) of their residuals have equal areas, and those formed by the circumcenters are congruent.

1. Isotomic inscribed triangles

The starting point of this investigation was the interesting observation that if we consider the points of tangency of the sides of a triangle with its incircle and excircles, we have two triangles of equal areas.

Figure 1

In Figure 1, $X, Y, Z$ are the points of tangency of the incircle with the sides $BC$, $CA$, $AB$ of triangle $ABC$, and $X', Y', Z'$ those with the corresponding excircles. In [2], $XYZ$ and $X'Y'Z'$ are called the intouch and extouch triangles of $ABC$ respectively. That these two triangles have equal areas is best explained by the fact that each pair of points $X, X'; Y, Y'; Z, Z'$ are isotomic on their respective sides, i.e.,

$$BX = X'C, \quad CY = Y'A, \quad AZ = Z'B.$$  \hspace{1cm} (1)
We shall say that \( XYZ \) and \( X'Y'Z' \) are isotomic inscribed triangles. The following basic proposition follows from simple calculations with barycentric coordinates.

**Proposition 1.** *Isotomic inscribed triangles have equal areas.*

**Proof.** Let \( X, Y, Z \) be points on the sidelines \( BC, CA, AB \) dividing the sides in the ratios

\[
\]

In terms of barycentric coordinates with respect to \( ABC \), we have

\[
X = (1 - x)B + xC, \quad Y = (1 - y)C + yA, \quad Z = (1 - z)A + zB. \tag{2}
\]

The area of triangle \( XYZ \), in terms of the area \( \triangle \) of \( ABC \), is

\[
\triangle XYZ = \begin{vmatrix} 0 & 1 - x & x \\ y & 0 & 1 - y \\ 1 - z & z & 0 \end{vmatrix} \triangle
= (1 - (x + y + z) + (xy + yz + zx)) \triangle
= (xyz + (1 - x)(1 - y)(1 - z)) \triangle. \tag{3}
\]

See, for example, [4, Proposition 1]. If \( X', Y', Z' \) are points satisfying (1), then

\[
BX' : X'C = 1 - x : x, \quad CY' : Y'A = 1 - y : y, \quad AZ' : Z'B = 1 - z : z, \tag{4}
\]

and

\[
X' = xB + (1 - x)C, \quad Y' = yC + (1 - y)A, \quad Z' = zA + (1 - z)B. \tag{5}
\]

The area of triangle \( X'Y'Z' \) can be obtained from (3) by replacing \( x, y, z \) by \( 1 - x, 1 - y, 1 - z \) respectively. It is clear that this results in the same expression. This completes the proof of the proposition. \( \square \)

**Proposition 2.** *The centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle.*

**Proof.** The expressions in (2) allow one to determine the centroid of triangle \( XYZ \) easily. This is the point

\[
G_{XYZ} = \frac{1}{3}(X + Y + Z) = \frac{(1 + y - z)A + (1 + z - x)B + (1 + x - y)C}{3}. \tag{6}
\]

On the other hand, with the coordinates given in (5), the centroid of triangle \( X'Y'Z' \) is

\[
G_{X'Y'Z'} = \frac{1}{3}(X' + Y' + Z') = \frac{(1 - y + z)A + (1 - z + x)B + (1 - x + y)C}{3}. \tag{7}
\]

It follows easily that

\[
\frac{1}{2}(G_{XYZ} + G_{X'Y'Z'}) = \frac{1}{3}(A + B + C) = G,
\]

the centroid of triangle \( ABC \). \( \square \)
Corollary 3. The intouch and extouch triangles have equal areas, and the midpoint of their centroids is the centroid of triangle $ABC$.

Proof. These follow from the fact that the intouch triangle $XYZ$ and the extouch triangle $X'Y'Z'$ are isotomic, as is clear from the following data, where $a$, $b$, $c$ denote the lengths of the sides $BC$, $CA$, $AB$ of triangle $ABC$, and $s = \frac{1}{2}(a+b+c)$.

\begin{align*}
BX &= X'C = s - b, & BX' &= XC = s - c, \\
CY &= Y'A = s - c, & CY' &= YA = s - a, \\
AZ &= Z'B = s - a, & AZ' &= ZB = s - b.
\end{align*}

$\square$

In fact, we may take $x = \frac{s - b}{a}$, $y = \frac{s - c}{b}$, $z = \frac{s - a}{c}$, and use (3) to obtain

$$\triangle XYZ = \triangle X'Y'Z' = \frac{2(s - a)(s - b)(s - c)}{abc} \triangle.$$

Let $R$ and $r$ denote respectively the circumradius and inradius of triangle $ABC$. Since $\triangle = rs$ and

$$R = \frac{abc}{4\triangle}, \quad r^2 = \frac{(s - a)(s - b)(s - c)}{s},$$

we have

$$\triangle XYZ = \triangle X'Y'Z' = \frac{r}{2R} \cdot \triangle.$$

If we denote by $A_b$ and $A_c$ the points of tangency of the line $BC$ with the $B$- and $C$-excircles, it is easy to see that $A_b$ and $A_c$ are isotomic points on $BC$. In fact,

$$BA_b = A_cC = s, \quad BA_c = A_bC = -(s - a).$$

Similarly, the other points of tangency $B_c, B_a, C_a, C_b$ form pairs of isotomic points on the lines $CA$ and $AB$ respectively. See Figure 1.

Corollary 4. The triangles $A_bB_cC_a$ and $A_cB_aC_b$ have equal areas. The centroids of these triangles are symmetric with respect to the centroid $G$ of triangle $ABC$.

These follow because $A_bB_cC_a$ and $A_cB_aC_b$ are isotomic inscribed triangles. Indeed,

$$BA_b : A_bC = s : -(s - a) = 1 + \frac{s - a}{a} : -\frac{s - a}{a} = CA_c : A_cB,$$

$$CB_c : B_cA = s : -(s - b) = 1 + \frac{s - b}{b} : -\frac{s - b}{b} = AB_a : B_aC,$$

$$AC_a : C_aB = s : -(s - c) = 1 + \frac{s - c}{c} : -\frac{s - c}{c} = BC_b : C_bA.$$

Furthermore, the centroids of the four triangles $XYZ$, $X'Y'Z'$, $A_bB_cC_a$ and $A_cB_aC_b$ form a parallelogram. See Figure 2.
2. Triangles of residual centroids

For an inscribed triangle $XYZ$, we call the triangles $AYZ$, $BZX$, $CXY$ its residuals. From (2, 5), we easily determine the centroids of these triangles.

\[ G_{AYZ} = \frac{1}{3}((2 + y - z)A + zB + (1 - y)C), \]
\[ G_{BZX} = \frac{1}{3}((1 - z)A + (2 + z - x)B + xC), \]
\[ G_{CXY} = \frac{1}{3}(yA + (1 - x)B + (2 + x - y)C). \]

We call these the residual centroids of the inscribed triangle $XYZ$.

The following two propositions are very easily to established, by making the interchanges $(x, y, z) \leftrightarrow (1 - x, 1 - y, 1 - z)$.

**Proposition 5.** The triangles of residual centroids of isotomic inscribed triangles have equal areas.

**Proof.** From the coordinates given above, we obtain the area of the triangle of residual centroids as

\[
\frac{1}{27} \begin{vmatrix}
2 + y - z & z & 1 - y \\
1 - z & 2 + z - x & x \\
y & 1 - x & 2 + x - y
\end{vmatrix} \Delta
\]
\[
= \frac{1}{9}(3 - x - y - z + xy + yz + zx)\Delta
\]
\[
= \frac{1}{9}(2 + xyz + (1 - x)(1 - y)(1 - z))\Delta
\]
By effecting the interchanges \((x, y, z) \leftrightarrow (1 - x, 1 - y, 1 - z)\), we obtain the area of the triangle of residual centroids of the isotomic inscribed triangle \(X'Y'Z'\). This clearly remains unchanged. \(\square\)

**Proposition 6.** Let \(XYZ\) and \(X'Y'Z'\) be isotomic inscribed triangles of \(ABC\). The centroids of the following five triangles are collinear:

- \(G\) of triangle \(ABC\),
- \(G_{XYZ}\) and \(G_{X'Y'Z'}\) of the inscribed triangles,
- \(\tilde{G}\) and \(\tilde{G}'\) of the triangles of their residual centroids.

Furthermore,
\[
G_{XYZ}\tilde{G} : \tilde{G}G : GG' : \tilde{G}'G_{X'Y'Z'} = 1 : 2 : 2 : 1.
\]

**Proof.** The centroid \(\tilde{G}\) is the point
\[
\tilde{G} = \frac{1}{9}((3 + 2y - 2z)A + (3 + 2z - 2x)B + (3 + 2x - 2y)C).
\]

We obtain the centroid \(\tilde{G}'\) by interchanging \((x, y, z) \leftrightarrow (1 - x, 1 - y, 1 - z)\). From these coordinates and those given in (6,7), the collinearity is clear, and it is easy to figure out the ratios of division. \(\square\)

3. Triangles of residual orthocenters

**Proposition 7.** The triangles of residual orthocenters of isotomic inscribed triangles have equal areas.

See Figure 4. This is an immediate corollary of the following proposition (see Figure 5), which in turn is a special case of a more general situation considered in Proposition 8 below.
Proposition 8. An inscribed triangle and its triangle of residual orthocenters have equal areas.

Proposition 9. Given a triangle $ABC$, if pairs of parallel lines $L_{1B}, L_{1C}$ through $B, C$, $L_{2C}, L_{2A}$ through $C, A$, and $L_{3A}, L_{3B}$ through $A, B$ are constructed, and if

$$P_a = L_{2C} \cap L_{3B}, \quad P_b = L_{3A} \cap L_{1C}, \quad P_c = L_{1B} \cap L_{2A},$$

then the triangle $P_aP_bP_c$ has the same area as triangle $ABC$.

Proof: We write $Y = L_{2C} \cap L_{3A}$ and $Z = L_{2A} \cap L_{3B}$. Consider the parallelogram $AZP_aY$ in Figure 6. If the points $B$ and $C$ divide the segments $ZP_a$ and $YP_a$ in the ratios

$$ZB : BP_a = v : 1 - v, \quad YC : CP_a = w : 1 - w,$$

then it is easy to see that

$$\text{Area}(ABC) = \frac{1 + vw}{2} \cdot \text{Area}(AZP_aY). \quad (8)$$

Now, $P_b$ and $P_c$ are points on $AY$ and $AZ$ such that $BP_c$ and $CP_b$ are parallel. If

$$YP_b : P_bA = v' : 1 - v', \quad ZP_c : P_cA = w' : 1 - w',$$
then from the similarity of triangles $BP_c$ and $PbYC$, we have

$$ZB : ZP_c = YP_b : YC.$$  

This means that $v : w' = v' : w$ and $v'w' = vw$. Now, in the same parallelogram $AZP_aY$, we have

$$\text{Area}(P_aP_bP_c) = \frac{1 + v'w'}{2} \cdot \text{Area}(AZP_aY).$$  

From this we conclude that $P_aP_bP_c$ and $ABC$ have equal areas. \hfill $\square$

4. Triangles of residual circumcenters

Consider the circumcircles of the residuals of an inscribed triangle $XYZ$. By Miquel’s theorem, the circles $AYZ$, $BZX$, and $CXY$ have a common point. Furthermore, the centers $O_a$, $O_b$, $O_c$ of these circles form a triangle similar to $ABC$. See, for example, [1, p.134]. We prove the following interesting theorem.

**Theorem 10.** The triangles of residual circumcenters of the isotomic inscribed triangles are congruent.

We prove this theorem by calculations.

**Lemma 11.** Let $X, Y, Z$ be points on $BC$, $CA$, $AB$ such that

$$BX : XC = w : v, \quad CY : YA = u : w, \quad AZ : ZB = v : u.$$

The distance between the circumcenters $O_b$ and $O_c$ is the hypotenuse of a right triangle with one side $\frac{a}{2}$ and another side

$$\frac{(v - w)(u_b + v)(u_c + w)a^2 + (v + w)(w - u_c)(u_b + v)b^2 + (v + w)(w + u_c)(u_b - v)c^2}{8\Delta(u_b + v)(v + w)(w + u_c)} \cdot a.$$  

(9)
Proof. The distance between \(O_b\) and \(O_c\) along the side \(BC\) is clearly \(\frac{a}{2}\). We calculate their distance along the altitude on \(BC\). The circumradius of \(BZX\) is clearly \(R_b = \frac{ZX}{2 \sin B}\). The distance of \(O_b\) above \(BC\) is

\[
R_b \cos BZX = \frac{ZX \cos BZX}{2 \sin B} = \frac{2BZ \cdot ZX \cos BZX}{4BZ \sin B} = \frac{BZ^2 + X^2 - BX^2}{4BZ \sin B}
\]

\[
= \frac{BZ - BX \cos B}{2 \sin B} = \frac{c(BZ - BX \cos B)}{4\Delta} \cdot a
\]

\[
= \frac{c\left(\frac{u_b}{u_b+v}c - \frac{w}{v+w}a \cos B\right)}{4\Delta} \cdot a
\]

\[
= \frac{u_b(v+w)2c^2 - w(u_b + v)(c^2 + a^2 - b^2)}{8\Delta(u_b + v)(v+w)} \cdot a
\]

\[
= \frac{-(u_c + w)v(a^2 - c^2) + (2u_c w + u_c v - vw)b^2}{8\Delta(u_c + w)(v+w)} \cdot a
\]

By making the interchanges \(b \leftrightarrow c, v \leftrightarrow w, \) and \(u_b \leftrightarrow u_c,\) we obtain the distance of \(O_c\) above the same line as

\[
\frac{-(u_c + w)v(a^2 - c^2) + (2u_c w + u_c v - vw)b^2}{8\Delta(u_c + w)(v+w)} \cdot a.
\]

The difference between these two is the expression given in (9) above. \(\Box\)

Consider now the isotomic inscribed triangle \(X'Y'Z'.\) We have

\[
BX' : X'C = v : w,
\]

\[
CY' : Y'A = w : u_c = \frac{vw}{u_c} : v,
\]

\[
AZ' : Z'B = u_b : v = w : \frac{vw}{u_b}.
\]

Let \(O'_b\) and \(O'_c\) be the circumcenters of \(BZX'\) and \(CX'Y'.\) By making the following interchanges

\[
v \leftrightarrow w, \quad u_b \leftrightarrow \frac{vw}{u_b}, \quad u_c \leftrightarrow \frac{vw}{u_c}
\]

in (9), we obtain the distance between \(O'_b\) and \(O'_c\) along the altitude on \(BC\) as

\[
\frac{(w - v)(\frac{vw}{u_b} + w)(\frac{vw}{u_c} + v)a^2 + (v + w)(v - \frac{vw}{u_b})(\frac{vw}{u_c} + w)b^2 + (v + w)(v + \frac{vw}{u_c})(\frac{vw}{u_b} - w)c^2}{8\Delta(u_b + w)(v + w)(v + \frac{vw}{u_c})} \cdot a
\]

\[
= \frac{(w - v)(v + u_b)(w + u_c)a^2 + (v + w)(u_c - w)(v + u_b)b^2 + (v + w)(w + u_c)(v - u_c)c^2}{8\Delta(v + u_b)(v + w)(u_c + w)} \cdot a.
\]

Except for a reversal in sign, this is the same as (9).
From this we easily conclude that the segments $O_bO_c$ and $O'_bO'_c$ are congruent. The same reasoning also yields the congruences of $O_cO_a$, $O'_cO'_a$, and of $O_aO_b$, $O'_aO'_b$. It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. This completes the proof of Theorem 9.

5. Isotomic conjugates

Let $XYZ$ be the cevian triangle of a point $P$, i.e., $X$, $Y$, $Z$ are respectively the intersections of the line pairs $AP$, $BC$; $BP$, $CA$; $CP$, $AB$. By the residual centroids (respectively orthocenters, circumcenters) of $P$, we mean those of its cevian triangle. If we construct points $X'$, $Y'$, $Z'$ satisfying (1), then the lines $AX'$, $BY'$, $CZ'$ intersect at a point $P'$ called the isotomic conjugate of $P$. If the point $P$ has homogeneous barycentric coordinates $(x : y : z)$, then $P'$ has homogeneous barycentric coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. All results in the preceding sections apply to the case when $XYZ$ and $X'Y'Z'$ are the cevian triangles of two isotomic conjugates. In particular, in the case of residual circumcenters in §4 above, if $XYZ$ is the cevian triangle of $P$ with homogeneous barycentric coordinates $(u : v : w)$, then

$$BX : XC = w : v, \quad CY : YA = u : w, \quad AZ : ZB = v : u.$$ 

By putting $u_b = u_c = u$ in (9) we obtain a necessary and sufficient condition for the line $O_bO_c$ to be parallel to $BC$, namely,

$$(v-w)(u+v)(u+w)a^2+(v+w)(w-u)(u+v)b^2+(v+w)(w+u)(u-v)c^2 = 0.$$ 

This can be reorganized into the form

$$(b^2+c^2-a^2)w(a^2-b^2)w(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$ 

This is the equation of the Lucas cubic, consisting of points $P$ for which the line joining $P$ to its isotomic conjugate $P'$ passes through the orthocenter $H$. The symmetry of this equation leads to the following interesting theorem.

**Theorem 12.** The triangle of residual circumcenters of $P$ is homothetic to $ABC$ if and only if $P$ lies on the Lucas cubic.

It is well known that the Lucas cubic is the locus of point $P$ whose cevian triangle is also the pedal triangle of a point $Q$. In this case, the circumcircles of $AYZ$, $BZX$ and $CXY$ intersect at $Q$, and the circumcenters $O_a$, $O_b$, $O_c$ are the midpoints of the segments $AQ$, $BQ$, $CQ$. The triangle $O_aO_bO_c$ is homothetic to $ABC$ at $Q$.

For example, if $P$ is the Gergonne point, then $O_aO_bO_c$ is homothetic to $ABC$ at the incenter $I$. The isotomic conjugate of $P$ is the Nagel point, and $O'_aO'_bO'_c$ is homothetic to $ABC$ at the reflection of $I$ in the circumcenter $O$.

**References**


Mario Dalcín: Caribes 2364, C.P.11.600, Montevideo, Uruguay

E-mail address: filomate@adinet.com.uy
Abstract. We give an easy construction of points $A_a$, $B_a$, $C_a$ on the sides of a triangle $ABC$ such that the figure $M$ path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. We study the configuration consisting of the three figures $M$ of a triangle, and define an interesting mapping of triangle centers associated with such an $M$-configuration.

1. Introduction

Given a triangle $ABC$, we consider points $A_a$ on the line $BC$, $B_a$ on the half line $CA$, and $C_a$ on the half line $BA$ such that $BC_a = C_aA_a = A_aB_a = B_aC$. We shall refer to $BC_aA_aB_aC$ as $M_a$, because it looks like the letter $M$ when triangle $ABC$ is acute-angled. See Figures 1a. Figure 1b illustrates the case when the triangle is obtuse-angled. Similarly, we also have $M_b$ and $M_c$. The three figures $M_a, M_b, M_c$ constitute the $M$-configuration of triangle $ABC$. See Figure 2.

Proposition 1. The lines $AA_a$, $BB_a$, $CC_a$ concur at the point with homogeneous barycentric coordinates

$$\left( \frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C} \right).$$

Proof. Let $l_a$ be the length of $BC_a = C_aA_a = A_aB_a = B_aC$. It is clear that the directed length $BA_a = 2l_a \cos B$ and $A_aC = 2l_a \cos C$, and $BA_a : A_aC = \cos B : \cos C$. For the same reason, $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. It follows by Ceva’s theorem that the lines $AA_a$, $BB_a$, $CC_a$ concur at the point with homogeneous barycentric coordinates given above.\footnote{This point appears in [3] as $X_{92}$.}
Remark. Since \( 2l_a \cos B + 2l_a \cos C = a = 2R \sin A \), where \( R \) is the circumradius of triangle \( ABC \),

\[
l_a = \frac{a}{2(\cos B + \cos C)} = \frac{R \sin A}{\cos B + \cos C} = \frac{R \cos \frac{A}{2}}{\cos \frac{B-C}{2}}. \tag{1}
\]

For later use, we record the absolute barycentric coordinates of \( A_a, B_a, C_a \) in terms of \( l_a \):

\[
A_a = \frac{2l_a}{a} (\cos C \cdot B + \cos B \cdot C),
B_a = \frac{1}{b} (l_a \cdot A + (b - l_a)C),
C_a = \frac{1}{c} (l_a \cdot A + (c - l_a)B). \tag{2}
\]

2. Construction of \( M_a \)

Proposition 2. Let \( A' \) be the intersection of the bisector of angle \( A \) with the circumcircle of triangle \( ABC \).

(a) \( A_a \) is the intersection of \( BC \) with the parallel to \( AA' \) through the orthocenter \( H \).

(b) \( B_a \) (respectively \( C_a \)) is the intersection of \( CA \) (respectively \( BA \)) with the parallel to \( CA' \) (respectively \( BA' \)) through the circumcenter \( O \).

Proof. (a) The line joining \( A_a = (0 : \cos C : \cos B) \) to \( H = \left( \frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C} \right) \) has equation

\[
\begin{vmatrix}
0 & \cos C & \cos B \\
\frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \\
x & y & z
\end{vmatrix} = 0.
\]
This simplifies to
\[-(b - c)x \cos A + a(y \cos B - z \cos C) = 0.\]

It has infinite point
\[
(-a(\cos B + \cos C) : a \cos C - (b - c) \cos A : (b - c) \cos A + a \cos B)
\]
\[= (-a(\cos B + \cos C) : b(1 - \cos A) : c(1 - \cos A)).\]

It is clear that this is the same as the infinite point \(-(b + c) : b : c\), which is on the line joining \(A\) to the incenter.

(b) Let \(M\) be the midpoint of \(BC\), and \(Y, Z\) the pedals of \(B_a, C_a\) on \(BC\). See Figure 4. We have

\[
OM = \frac{a}{2} \cot A = l_a(\cos B + \cos C) \cot A,
\]

\[
C_a Z = l_a \sin B,
\]

\[
MZ = \frac{a}{2} - l_a \cos B = l_a(\cos B + \cos C) - l_a \cos B = l_a \cos C.
\]

From this the acute angle between the line \(C_aO\) and \(BC\) has tangent ratio

\[
\frac{C_a Z - OM}{MZ} = \frac{\sin B - (\cos B + \cos C) \cot A}{\cos C} = \frac{\sin B \sin A - (\cos B + \cos C) \cos A}{\cos C \sin A} = \frac{-\cos(A + B) - \cos C \cos A}{\cos C \sin A} = \frac{\cos C(1 - \cos A)}{\cos C \sin A} = \frac{1 - \cos A \sin A}{\sin A} = \tan \frac{A}{2}.
\]

It follows that \(C_aO\) makes an angle \(\frac{A}{2}\) with the line \(BC\), and is parallel to \(BA'\).

The same reasoning shows that \(B_bO\) is parallel to \(CA'\). \(\square\)
3. Circumcenters in the M-configuration

Note that $\angle B_aA_aC_a = \angle A$. It is clear that the circumcircles of $B_aA_aC_a$ and $B_aAC_a$ are congruent. The circumradius is

$$R_a = \frac{l_a}{2 \sin \left( \frac{\pi}{2} - \frac{A}{2} \right)} = \frac{l_a}{2 \cos \frac{A}{2}} = \frac{R}{2 \cos \frac{B - C}{2}}$$

from (1).

**Proposition 3.** The circumcircle of triangle $AB_aC_a$ contains (i) the circumcenter $O$ of triangle $ABC$, (ii) the orthocenter $H_a$ of triangle $A_aB_aC_a$, and (iii) the midpoint of the arc $BAC$.

**Proof.** (i) is an immediate corollary of Proposition 2(b) above.

(ii) Let $H_a$ be the orthocenter of triangle $A_aB_aC_a$. It is clear that

$$\angle B_aH_aC_a = \pi - \angle B_aA_aC_a = \pi - \angle BAC = \pi - \angle C_aAB_a.$$  

It follows that $H_a$ lies on the circumcircle of $AB_aC_a$. See Figure 5. Since the triangle $A_aB_aC_a$ is isosceles, $B_aH_a = C_aH_a$, and the point $H_a$ lies on the bisector of angle $A$.

(iii) Let $A''$ be the midpoint of the arc $BAC$. By a simple calculation, $\angle AA''O = \pi - \frac{1}{2} |B - C|$. Also, $\angle AC_aO = \frac{\pi}{2} + \frac{1}{2} |B - C|$.  

This shows that $A''$ also lies on the circle $AB_aOC_a$.  

The points $B_a$ and $C_a$ are therefore the intersections of the circle $OAA''$ with the sidelines $AC$ and $AB$. This furnishes another simple construction of the figure $M_a$.

\footnote{This is $C + \frac{1}{2}$ if $C \geq B$ and $B + \frac{1}{2}$ otherwise.}
Remarks. (1) If we take into consideration also the other figures $M_b$ and $M_c$, we have three triangles $AB_aC_a, BC_bA_b, CA_cB_c$ with their circumcircles intersecting at $O$.

(2) We also have three triangles $A_A, B_A, C_A$, $A_B, B_B, C_B$, $A_C, B_C, C_C$ with their orthocenters forming a triangle perspective with $ABC$ at the incenter $I$.

\textbf{Proposition 4.} The circumcenter $O_a$ of triangle $A_aB_aC_a$ is equidistant from $O$ and $H$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Figure 6}
\end{figure}

\textbf{Proof.} Construct the circle through $O$ and $H$ with center $Q$ on the line $BC$. We prove that the midpoint $P$ of the arc $OH$ on the opposite side of $Q$ is the circumcenter $O_a$ of triangle $A_aB_aC_a$. See Figure 6. It will follow that $O_a$ is equidistant from $O$ and $H$. Let $N$ be the midpoint of $OH$. Suppose the line $PQ$ makes an angle $\varphi$ with $BC$. Let $X, Y, M$ be the pedals of $H, N, O$ on the line $BC$.

Since $H, X, Q, N$ are concyclic, and the diameter of the circle containing them is $QH = \frac{NX}{\sin \varphi} = \frac{R}{2\sin \varphi}$. This is the radius of the circle $OPH$.

By symmetry, the circle $OPH$ contains the reflection $H'$ of $H$ in the line $BC$.

\[ \angle HH'P = \frac{1}{2} \angle HQP = \frac{1}{2} \angle HQN = \frac{1}{2} \angle HXN = \frac{1}{2} |B - C|. \]

Therefore, the angle between $HH'$ and $BC$ is $\frac{\pi}{2} - \frac{1}{2} |B - C|$. It is obvious that the angle between $A_aO_a$ and $BC$ is the same. But from Proposition 2(a), the angle between $HA_a$ and $BC$ is the same too, so is the angle between the reflection $HA_a$ and $BC$. From these we conclude that $H', A_a, O_a$ and $P$ are collinear. Now, let $Z$ be the pedal of $P$ on $BC$.

\[ A_aP = \frac{PZ}{\cos \frac{1}{2}(B - C)} = \frac{QP \sin \varphi}{\cos \frac{1}{2}(B - C)} = \frac{R}{2 \cos \frac{1}{2}(B - C)} = R_a. \]

Therefore, $P$ is the circumcenter $O_a$ of triangle $A_aB_aC_a$. \hfill \Box
Applying this to the other two figures $M_b$ and $M_c$, we obtain the following remarkable theorem about the $M$-configuration of triangle $ABC$.

**Theorem 5.** The circumcenters of triangles $A_aB_aC_a$, $A_bB_bC_b$, and $A_cB_cC_c$ are collinear. The line containing them is the perpendicular bisector of the segment $OH$.

\[
\sin 3A \sin A x + \sin 3B \sin B y + \sin 3C \sin C z = 0.
\]

**4. A central mapping**

Let $P$ be a triangle center in the sense of Kimberling [2, 3], given in homogeneous barycentric coordinates \((f(a, b, c) : f(b, c, a) : f(c, a, b))\) where \(f = f_P\) satisfies \(f(a, b, c) = f(a, c, b)\). If the reference triangle $ABC$ is isosceles, say, with $AB = AC$, then $P$ lies on the perpendicular bisector of $BC$ and has coordinates of the form \((g_P : 1 : 1)\). The coordinate $g$ depends only on the shape of the isosceles triangle, and we express it as a function of the base angle. We shall call $g = g_P$ the *isoscelized form* of the triangle center function $f_P$. Let $P^*$ denote the isogonal conjugate of $P$.

**Lemma 6.** $g^*_P(B) = \frac{4 \cos^2 B}{g_P(B)}$.

**Proof.** If $P = (g_P(B) : 1 : 1)$ for an isosceles triangle $ABC$ with $B = C$, then

\[
P^* = \left( \frac{\sin^2 A}{g_P(B)} : \sin^2 B : \sin^2 B \right) = \left( \frac{4 \cos^2 B}{g_P(B)} : 1 : 1 \right)
\]
since \( \sin^2 A = \sin^2(\pi - 2B) = \sin^2 2B = 4 \sin^2 B \cos^2 B \).

Here are some examples.

<table>
<thead>
<tr>
<th>Center</th>
<th>( f_P )</th>
<th>( g_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>centroid</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>incenter</td>
<td>( a )</td>
<td>( 2 \cos B )</td>
</tr>
<tr>
<td>circumcenter</td>
<td>( a^2(b^2 + c^2 - a^2)/c )</td>
<td>(-2 \cos 2B)</td>
</tr>
<tr>
<td>orthocenter</td>
<td>( 1/(b^2 + c^2 - a^2) )</td>
<td>(-2 \cos^2 B/\cos 2B)</td>
</tr>
<tr>
<td>symmedian point</td>
<td>( a^2 )</td>
<td>( 4 \cos^2 B )</td>
</tr>
<tr>
<td>Gergonne point</td>
<td>( 1/s-a )</td>
<td>( \cos B/1-\cos B )</td>
</tr>
<tr>
<td>Nagel point</td>
<td>( s-a )</td>
<td>( 1/\cos B )</td>
</tr>
<tr>
<td>Mittenpunkt</td>
<td>( a(s-a) )</td>
<td>( 2(1-\cos B) )</td>
</tr>
<tr>
<td>Spieker point</td>
<td>( b+c )</td>
<td>( 1+2 \cos B/2 )</td>
</tr>
<tr>
<td>( X_{55} )</td>
<td>( a^2(s-a) )</td>
<td>( 4 \cos B(1-\cos B) )</td>
</tr>
<tr>
<td>( X_{56} )</td>
<td>( a^2/s-a )</td>
<td>( 4 \cos^3 B/1-\cos B )</td>
</tr>
<tr>
<td>( X_{57} )</td>
<td>( a/s-a )</td>
<td>( 2 \cos^4 B/1-\cos B )</td>
</tr>
</tbody>
</table>

Consider a triangle center given by a triangle center function with isocelized form \( g = g_P \). The triangle center of the isosceles triangle \( C_aBA_a \) is the point \( P_{a,b} \) with coordinates \( (g(B) : 1 : 1) \) relative to \( C_aBA_a \). Making use of the absolute barycentric coordinates of \( A_a, B_a, C_a \) given in (2), it is easy to see that this is the point

\[
P_{a,b} = \left( \frac{g(B)l_a}{c} : \frac{g(B)(c - l_a)}{c} + 1 + \frac{2l_a}{a} \cos C : \frac{2l_a}{a} \cos B \right).
\]

The same triangle center of the isosceles triangle \( B_aA_aC \) is the point

\[
P_{a,c} = \left( \frac{g(C)l_a}{b} : \frac{2l_a}{a} \cos C : \frac{g(C)(b - l_a)}{b} + \frac{2l_a}{a} \cos B + 1 \right).
\]

It is clear that the lines \( BP_{a,b} \) and \( CP_{a,c} \) intersect at the point

\[
P_a = \left( \frac{g(B)g(C)l_a^2}{bc} : \frac{2g(B)l_a^2}{ca} \cos C : \frac{2g(C)l_a^2}{ab} \cos B \right)
\]

\[
= (ag(B)g(C) : 2bg(B) \cos C : 2cg(C) \cos B)
\]

\[
= \left( \frac{a(g(B)g(C))}{2 \cos B \cos C} : \frac{b(g(B))}{\cos B} : \frac{c(g(C))}{\cos C} \right).
\]

Figure 8 illustrates the case of the Gergonne point.

In the M-configuration, we may also consider the same triangle center (given in isocelized form \( g_P \) of the triangle center function) in the isosceles triangles. These are the point \( P_{b,c}, P_{b,a}, P_{c,a}, P_{c,b} \). The pairs of lines \( CP_{b,c}, AP_{b,a} \) intersecting at \( P_b \) and \( AP_{c,a}, BP_{c,b} \) intersecting at \( P_c \). The coordinates of \( P_b \) and \( P_c \) can be
written down easily from those of \( P_a \). From these coordinates, we easily conclude that \( P_aP_bP_c \) is perspective with triangle \( ABC \) at the point

\[
\Phi(P) = \left( \frac{ag_P(A)}{\cos A} : \frac{bg_P(B)}{\cos B} : \frac{cg_P(C)}{\cos C} \right)
= \left( g_P(A) \tan A : g_P(B) \tan B : g_P(C) \tan C \right).
\]

**Proposition 7.** \( \Phi(P^*) = \Phi(P)^* \).

**Proof.** We make use of Lemma 6.

\[
\Phi(P^*) = \left( g_{P^*}(A) \tan A : g_{P^*}(B) \tan B : g_{P^*}(C) \tan C \right)
= \left( \frac{4 \cos^2 A}{g_P(A)} \tan A : \frac{4 \cos^2 B}{g_P(B)} \tan B : \frac{4 \cos^2 C}{g_P(C)} \tan C \right)
= \left( \frac{\sin^2 A}{g_P(A) \tan A} : \frac{\sin^2 B}{g_P(B) \tan B} : \frac{\sin^2 C}{g_P(C) \tan C} \right)
= \Phi(P)^*.
\]

\( \square \)

We conclude with some examples.
The M-configuration of a triangle

<table>
<thead>
<tr>
<th>P</th>
<th>(\Phi(P))</th>
<th>(P^*)</th>
<th>(\Phi(P^<em>) = \Phi(P)^</em>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>incenter</td>
<td>incenter</td>
<td>symmedian point</td>
<td>circumcenter</td>
</tr>
<tr>
<td>centroid</td>
<td>orthocenter</td>
<td>incenter</td>
<td>X_{68}</td>
</tr>
<tr>
<td>circumcenter</td>
<td>(X_{24})</td>
<td>orthocenter</td>
<td>(X_{55})</td>
</tr>
<tr>
<td>Gergonne point</td>
<td>Nagel point</td>
<td>(X_{56})</td>
<td>(X_{56})</td>
</tr>
<tr>
<td>Nagel point</td>
<td>(X_{1118})</td>
<td>(X_{56})</td>
<td>(X_{1259} = X_{1118}^*)</td>
</tr>
<tr>
<td>Mittenpunkt</td>
<td>(X_{34})</td>
<td>(X_{57})</td>
<td>(X_{78} = X_{34}^*)</td>
</tr>
</tbody>
</table>

For the Spieker point, we have

\[
\Phi(X_{10}) = \left(\frac{\tan A}{1 + 2\cos A} : \frac{\tan B}{1 + 2\cos B} : \frac{\tan C}{1 + 2\cos C}\right) = \left(\frac{1}{a(b^2 + c^2 - a^2)(b^2 + c^2 - a^2 + bc)} : \cdots : \cdots\right).
\]

This triangle center does not appear in the current edition of [3].

Remark. For \(P = X_8\), the Nagel point, the point \(P_a\) has an alternative description. Antreas P. Hatzipolakis [1] considered the incircle of triangle \(ABC\) touching the sides \(CA\) and \(AB\) at \(Y\) and \(Z\) respectively, and constructed perpendiculars from \(Y, Z\) to \(BC\) intersecting the incircle again at \(Y'\) and \(Z'\). See Figure 9. It happens that \(B, Z', P_{a,b}\) are collinear; so are \(C, Y', P_{a,c}\). Therefore, \(BZ'\) and \(CY'\) intersect at \(P_a\). The coordinates of \(Y'\) and \(Z'\) are

\[
Y' = (a^2(b + c - a)(c + a - b) : (a^2 + b^2 - c^2)^2 : (b + c)^2(a + b - c)(c + a - b)),
\]

\[
Z' = (a^2(b + c - a)(a + b - c) : (b + c)^2(c + a - b)(a + b - c) : (a^2 - b^2 + c^2)^2).
\]
The lines $BZ'$ and $CY'$ intersect at

$$P_a = \left( \frac{a^2(b + c - a)}{c + a - b} : \frac{(a^2 + b^2 - c^2)^2}{a + b - c} : \frac{(a^2 - b^2 + c^2)^2}{a + b - c} \right)$$

$$= \left( \frac{a^2(b + c - a)}{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)^2} : \frac{1}{(c + a - b)(a^2 - b^2 + c^2)^2} : \frac{1}{(a + b - c)(a^2 + b^2 - c^2)^2} \right).$$

It was in this context that Hatzopolakis constructed the triangle center

$$X_{1118} = \left( \frac{1}{(b + c - a)(b^2 + c^2 - a^2)^2} : \cdots : \cdots \right).$$

References


Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex.geom@mtu-net.ru
Rectangles Attached to Sides of a Triangle

Nikolaos Dergiades and Floor van Lamoen

Abstract. We study the figure of a triangle with a rectangle attached to each side. In line with recent publications on special cases we find concurrencies and study homothetic triangles. Special attention is given to the cases in which the attached rectangles are similar, have equal areas and have equal perimeters, respectively.

1. Introduction

In recent publications [3, 4, 10, 11, 12] the configurations have been studied in which rectangles or squares are attached to the sides of a triangle. In these publications the rectangles are all similar. In this paper we study the more general case in which the attached rectangles are not necessarily similar. We consider a triangle \( \triangle ABC \) with attached rectangles \( \triangle BCA_c, \triangle CAB_b \) and \( \triangle ABC_a \). Let \( u \) be the length of \( CA \), positive if \( A \) and \( A_c \) are on opposite sides of \( BC \), otherwise negative. Similarly let \( v \) and \( w \) be the lengths of \( AB \) and \( BC \). We describe the shapes of these rectangles by the ratios

\[
U = \frac{a}{u}, \quad V = \frac{b}{v}, \quad W = \frac{c}{w}.
\]

(1)

The vertices of these rectangles are \(^1\)

\[
A_b = (-a^2 : S_C + SU : S_B), \quad A_c = (-a^2 : S_C : S_B + SU),
\]

\[
B_a = (S_C + SV : -b^2 : S_A), \quad B_c = (S_C : -b^2 : S_A + SV),
\]

\[
C_a = (S_B + SW : S_A : -c^2), \quad C_b = (S_B : S_A + SW : -c^2).
\]

Consider the flank triangles \( \triangle AB_bC_a, \triangle A_bBC_b \) and \( \triangle A_cB_cC \). With the same reasoning as in [10], or by a simple application of Ceva’s theorem, we can see that the triangle \( \triangle H_aH_bH_c \) of orthocenters of the flank triangles is perspective to \( \triangle ABC \) with perspector

\[
P_1 = \left( \frac{a}{u}, \frac{b}{v}, \frac{c}{w} \right) = (U : V : W).
\]

\[\)
See Figure 1. On the other hand, the triangle $O_aO_bO_c$ of circumcenters of the flank triangles is clearly homothetic to $ABC$, the homothetic center being the point

$$P_2 = (au : bv : cw) = \left( \frac{a^2}{U} : \frac{b^2}{V} : \frac{c^2}{W} \right).$$

Clearly, $P_1$ and $P_2$ are isogonal conjugates.

Now the perpendicular bisectors of $B_aC_a$, $A_bC_b$ and $A_cB_c$ pass through $O_a$, $O_b$ and $O_c$ respectively and are parallel to $AP_1$, $BP_1$ and $CP_1$ respectively. This shows that these perpendicular bisectors concur in a point $P_3$ on $P_1P_2$ satisfying

$$P_2P_1 : P_1P_3 = 2S : au + bv + cw,$$

where $S$ is twice the area of $ABC$. See Figure 2. More explicitly,

$$P_3 = (-a^2VW(V + W) + U^2(b^2W + c^2V) + 2SU^2VW \quad : \quad -b^2WU(W + U) + V^2(c^2U + a^2W) + 2SUV^2W)$$

$$: \quad -c^2UV(U + V) + W^2(a^2V + b^2U) + 2SUVW^2)$$

This concurrency generalizes a similar result by Hoehn in [4], and was mentioned by L. Lagrangia [9]. It was also a question in the Bundeswettbewerb Mathematik Deutschland (German National Mathematics Competition) 1996, Second Round.

From the perspectivity of $ABC$ and the orthocenters of the flank triangles, we see that $ABC$ and the triangle $A'B'C'$ enclosed by the lines $B_aC_a$, $A_bC_b$ and $A_cB_c$ are orthologic. This means that the lines from the vertices of $A'B'C'$ to the corresponding sides of $ABC$ are concurrent as well. The point of concurrency is the reflection of $P_1$ in $O$, i.e.,
\[ P_4 = (-S_{BC}U + a^2S_A(V + W) : \cdots : \cdots). \quad (5) \]

**Remark.** We record the coordinates of \( A' \). Those of \( B' \) and \( C' \) can be written down accordingly.

\[
A' = -(a^2S(U + V + W) + (a^2V + S_CU)(a^2W + S_BU))
\]

\[
:S_C S(U + V + W) + (b^2U + S_CV)(a^2W + S_BU)
\]

\[
:S_B S(U + V + W) + (a^2V + S_CU)(c^2U + S_BU)).
\]

2. **Special cases**

We are mainly interested in three special cases.

2.1. *The similarity case.* This is the case when the rectangles are similar, i.e., \( U = V = W = t \) for some \( t \). In this case, \( P_1 = G \), the centroid, and \( P_2 = K \), the symmedian point. As \( t \) varies,

\[
P_3 = (b^2 + c^2 - 2a^2 + 2St : c^2 + a^2 - 2b^2 + 2St : a^2 + b^2 - 2e^2 + 2St)
\]

traverses the line \( GK \). The point \( P_4 \), being the reflection of \( G \) in \( O \), is \( X_{376} \) in [7]. The triangle \( M_AM_BM_C \) is clearly perspective with \( ABC \) at the orthocenter \( H \).

More interestingly, it is also perspective with the medial triangle at

\[
((S_A + St)(a^2 + 2St) : (S_B + St)(b^2 + 2St) : (S_C + St)(c^2 + 2St)),
\]
which is the complement of the Kiepert perspector

\[
\left( \frac{1}{S_A + S_t} : \frac{1}{S_B + S_t} : \frac{1}{S_C + S_t} \right).
\]

It follows that as \( t \) varies, this perspector traverses the Kiepert hyperbola of the medial triangle. See [8].

The case \( t = 1 \) is the Pythagorean case, when the rectangles are squares erected externally. The perspector of \( M_aM_bM_c \) and the medial triangle is the point

\[
O_1 = \left( 2a^4 - 3a^2(b^2 + c^2) + (b^2 - c^2)^2 - 2(b^2 + c^2)S : \cdots : \cdots \right),
\]

which is the center of the circle through the centers of the squares. See Figure 3. This point appears as \( X_{641} \) in [7].

\[\text{Figure 3}\]

2.2. **The equiareal case.** When the rectangles have equal areas \( \frac{T}{2} \), i.e., \((U,V,W) = \left( \frac{2a^2}{T}, \frac{2b^2}{T}, \frac{2c^2}{T} \right) \), it is easy to see that \( P_1 = K \), \( P_2 = G \), and

\[
P_4 = (a^2(-S_{BC} + S_A(b^2 + c^2)) : \cdots : \cdots)
= (a^2(a^4 + 2a^2(b^2 + c^2) - (3b^4 + 2b^2c^2 + 3c^4)) : \cdots : \cdots)
\]

is the reflection of \( K \) in \( O \). \(^2\) The special equiareal case is when \( T = S \), the rectangles having the same area as triangle \( ABC \). See Figure 4. In this case,

\[
P_3 = (6a^2 - b^2 - c^2 : 6b^2 - c^2 - a^2 : 6c^2 - a^2 - b^2).
\]

\(^2\)This point is not in the current edition of [7].
2.3. The isoperimetric case. This is the case when the rectangles have equal perimeters \(2p\), i.e., \((u,v,w) = (p-a, p-b, p-c)\). The special isoperimetric case is when \(p = s\), the semiperimeter, the rectangles having the same perimeter as triangle \(ABC\). In this case, \(P_1 = X_{57}\), \(P_2 = X_9\), the Mittenpunkt, and

\[
P_3 = (a(bc(2a^2 - a(b + c) - (b - c)^2) + 4(s - b)(s - c)S) : \cdots : \cdots),
\]

\[
P_4 = (a(a^6 - 2a^5(b + c) - a^4(b^2 - 10bc + c^2) + 4a^3(b + c)(b^2 - bc + c^2) - a^2(b^4 + 8b^3c - 2b^2c^2 + 8c^3b + c^4) - 2a(b + c)(b - c)(b^2 + c^2) + (b + c)^2(b - c)^4) : \cdots : \cdots).
\]

These points can be described in terms of division ratios as follows.\(^3\)

\[
P_3 X_{57} : X_{57} X_9 = 4R + r : 2s,
\]

\[
P_4 I : IX_{57} = 4R : r.
\]

3. A pair of homothetic triangles

Let \(A_1\), \(B_1\) and \(C_1\) be the centers of the rectangles \(BCA, Ab, CAB, bC\) and \(ABC, Cb, CB\) respectively, and \(A_2B_2C_2\) the triangle bounded by the lines \(B_1C_b, C_bA_c\) and \(A_bB_a\). Since, for instance, segments \(B_1C_b\) and \(B_cC_b\) are homothetic through

\(^3\)These points are not in the current edition of [7].
A, the triangles $A_1 B_1 C_1$ and $A_2 B_2 C_2$ are homothetic. See Figure 5. Their homothetic center is the point

$$P_5 = (-a^2 S_A (V + W) + U (S_B + SW)(S_C + SV) : \cdots : \cdots).$$

For the Pythagorean case with squares attached to triangles, i.e., $U = V = W = 1$, Toshio Seimiya and Peter Woo [12] have proved the beautiful result that the areas $\Delta_1$ and $\Delta_2$ of $A_1 B_1 C_1$ and $A_2 B_2 C_2$ have geometric mean $\Delta$. See Figure 5. We prove a more general result by computation using two fundamental area formulae.

**Proposition 1.** For $i = 1, 2, 3$, let $P_i$ be finite points with homogeneous barycentric coordinates $(x_i : y_i : z_i)$ with respect to triangle $ABC$. The oriented area of the triangle $P_1 P_2 P_3$ is

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)(x_3 + y_3 + z_3)} \cdot \Delta.$$

A proof of this proposition can be found in [1, 2].

**Proposition 2.** For $i = 1, 2, 3$, let $\ell_i$ be a finite line with equation $p_i x + q_i y + r_i z = 0$. The oriented area of the triangle bounded by the three lines $\ell_1, \ell_2, \ell_3$ is

$$\frac{\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}^2}{D_1 \cdot D_2 \cdot D_3} \cdot \Delta.$$
where

\[
D_1 = \begin{vmatrix} 1 & 1 & 1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} p_1 & q_1 & r_1 \\ 1 & 1 & 1 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix}.
\]

A proof of this proposition can be found in [5].

**Theorem 3.** $\frac{\Delta_1 \Delta_2}{\Delta^2} = \frac{(U + V + W - UVW)^2}{4(UVW)^2}.$

**Proof.** The coordinates of $A_1, B_1, C_1$ are

\[
A_1 = (-a^2 : S_C + SU : S_B + SU),
\]

\[
B_1 = (S_C + SV : -b^2 : S_A + SV),
\]

\[
C_1 = (S_B + SW : S_A + SW : -c^2).
\]

By Proposition 1, the area of triangle $A_1B_1C_1$ is

\[
\Delta_1 = \frac{S(U + V + W + UVW) + (a^2VW + b^2WU + c^2UV)}{4SUVW} \cdot \Delta. \tag{6}
\]

The lines $B_C C_b, C_A A_c, A_B B_a$ have equations

\[
(S(1 - VW) - S_A(V + W))x + (S + S_B V)y + (S + S_CW)z = 0,
\]

\[
(S + S_AU)x + (S(1 - WU) - S_B(W + U))y + (S + S_CW)z = 0,
\]

\[
(S + S_AU)x + (S + S_B V)y + (S(1 - UV) - S_C(U + V))z = 0.
\]

By Proposition 2, the area of the triangle bounded by these lines is

\[
\Delta_2 = \frac{S(U + V + W - UVW)^2}{UVW(S(U + V + W + UVW) + (a^2VW + b^2WU + c^2UV))} \cdot \Delta. \tag{7}
\]

From (6, 7), the result follows. \[\square\]

**Remarks.** (1) The ratio of homothety is

\[
\frac{-S(U + V + W - UVW)}{2(S(U + V + W + UVW) + (a^2VW + b^2WU + c^2UV))}.
\]

(2) We record the coordinates of $A_2$ below. Those of $B_2$ and $C_2$ can be written down accordingly.

\[
A_2 = (-a^2((S + S_AU)(V + W) + SU(1 - VW)) + (S_B + SW)(S_C + SV)U^2
\]

\[
: (S + S_AU)(SUV + S_C(U + V + W))
\]

\[
: (S + S_AU)(SWU + S_B(U + V + W)).
\]

From the coordinates of $A_2B_2C_2$ we see that this triangle is perspective to $ABC$ at the point

\[
P_6 = \left(\frac{1}{S_A(U + V + W) + SVW}, \ldots : \ldots \right).
\]
4. Examples

4.1. The similarity case. If the rectangles are similar, $U = V = W = t$, then

$$P_6 = \left( \frac{1}{3S_A + St} : \frac{1}{3S_B + St} : \frac{1}{3S_C + St} \right)$$

traverses the Kiepert hyperbola. In the Pythagorean case, the homothetic center $P_5$ is the point

$$((S_B - S)(S_C - S) - 4S_{BC} : (S_C - S)(S_A - S) - 4S_{CA} : (S_A - S)(S_B - S) - 4S_{AB}).$$

Figure 6

4.2. The equiareal case. For $(U, V, W) = \left( \frac{2a^2}{T}, \frac{2b^2}{T}, \frac{2c^2}{T} \right)$, we have

$$P_6 = \left( \frac{1}{T(a^2 + b^2 + c^2)S_A + 2Sb^2c^2} : \ldots : \ldots \right).$$

This traverses the Jerabek hyperbola as $T$ varies. When the rectangles have the same area as the triangle, the homothetic center $P_5$ is the point

$$a^2((a^2 + 3b^2 + 3c^2)^2 - 4(4b^4 - b^2c^2 + 4c^4)) : \ldots : \ldots).$$

5. More homothetic triangles

Let $C_A, C_B$ and $C_C$ be the circumcircles of rectangles $BCA, Ab, CAB, Bc$ and $ABC, Ac$ respectively. See Figure 7. Since the circle $C_A$ passes through $B$ and $C$, its equation is of the form

$$a^2yz + b^2zx + c^2xy - px(x + y + z) = 0.$$
Since the same circle passes through $A_b$, we have $p = \frac{S_{A^U} + S}{U} = S_A + \frac{S}{U}$. By the same method we derive the equations of the three circles:

\[
\begin{align*}
    a^2yz + b^2zx + c^2xy &= (S_A + \frac{S}{U})x(x + y + z), \\
    a^2yz + b^2zx + c^2xy &= (S_B + \frac{S}{V})y(x + y + z), \\
    a^2yz + b^2zx + c^2xy &= (S_C + \frac{S}{W})z(x + y + z).
\end{align*}
\]

From these, the radical center of the three circles is the point

\[
J = \left( \frac{1}{S_A + \frac{S}{U}} : \frac{1}{S_B + \frac{S}{V}} : \frac{1}{S_C + \frac{S}{W}} \right) = \left( \frac{U}{S_AU + S} : \frac{V}{S_BV + S} : \frac{W}{S_CW + S} \right).
\]

Note that the isogonal conjugate of $J$ is the point

\[
J^* = \left( a^2S_A + S : \frac{a^2}{U} \cdot b^2S_B + S : \frac{b^2}{V} \cdot c^2S_C + S : \frac{c^2}{W} \right).
\]

It lies on the line joining $O$ to $P_2$. In fact,

\[
P_2J^* : J^*O = 2S : au + bv + cw = P_2P_1 : P_1P_3.
\]
The circles \( C_B \) and \( C_C \) meet at \( A \) and a second point \( A_3 \), which is the reflection of \( A \) in \( B_1C_1 \). See Figure 8. In homogeneous barycentric coordinates,

\[
A_3 = \left( \frac{V + W}{S_A(V + W) - S(1 - V W)} : \frac{V}{S_B V + S} : \frac{W}{S_C W + S} \right).
\]

Similarly we have points \( B_3 \) and \( C_3 \). Clearly, the radical center \( J \) is the perspector of \( ABC \) and \( A_3B_3C_3 \).

**Proposition 4.** The triangles \( ABC \) and \( A_2B_2C_2 \) are orthologic. The perpendiculars from the vertices of one triangle to the corresponding lines of the other triangle concur at the point \( J \).

**Proof.** As \( C_1B_1 \) bisects \( AA_3 \), we see \( A_3 \) lies on \( B_C C_b \) and \( AJ \perp B_C C_b \). Similarly, we have \( BJ \perp C_A A_c \) and \( CJ \perp A_B B_a \). The perpendiculars from \( A, B, C \) to the corresponding sides of \( A_2B_2C_2 \) concur at \( J \).

On the other hand, the points \( B, C_3, B_3, C \) are concyclic and \( B_3C_3 \) is antiparallel to \( BC \) with respect to triangle \( JBC \). The quadrilateral \( JB_3A_2C_3 \) is cyclic, with \( JA_2 \) as a diameter. It is known that every perpendicular to \( JA_2 \) is antiparallel to
with respect to triangle $JB_3C_3$. Hence, $A_2J \perp BC$. Similarly, $B_2J \perp CA$ and $C_2J \perp AB$.

It is clear that the perpendiculars from $A_3$, $B_3$, $C_3$ to the corresponding sides of triangle $A_2B_2C_2$ intersect at $J$. Hence, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are orthologic.

**Proposition 5.** The perpendiculars from $A_2$, $B_2$, $C_2$ to the corresponding sides of $A_3B_3C_3$ meet at the reflection of $J$ in the circumcenter $O_3$ of triangle $A_3B_3C_3$.

**Proof.** Since triangle $A_3B_3C_3$ is the pedal triangle of $J$ in $A_2B_2C_2$, and $A_2J$ passes through the circumcenter of triangle $A_2B_3C_3$, the perpendicular from $A_2$ to $B_3C_3$ passes through the orthocenter of $A_2B_3C_3$ and is isogonal to $A_2J$ in triangle $A_2B_2C_2$. This line therefore passes through the isogonal conjugate of $J$ in $A_2B_2C_2$. We denote this point by $J'$. Similarly, the perpendiculars from $B_2$, $C_2$ to the sides $C_3A_3$ and $A_3B_3$ pass through $J'$. The circumcircle of $A_3B_3C_3$ is the pedal circle of $J$. Hence, its circumcenter $O_3$ is the midpoint of $JJ'$. It follows that $J'$ is the reflection of $J$ in $O_3$.

**Remark.** The point $J$ and the circumcenters $O$ and $O_3$ of triangles $ABC$ and $A_3B_3C_3$ are collinear. This is because $|JA \cdot JA_3| = |JB \cdot JB_3| = |JC \cdot JC_3|$, say, $= d^2$, and an inversion in the circle $(J, d)$ transforms $ABC$ into $A_3B_3C_3$ or its reflection in $J$.

**Theorem 6.** The perpendicular bisectors of $B_aC_b$, $C_aA_c$, $A_bB_a$ are concurrent at a point which is the reflection of $J$ in the circumcenter $O_3$ of triangle $A_1B_1C_1$.

**Proof.** Let $M_1$ and $M_a$ be the midpoints of $B_1C_1$ and $B_aC_b$ respectively. Note that $M_1$ is also the midpoint of $AM_a$. Also, let $O_1$ be the circumcenter of $A_1B_1C_1$, and the perpendicular bisector of $B_aC_b$ meet $JO_1$ at $N$. See Figure 8. Consider the trapezium $AM_aNJ$. Since $O_1M_1$ is parallel to $AJ$, we conclude that $O_1$ is the midpoint of $JN$. Similarly the perpendicular bisectors of $C_aA_c$, $A_bB_a$ pass through $N$, which is the reflection of $J$ in $O_1$.

We record the coordinates of $O_1$:

$$((e^2U^2V - a^2VW(V + W) + b^2WU(W + U) + UVW((S_A + 3S_B)UV + (S_A + 3S_C)UW))S$$
$$+ c^2SBU^2V^2 + b^2SCU^2W^2 - a^4V^2W^2$$
$$+ (S^2 + S_{BC})U^2V^2W^2 + 4S^2U^2W^2$$
$$: \cdots : \cdots)$$

In the Pythagorean case, the coordinates of $O_1$ are given in §2.1.

6. More triangles related to the attached rectangles

Write $U = \tan \alpha$, $V = \tan \beta$, and $W = \tan \gamma$ for angles $\alpha$, $\beta$, $\gamma$ in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. The point $A_4$ for which the swing angles $CBA_4$ and $BCA_4$ are $\beta$ and $\gamma$
respectively has coordinates
\[
(-a^2 : S_C + S \cdot \cot \gamma : S_B + S \cdot \cot \beta) = \left( -a^2 : S_C + \frac{S}{W} : S_B + \frac{S}{V} \right).
\]

It is clear that this point lies on the line \( AJ \). See Figure 9. If \( B_4 \) and \( C_4 \) are analogously defined, the triangles \( A_4B_4C_4 \) and \( ABC \) are perspective at \( J \).

Note that \( A_3, B, A_4, C \) are concyclic since \( \angle A_4BC = \beta = \angle ABcV = \angle A_4A_3C \).

Let \( d_1 = B_cC_b, d_2 = C_aA_c, d_3 = A_bB_a, d'_1 = AA_4, d'_2 = BB_4, d'_3 = CC_4 \).

**Proposition 7.** The ratios \( \frac{d_i}{d'_i}, i = 1, 2, 3 \), are independent of triangle \( ABC \). More precisely,
\[
\frac{d_1}{d'_1} = \frac{1}{V} + \frac{1}{W}, \quad \frac{d_2}{d'_2} = \frac{1}{W} + \frac{1}{U}, \quad \frac{d_3}{d'_3} = \frac{1}{U} + \frac{1}{V}.
\]

**Proof.** Since \( AA_4 \perp C_bB_c \), the circumcircle of the cyclic quadrilateral \( A_3BA_4C \) meets \( C_bB_c \) besides \( A_3 \) at the antipode \( A_5 \) of \( A_4 \). See Figure 10. Let \( f, g, h \) denote, for vectors, the compositions of a rotation by \( \frac{\pi}{2} \), and homotheties of ratios
Then
\[ g(AA_4) = g(AC) + g(CA_4) = CB_5 + A_5C = A_5B_5, \]
and \( \frac{A_5B_5}{AA_4} = \frac{1}{V}. \) Similarly, \( h(AA_4) = C_bA_5, \) and \( \frac{C_bA_5}{AA_4} = \frac{1}{W}. \) It follows that
\[ \frac{A_5}{A_4} = \frac{1}{V} + \frac{1}{W}. \]

The coordinates of \( A_5 \) can be seen immediately: Since \( A_4A_5 \) is a diameter of the circle \((A_4BC),\) we see that \( \angle BCA_5 = -\frac{\pi}{2} + \angle BCA_4, \) and
\[ A_5 = (-a^2 : S_C - SW : S_B - SV). \]
Similarly, we have the coordinates of \( B_5 \) and \( C_5. \) From these, it is clear that \( A_5B_5C_5 \) and \( ABC \) are perspective at
\[ P_7 = \left( \frac{1}{S_A - SU} : \frac{1}{S_B - SV} : \frac{1}{S_C - SW} \right) = \left( \frac{1}{\cot A - U} : \frac{1}{\cot B - V} : \frac{1}{\cot C - W} \right). \]

For example, in the similarity case it is obvious from the above proof that the points \( A_5, B_5, C_5 \) are the midpoints of \( B_cC_b, C_aA_c, A_bB_a. \) Clearly in the Pythagorean case, the points \( A_4, B_4, C_4 \) coincide with \( A_1, B_1, C_1 \) respectively.
In this case, $J$ is the Vecten point and from the above proof we have $d_1 = 2d'_1$, $d_2 = 2d'_2$, $d_3 = 2d'_3$ and $P_7 = X_{486}$.

7. Another interesting special case

If $\alpha + \beta + \gamma = \pi$, then $U + V + W = UVW$. From Theorem 3 we conclude that $\Delta_2 = 0$, and the points $A_2$, $B_2$, $C_2$, $A_3$, $B_3$, $C_3$ coincide with $J$, which now is the common point of the circumcircles of the three rectangles. Also, the points $A_4$, $B_4$, $C_4$ lie on the circles $C_A$, $C_B$, $C_C$ respectively.

In Figure 11 we illustrate the case $\alpha = \beta = \gamma = \frac{\pi}{3}$. In this case, $J$ is the Fermat point. The triangles $BCA_4$, $CAB_4$, $ABC_4$ are the Fermat equilateral triangles, and the angles of the lines $AA_4$, $BB_4$, $CC_4$, $B_cC_b$, $C_aA_c$, $A_bB_a$ around $J$ are $\frac{\pi}{6}$. The points $A_5$, $B_5$, $C_5$ are the mid points of $B_cC_b$, $C_aA_c$, $A_bB_a$. Also, $d'_1 = d'_2 = d'_3$, and $d_1 = d_2 = d_3 = \frac{2\sqrt{3}}{3}d'_1$. In this case, $P_7$ is the second Napoleon point, the point $X_{18}$ in [7].
Rectangles attached to the sides of a triangle

References


Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
*E-mail address*: ndergiades@yahoo.gr

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
*E-mail address*: f.v.lamoen@wxs.nl
A Generalization of the Lemoine Point

Charles Thas

Abstract. It is known that the Lemoine point $K$ of a triangle in the Euclidean plane is the point of the plane where the sum of the squares of the distances $d_1$, $d_2$, and $d_3$ to the sides of the triangle takes its minimal value. There are several ways to generalize the Lemoine point. First, we can consider $n \geq 3$ lines $u_1, \ldots, u_n$ instead of three in the Euclidean plane and search for the point which minimalizes the expression $d_1^2 + \cdots + d_n^2$, where $d_i$ is the distance to the line $u_i$, $i = 1, \ldots, n$. Second, we can work in the Euclidean $m$-space $R^m$ and consider $n$ hyperplanes in $R^m$ with $n \geq m + 1$. In this paper a combination of these two generalizations is presented.

1. Introduction

Let us start with a triangle $A_1A_2A_3$ in the Euclidean plane $R^2$ and suppose that its sides $a_1 = A_2A_3$, $a_2 = A_3A_1$, and $a_3 = A_1A_2$ have length $l_1$, $l_2$, and $l_3$, respectively. The easiest way to deal with the Lemoine point $K$ of the triangle is to work with trilinear coordinates with regard to $A_1A_2A_3$ (also called normal coordinates). See [1, 5, 6]. These are homogeneous projective coordinates $(x_1, x_2, x_3)$ such that $A_1$, $A_2$, $A_3$, and the incenter $I$ of the triangle, have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, respectively. If $(a_1^i, a_2^i)$ are the non-homogeneous coordinates $(x, y)$ of the point $A_i$ with respect to an orthonormal coordinate system in $R^2$, $i = 1, 2, 3$, then the relationship between homogeneous cartesian coordinates $(x, y, z)$ and trilinear coordinates $(x_1, x_2, x_3)$ is given by

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
l_1a_1^1 & l_2a_1^2 & l_3a_1^3 \\
l_1a_2^1 & l_2a_2^2 & l_3a_2^3 \\
l_1 & l_2 & l_3
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
$$

This follows from the fact that the position vector of the incenter $I$ of $A_1A_2A_3$ is given by

$$
\vec{r} = \frac{l_1\vec{r}_1 + l_2\vec{r}_2 + l_3\vec{r}_3}{l_1 + l_2 + l_3},
$$

with $\vec{r}_i$ the position vector of $A_i$. Remark also that $z = 0$ corresponds with $l_1x_1 + l_2x_2 + l_3x_3 = 0$, which is the equation in trilinear coordinates of the line at infinity.
of $R^2$. If $(x_1, x_2, x_3)$ are normal coordinates of any point $P$ of $R^2$ with regard to $A_1A_2A_3$, then the so-called absolute normal coordinates of $P$ are

$$(d_1, d_2, d_3) = \left( \frac{2Fx_1}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_2}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_3}{l_1x_1 + l_2x_2 + l_3x_3} \right),$$

where $F$ is the area of $A_1A_2A_3$. It is well known that $d_i$ is the relative distance from $P$ to the side $a_i$ of the triangle ($d_i$ is positive or negative, according as $P$ lies at the same side or opposite side as $A_i$, with regard to $a_i$).

Next, consider the locus of the points of $R^2$ for which $d_1^2 + d_2^2 + d_3^2 = k$, with $k$ a given value. In trilinear coordinates this locus is given by

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - k(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \quad (1)$$

For variable $k$, we get a pencil of homothetic ellipses (they all have the same points at infinity, the same asymptotes, the same center and the same axes), and the center of these ellipses is the Lemoine point $K$ of the triangle $A_1A_2A_3$. A straightforward calculation gives that $(l_1, l_2, l_3)$ are trilinear coordinates of $K$ and the minimal value of $d_1^2 + d_2^2 + d_3^2$ reached at $K$ is $\frac{4F^2}{l_1^2 + l_2^2 + l_3^2}$.

Remark also that $K$ is the singular point of the degenerate ellipse of the pencil $(1)$ corresponding with $k = \frac{1}{l_1^2 + l_2^2 + l_3^2}$ (set $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$).

More properties and constructions of the Lemoine point $K$ can be found in [1]. And in [3] and [7] constructions for the axes of the ellipses $(1)$ are given, while [7] contains a lot of generalizations.

Next, the foregoing can immediately be generalized to higher dimensions as follows. Consider in the Euclidean $m$-space $R^m$ ($m \geq 2$), $m + 1$ hyperplanes not through a point and no two parallel; this determines an $m$-simplex with vertices $A_1, \ldots, A_{m+1}$. Let us denote the $(m - 1)$-dimensional volume of the “face” $a_i$ with vertices $A_1, \ldots, A_i, \ldots, A_{m+1}$ by $F_i$, $i = 1, \ldots, m + 1$. Then the position vector of the incenter $I$ of $A_1A_2 \ldots A_{m+1}$ (center of the hypersphere of $R^m$ inscribed in $A_1 \ldots A_{m+1}$) is given by

$$\vec{r} = \frac{F_1\vec{r}_1 + F_2\vec{r}_2 + \cdots + F_{m+1}\vec{r}_{m+1}}{F_1 + F_2 + \cdots + F_{m+1}},$$

where $\vec{r}_i$ is the position vector of $A_i$, and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1, \ldots, A_{m+1}$ are homogeneous projective coordinates such that $A_i, \ldots, A_{m+1}$, and $I$, have coordinates $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, and $(1, 1, \ldots, 1)$, respectively. If $(a_1^1, a_2^1, \ldots, a_i^1)$ are cartesian coordinates (with respect to an orthonormal coordinate system) of $A_i$, $i = 1, \ldots, m + 1$, the coordinate transformation between homogeneous cartesian coordinates $(a_1, \ldots, z_{m+1})$ and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots A_{m+1}$ is given by
and

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_m \\
  z_{m+1}
\end{pmatrix} =
\begin{pmatrix}
  F_1a_1 \cdot \partial F_1a_1 \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  F_1a_2 \cdot \partial F_1a_2 \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  \vdots \\
  F_1a_m \cdot \partial F_1a_m \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  F_1 \cdot \partial F_1 \cdot \partial \cdots \cdot \partial F_m+1
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m \\
  x_{m+1}
\end{pmatrix}.
\]

In normal coordinates the hyperplane at infinity of \( R^m \) has the equation \( F_1x_1 + \cdots + F_{m+1}x_{m+1} = 0 \). Absolute normal coordinates of a point \( P \) of \( R^m \) with respect to \( A_1, A_2, \ldots, A_{m+1} \) are \( d_i = F_1x_1 + \cdots + F_{m+1}x_{m+1} \), \( i = 1, \ldots, m+1 \), where \( F \) is the \( m \)-dimensional volume of \( A_1A_2 \cdots A_{m+1} \) and \( d_i \) is the relative distance from \( P \) to the face \( a_i \) \( (d_i \) is positive or negative, according as \( P \) lies at the same side or at the opposite face as \( A_i \), with regard to \( a_i \)). Remark that \( F_1d_1 + \cdots + F_{m+1}d_{m+1} = mF \).

The locus of the points of \( R^m \) for which \( d_1^2 + \cdots + d_{m+1}^2 = k \) now determines a pencil of hyperquadrics (hyperellipsoids) with equation

\[
x_1^2 + x_2^2 + \cdots + x_{m+1}^2 - k(F_1x_1 + \cdots + F_{m+1}x_{m+1})^2 = 0 \tag{2}
\]

and all these (homothetic) hyperellipsoids have the same axes, the same points at infinity and the same center \( K \), which we call the Lemoine point of \( A_1 \cdots A_{m+1} \) and which obviously has normal coordinates \( (F_1, F_2, \ldots, F_{m+1}) \). The minimal value of \( d_1^2 + \cdots + d_{m+1}^2 \), reached at \( K \) is given by \( \frac{m^2F^2}{F_1^2 + \cdots + F_{m+1}^2} \). Remark that \( K \) is the singular point of the singular hyperquadric (hypercone) corresponding in the pencil (2) with the value \( k = \frac{1}{F_1^2 + \cdots + F_{m+1}^2} \).

**Remark.** Some characterizations and constructions of the Lemoine point \( K \) of a triangle in the plane \( R^2 \) are no longer valid in higher dimensions. For instance, \( K \) is the perspective center of the triangle \( A_1A_2A_3 \) and the triangle \( A_1'A_2'A_3' \) whose sides are the tangents of the circumscribed circle of \( A_1A_2A_3 \) at \( A_1, A_2, \) and \( A_3 \) (in trilinear coordinates the circumcircle has equation \( l_1x_2x_3 + l_2x_3x_1 + l_3x_1x_2 = 0 \)). This construction is, in general, not correct in \( R^3 \) : a tetrahedron \( A_1A_2A_3A_4 \) and its so called tangential tetrahedron, which is the tetrahedron \( A_1A_2A_3A_4' \) consisting of the tangent planes of the circumscribed sphere of \( A_1A_2A_3A_4 \) at \( A_1, A_2, A_3, \) and \( A_4 \), are, in general, not perspective. If they are perspective, the tetrahedron is a special one, an *isodynamic* tetrahedron in which the three products of the three pairs of opposite edges are equal. The lines joining the vertices of an isodynamic tetrahedron to the Lemoine points of the respective opposite faces have a point in common and this common point is the perspective center of the isodynamic tetrahedron and its tangential tetrahedron (see [2]). It is not difficult to prove that this point of an isodynamic tetrahedron coincides with the Lemoine point \( K \) of the tetrahedron obtained with our definition of “Lemoine point”.

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_m \\
  z_{m+1}
\end{pmatrix} =
\begin{pmatrix}
  F_1a_1 \cdot \partial F_1a_1 \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  F_1a_2 \cdot \partial F_1a_2 \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  \vdots \\
  F_1a_m \cdot \partial F_1a_m \cdot \partial \cdots \cdot \partial F_m+1 \cdot a_m+1 \\
  F_1 \cdot \partial F_1 \cdot \partial \cdots \cdot \partial F_m+1
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m \\
  x_{m+1}
\end{pmatrix}.
\]
2. The main theorem

First we give some notations. Consider \( n \) hyperplanes, denoted by \( u_1, \ldots, u_n \) in the Euclidean space \( \mathbb{R}^m \) \((m \geq 2, n \geq m + 1)\), in general position (this means: no two are parallel and no \( m + 1 \) are concurrent). The “figure” consisting of these \( n \) hyperplanes is called an \( n \)-hyperplane (examples: for \( m = 2, n = 3 \) it determines a triangle in \( \mathbb{R}^2 \), for \( m = 2, n = 4 \) it is a quadrilateral in \( \mathbb{R}^2 \), and for \( m = 3, n = 4 \) it is a tetrahedron in \( \mathbb{R}^3 \)). The Lemoine point \( K \) of this \( n \)-hyperplane is, by definition, the point of \( \mathbb{R}^m \) for which the sum of the squares of the distances to the \( n \) hyperplanes \( u_1, \ldots, u_n \) is minimal. The uniqueness of \( K \) follows from the proof of the next theorem.

Next, \( K^i \) is the Lemoine point of the \((n - 1)\)-hyperplane \( u_1u_2 \ldots \hat{u}_i \ldots u_n \), \( i = 1, \ldots, n \). And \( K^{rs} = K^{sr} \) is the Lemoine point of the \((n - 2)\)-hyperplane \( u_1u_2 \ldots \hat{u}_r \ldots \hat{u}_s \ldots u_n \), with \( r, s = 1, \ldots, n, r \neq s \) (only defined if \( n > m + 1 \)).

Now, for an \((m + 1)\)-hyperplane or \( m \)-simplex in \( \mathbb{R}^m \) (a triangle in \( \mathbb{R}^2 \), a tetrahedron in \( \mathbb{R}^3 \), . . . ) we know the position (the normal coordinates) of the Lemoine point (see §1). The following theorem gives us a construction for the Lemoine point \( K \) of a general \( n \)-hyperplane in \( \mathbb{R}^m \) \((m \geq 2 \) and \( n > m + 1 \));

**Theorem 1.** Working with an \( n \)-hyperplane in \( \mathbb{R}^m \), we have, with the notations given above that \( K^i K \cap u_j = K^j K^j \cap u_j, i, j = 1, \ldots, n \) and \( n > m + 1 \).

**Proof.** In this proof, we work with cartesian coordinates \((x_1, \ldots, x_m)\) or homogeneous \((x_1, \ldots, x_{m+1})\) with respect to an orthonormal coordinate system in \( \mathbb{R}^m \). Suppose that the hyperplane \( u_r \) has equation \( a^1_rx_1 + a^2_rx_2 + \cdots + a^m_rx_m + a^{m+1}_r = 0 \), with \( (a^1_r)^2 + (a^2_r)^2 + \cdots + (a^m_r)^2 = 1 \), \( r = 1, \ldots, n \). Then the Lemoine point \( K \) of the \( n \)-hyperplane \( u_1u_2 \ldots u_n \) is the center of the hyperquadrics of the pencil with equation

\[
\mathcal{F}(x_1, \ldots, x_m) = \sum_{r=1}^{n} (a^1_rx_1 + a^2_rx_2 + \cdots + a^m_rx_m + a^{m+1}_r)^2 - k = 0, \quad (3)
\]

where \( k \) is a parameter. Indeed, since the coordinates of \( K \) minimize the expression \( \sum_{r=1}^{n} (a^1_rx_r + \cdots + a^{m+1}_r)^2 \), they are a (the) solution of \( \frac{\partial \mathcal{F}}{\partial x_1} = \frac{\partial \mathcal{F}}{\partial x_2} = \cdots = \frac{\partial \mathcal{F}}{\partial x_m} = 0 \). In homogeneous coordinates, (3) becomes

\[
\mathcal{F}(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n} (a^1_rx_1 + \cdots + a^{m+1}_rx_{m+1})^2 - kx_{m+1}^2 = 0. \quad (4)
\]

Next, the Lemoine point \( K^i \) of \( u_1u_2 \ldots \hat{u}_i \ldots u_n \) is the center of the hyperquadrics of the pencil given by (we use the same notation \( k \) for the parameter)

\[
\mathcal{F}^i(x_1, \ldots, x_{m+1}) = \sum_{r \neq i}^{n} (a^1(rx_1 + \cdots + a^{m+1}_rx_{m+1})^2 - kx_{m+1}^2 = 0. \quad (5)
\]

The diameter of the hyperquadrics (5), conjugate with respect to the direction of the \( i \)th hyperplane \( u_i \) has the equations (consider the polar hyperplanes of the
$m - 1$ points at infinity with coordinates $(a_i^2, -a_i^1, 0, \ldots, 0)$, $(a_i^3, 0, -a_i^1, 0, \ldots, 0)$, $(a_i^4, 0, 0, -a_i^1, 0, \ldots, 0), \ldots, (a_i^m, 0, \ldots, 0, -a_i^1, 0)$ of the hyperplane $u_i$):

$$
\begin{align*}
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})(a_i^{1}a_i^{2} - a_r^{2}a_i^{1}) = 0, \\
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})(a_i^{1}a_i^{3} - a_r^{3}a_i^{1}) = 0, \\
&\quad \vdots \\
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})(a_i^{1}a_i^{m} - a_r^{m}a_i^{1}) = 0.
\end{align*}
$$

But the first side of each of these equations becomes zero for $r = i$, and thus (6) gives us also the conjugate diameter with respect to the direction of $u_i$ of the hyperquadrics of the pencil (5). It follows that (6) determines the line $KK_i$.

Next, the Lemoine point $K^j_i$ is the center of the hyperquadrics of the pencil

$$
F^j(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})^2 - kx_{m+1}^2 = 0,
$$

and $K^{ji}$ is the center of the hyperquadrics:

$$
F^{ji}(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})^2 - kx_{m+1}^2 = 0.
$$

The diameter of the hyperquadrics (7), conjugate with respect to the direction of $u_i$ is given by

$$
\begin{align*}
&\sum_{r\neq j}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})(a_i^{1}a_i^{2} - a_r^{2}a_i^{1}) = 0 \\
&\quad \vdots \\
&\sum_{r\neq j}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1})(a_i^{1}a_i^{m} - a_r^{m}a_i^{1}) = 0.
\end{align*}
$$

And this gives us also the diameter of the hyperquadrics (8) conjugate with regard to the direction of $u_j$; in other words, (9) determines the line $K^j_iK^{ji}$.

Finally, the coordinates of the point $K^j_iK^{ji} \cap u_j$ are the solutions of the linear system

$$
\begin{align*}
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1}) = 0, \\
&\quad \vdots \\
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1}) = 0.
\end{align*}
$$

while the point $K^j_iK^{ji} \cap u_j$ is given by

$$
\begin{align*}
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1}) = 0, \\
&\quad \vdots \\
&\sum_{r=1}^{n}(a_i^{1}x_1 + \cdots + a_r^{m+1}x_{m+1}) = 0.
\end{align*}
$$

It is obvious that this gives the same point and the proof is complete. \hfill \Box

3. Applications

3.1. Let us first consider the easiest example for trying out our construction: the case where $m = 2$ and $n = 4$, or four lines $u_1, u_2, u_3, u_4$ in general position (they form a quadrilateral) in $R^2$. Using orthonormal coordinates $(x, y, z)$ in $R^2$, the homogeneous equation of $u_r$ is $a_r^2x + b_r^1y + c_r^1z = 0$ with $a_r^2 + b_r^2 = 1$, \.
$r = 1, 2, 3, 4$. Where lies the Lemoine point $K$ of the quadrilateral $u_1u_2u_3u_4$?

For instance $K^1$ is the Lemoine point of the triangle with sides (lines) $u_2, u_3, u_4$; $K^2$ of the triangle with sides $u_1, u_3, u_4$, and so on ... We may assume that we can construct the Lemoine point of a triangle. But which point is, for instance, the point $K^{12}$: it is the Lemoine point of the 2-side $u_3u_4$, i.e., it is the point $u_3 \cap u_4$.

Let us denote the six vertices of the quadriliteral as follows: $u_k \cap u_l = C$, $u_2 \cap u_3 = A$, $u_3 \cap u_4 = F$, $u_1 \cap u_4 = D$, $u_2 \cap u_4 = E$, and $u_1 \cap u_3 = B$, then $K^{12} = K^{21} = F$, $K^{23} = D$, $K^{34} = C$, $K^{14} = A$, $K^{24} = B$, and $K^{13} = E$. Now, from $K^3K \cap u_j = K^3K^{j1} \cap u_j$, we find, for instance for $i = 1$ and $j = 2$:

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 = K^2F \cap u_2$$

and for $i = 2$ and $j = 1$: $K^2K \cap u_1 = K^1K^{12} \cap u_1 = K^1F \cap u_1$, with $K^1$ ($K^2$, resp.) the Lemoine point of the triangle AFE (of the triangle BFD, resp.). This allows us to construct the point $K$.

In particular, we can construct the diameters $KK^1, KK^2, KK^3$, and $KK^4$ of the ellipses of the pencil $\sum_{r=1}^{4}(a_rx + b_ry + c_rz)^2 = kz^2$, which are conjugate to the directions of the lines $u_1, u_2, u_3$, and $u_4$, respectively. In other words, we have four pairs of conjugate diameters of these ellipses: $(KK^1, KI^1_{\infty})$, where $I^1_{\infty}$ is the point at infinity of the line $u_i, i = 1, \ldots, 4$. From this, we can construct the axes of the conics of this bundle (in fact, two pairs of conjugate diameters are sufficient): consider any circle $C$ through $K$ and project the involution of conjugate diameters onto $C$; if $S$ is the center of this involution on $C$ and if the diameter of $C$ through $S$ intersects $C$ at the points $S_1$ and $S_2$, then $KS_1$ and $KS_2$ are the axes.

In the case of a triangle in $R^2$, constructions of the common axes of the ellipses determined by $d_1^2 + d_2^2 + d_3^2 = k$ with center the Lemoine point of the triangle, are given in [3] and [7]. In [3], J. Bilo proved that the axes are the perpendicular lines through $K$ on the Simson lines of the common points of the Euler line and the circumscribed circle of the triangle. And in [7], we proved that these axes are the orthogonal lines through $K$ which cut the sides of the triangle in pairs of points whose midpoints are three collinear points. Moreover [7] contains a lot of generalizations for pencils whose conics have any point $P$ of the plane as common center and whose common axes are constructed in the same way.

3.2. In the case $m = 2$ and $n \geq 4$, we can construct the $n$ diameters $KK^1, \ldots, KK^n$ of the ellipses $\sum_{r=1}^{n}(a_rx + b_ry + c_rz)^2 = kz^2$ which are conjugate to the directions of the $n$ lines $u_1, \ldots, u_n$.

3.3. The easiest example in space is the case where $m = 3$ and $n = 5$, or five planes in $R^3$. Assume that the planes have equations $a_rx + b_ry + c_rz + d_ru = 0$, with $a_r^2 + b_r^2 + c_r^2 = 1, r = 1, 2, \ldots, 5$. We look for the Lemoine point $K$ of the “5-plane” $u_1u_2u_3u_4u_5$ in $R^3$ and assume that we know the position of the Lemoine point of any tetrahedron in $R^3$ (we know its normal coordinates). The points $K^1$, \ldots, $K^5$ are the Lemoine points of the tetrahedra $u_2u_3u_4u_5, \ldots, u_1u_2u_3u_4$, respectively. And, for instance $K^{12}$ is the Lemoine point of the “3-plane” $u_3u_4u_5$, i.e., it
A generalization of the Lemoine point is the common point of these three planes $u_3, u_4,$ and $u_5$. Now, for instance from

$$K^1 K \cap u_2 = K^2 K^21 \cap u_2 \quad \text{and} \quad K^2 K \cap u_1 = K^1 K^{12} \cap u_1,$$

we can construct the lines $K^1 K$ and $K^2 K$, and thus the point $K$. In fact, we can construct the diameters $KK^1, \ldots, KK^5$ conjugate to the plane directions of $u_1, \ldots, u_5$, respectively, of the quadrics with center $K$ of the pencil given by $d_1^2 + \cdots + d_5^2 = k$ or

$$\sum_{r=1}^{5} (a_r x + b_r y + c_r z + d_r u)^2 = k u^2.$$

Finally, the construction of the point $K$ in the general case $n > m + 1, m \geq 2$ is obvious.

References


Charles Thas: Department of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, B-9000 Gent, Belgium

E-mail address: charles.thas@UGent.be
The Parasix Configuration and Orthocorrespondence

Bernard Gibert and Floor van Lamoen

Abstract. We introduce the parasix configuration, which consists of two congruent triangles. The conditions of these triangles to be orthologic with $ABC$ or a circumcevian triangle, to form a cyclic hexagon, to be equilateral or to be degenerate reveal a relation with orthocorrespondence, as defined in [1].

1. The parasix configuration

Consider a triangle $ABC$ of reference with finite points $P$ and $Q$ not on its sidelines. Clark Kimberling [2, §§9.7,8] has drawn attention to configurations defined by six triangles. As an example of such configurations we may create six triangles using the lines $\ell_a$, $\ell_b$ and $\ell_c$ through $Q$ parallel to sides $a$, $b$ and $c$ respectively. The triples of lines $(\ell_a, b, c)$, $(a, \ell_b, c)$ and $(a, b, \ell_c)$ bound three triangles which we refer to as the great paratriple. Figure 1a shows the $A$-triangle of the great paratriple. On the other hand, the triples $(a, \ell_b, \ell_c)$, $(\ell_a, b, \ell_c)$ and $(\ell_a, \ell_b, c)$ bound three triangles which we refer to as the small paratriple. See Figure 1b.

Clearly these six triangles are all homothetic to $ABC$, and it is very easy to find the homothetic images of $P$ in these triangles, $A_g$ in the $A$-triangle bounded by $(\ell_a, b, c)$ in the great paratriple, and $A_s$ in the $A$-triangle bounded by $(a, \ell_b, \ell_c)$ in the small paratriple; similarly for $B_g, C_g, B_s, C_s$. These six points form the parasix configuration of $P$ with respect to $Q$, or shortly Parasix($P, Q$). See Figure 2. If in homogeneous barycentric coordinates with reference to $ABC$, $P = (u : v : w)$ and $Q = (f : g : h)$, then these are the points

---

Figure 2. Parasix\((P,Q)\)

\[
A_g = (u(f + g + h) + f(v + w) : v(g + h) : w(g + h)),
\]
\[
B_g = (u(f + h) : g(u + w) + v(f + g + h) : w(f + h)),
\]
\[
C_g = (u(f + g) : v(f + g) : h(u + v) + w(f + g + h));
\]
\[
A_s = (uf : g(u + w) + v(f + g) : h(u + v) + w(f + h)),
\]
\[
B_s = (u(f + g) + f(v + w) : vg : h(u + v) + w(g + h)),
\]
\[
C_s = (u(f + h) + f(v + w) : g(u + w) + v(g + h) : wh).
\]

**Proposition 1.**

1. Triangles \(A_gB_gC_g\) and \(A_sB_sC_s\) are symmetric about the midpoint of segment \(PQ\).
2. The six points of a parasix configuration lie on a central conic.
3. The centroids of triangles \(A_gB_gC_g\) and \(A_sB_sC_s\) trisect the segment \(PQ\).

**Proof.** It is clear from the coordinates given above that the segments \(A_gA_s, B_gB_s, C_gC_s\), \(PQ\) have a common midpoint
\[
(f(u + v + w) + u(f + g + h) : \cdots : \cdots).
\]
The six points therefore lie on a conic with this common midpoint as center. For (3), it is enough to note that the centroids \(G_g\) and \(G_s\) of \(A_gB_gC_g\) and \(A_sB_sC_s\) are the points
\[
G_g = (2u(f + g + h) + f(u + v + w) : \cdots : \cdots),
\]
\[
G_s = (u(f + g + h) + 2f(u + v + w) : \cdots : \cdots).
\]
It follows that vectors \(\overrightarrow{PG_g} = \frac{1}{3} \overrightarrow{PQ}\) and \(\overrightarrow{PG_s} = \frac{2}{3} \overrightarrow{PQ}\). □
While Parasix\((P, Q)\) consists of the two triangles \(A_gB_gC_g\) and \(A_sB_sC_s\), we write \(\tilde{A}_g\tilde{B}_g\tilde{C}_g\) and \(\tilde{A}_s\tilde{B}_s\tilde{C}_s\) for the two corresponding triangles of Parasix\((Q, P)\). From (1) we easily derive their coordinates by interchanging the roles of \(f, g, h,\) and \(u, v, w\). Note that \(\tilde{G}_s = G_g\) and \(\tilde{G}_g = G_s\).

Let \(P_A\) and \(Q_A\) be the the points where \(AP\) and \(AQ\) meet \(BC\) respectively, and let \(AP : PP_A = t_P : 1 - t_P\) while \(AQ : QQ_A = t_Q : 1 - t_Q\). Then it is easy to see that

\[
AA_g : A_gP_A = \tilde{A}_gQ_A = t_pt_Q : 1 - t_pt_Q
\]

so that the line \(A_g\tilde{A}_g\) is parallel to \(BC\). By Proposition 1, \(A_s\tilde{A}_s\) is also parallel to \(BC\).

**Proposition 2.** (a) The lines \(A_g\tilde{A}_g, B_g\tilde{B}_g\) and \(C_g\tilde{C}_g\) bound a triangle homothetic to \(ABC\). The center of homothety is the point

\[
(f(u + v + w) + u(g + h) : g(u + v + w) + v(h + f) : h(u + v + w) + w(f + g)).
\]

The ratio of homothety is

\[
\frac{fu + gv + hw}{(f + g + h)(u + v + w)}.
\]

(b) The lines \(A_s\tilde{A}_s, B_s\tilde{B}_s\) and \(C_s\tilde{C}_s\) bound a triangle homothetic to \(ABC\) with center of homothety \((uf : vg : wh)\) \(^1\) The ratio of homothety is

\[
1 - \frac{fu + gv + hw}{(f + g + h)(u + v + w)}.
\]

---

\(^1\)This point is called the barycentric product of \(P\) and \(Q\). Another construction was given by P. Yiu in \([4]\). These homothetic centers are collinear with the midpoint of \(PQ\).
2. Parasix loci

We present a few line and conic loci associated with parasix configurations. For \( P = (u : v : w) \), we denote by

(i) \( L_P \) the trilinear polar of \( P \), which has equation

\[
\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0;
\]

(ii) \( C_P \) the circumconic with perspector \( P \), which has equation

\[
\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.
\]

2.1. Area of parasix triangles. The parasix triangles \( A_gB_gC_g \) and \( A_sB_sC_s \) have a common area

\[
\frac{ghu + hfv + fgw}{(f + g + h)^2(u + v + w)}.
\]

Proposition 3. (a) For a given \( Q \), the locus of \( P \) for which the triangles \( A_gB_gC_g \) and \( A_sB_sC_s \) have a fixed (signed) area is a line parallel to \( L_Q \).

(b) For a given \( P \), the locus of \( Q \) for which the triangles \( A_gB_gC_g \) and \( A_sB_sC_s \) have a fixed (signed) area is a conic homothetic to \( C_P \) at its center.

In particular, the parasix triangles degenerate into two parallel lines if and only if

\[
\frac{u}{f} + \frac{v}{g} + \frac{w}{h} = 0. \quad (*)
\]

This condition can be construed in two ways: \( P \in L_Q \), or equivalently, \( P \in C_P \). See §6.

2.2. Perspectivity with the pedal triangle.

Proposition 4. (a) Given \( P \), the locus of \( Q \) so that \( A_sB_sC_s \) is perspective to the pedal triangle of \( Q \) is the line\(^2\)

\[
\sum_{\text{cyclic}} S_A(S_Bv - S_Cw)(-uS_A + vS_B + wS_c)x = 0.
\]

This line passes through the orthocenter \( H \) and the point

\[
\left( \frac{1}{S_A(-uS_A + vS_B + wS_c)} : \cdots : \cdots \right),
\]

which can be constructed as the perspector of \( ABC \) and the cevian triangle of \( P \) in the orthic triangle.

---

\(^2\)Here we adopt J.H. Conway’s notation by writing \( S \) for twice the area of triangle \( ABC \) and

\[
S_A = S \cdot \cot A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = S \cdot \cot B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = S \cdot \cot C = \frac{a^2 + b^2 - c^2}{2}.
\]

These satisfy \( S_{AB} + S_{BC} + S_{CA} = S^2 \). The expressions \( S_{AB}, S_{BC}, S_{CA} \) stand for \( S_A S_B, S_B S_C, S_C S_A \) respectively.
2.3. **Parallelology.** A triangle is said to be parallelologic to a second triangle if the lines through the vertices of the triangle parallel to the corresponding opposite sides of the second triangle are concurrent.

**Proposition 5.** (a) Given \( P = (u : v : w) \), the locus of \( Q \) for which \( ABC \) is parallelologic to \( A_gB_gC_g \) (respectively \( A_sB_sC_s \)) is the line \((v + w)x + (w + u)y + (u + v)z = 0\), which can be constructed as the trilinear polar of the isotomic conjugate of the complement of \( P \).

(b) Given \( Q = (f : g : h) \), the locus of \( P \) for which \( ABC \) is parallelologic to \( A_gB_gC_g \) (respectively \( A_sB_sC_s \)) is the line \((g + h)x + (h + f)y + (f + g)z = 0\), which can be constructed as the trilinear polar of the isotomic conjugate of the complement of \( Q \).

2.4. **Perspectivity with \( ABC \).** Clearly \( A_gB_gC_g \) is perspective to \( ABC \) at \( P \). The perspectrix is the line \( gh(g + h)x + fh(f + h)y + fg(f + g)z = 0 \), parallel to the trilinear polar of \( Q \). Given \( P \), the locus of \( Q \) such that \( A_sB_sC_s \) is perspective to \( ABC \) is the cubic

\[(v + w)x(uy^2 - vz^2) + (u + v)y(uz^2 - wx^2) + (u + v)z(vx^2 - uy^2) = 0,
\]

which is the isocubic with pivot \((v+w : w+u : u+v)\) and pole \( P \). For \( P = K \), the symmedian point, this is the isogonal cubic with pivot \( X_{141} = (b^2 + c^2 : c^2 + a^2 : a^2 + b^2) \).

3. **Orthology.**

Some interesting loci associated with the orthology of triangles attracted our attention because of their connection with the orthocorrespondence defined in [1]. We recall that two triangles are orthologic if the perpendiculars from the vertices of one triangle to the opposite sides of the corresponding vertices of the other triangle are concurrent.

First, consider the locus of \( Q \), given \( P \), such that the triangles \( A_gB_gC_g \) and \( A_sB_sC_s \) are orthologic to \( ABC \). We can find this locus by simple calculation since this is also the locus such that \( A_gB_gC_g \) is perspective to the triangle of the infinite points of the altitudes, with coordinates

\[H_A^\infty = (-a^2, S_C, S_B), \quad H_B^\infty = (S_C, -b^2, S_A), \quad H_C^\infty = (S_B, S_A, -c^2).
\]

The lines \( A_gH_A^\infty, V_gH_B^\infty \) and \( C_gH_C^\infty \) concur if and only if \( Q \) lies on the line

\[(S_Bv - S_Cw)x + (S_Cw - S_Au)y + (S_Au - S_Bv)z = 0, \tag{3}
\]

which is the line through the centroid \( G \) and the orthocorrespondent of \( P \), namely, the point \( ^3\)

\[P^\perp = (u(-S_Au + S_Bv + S_Cw) + a^2vw : \cdot : \cdot )\.
\]

The line (3) is the orthocorrespondent of the line \( HP \). See [1, §2.4].

\[^3\text{The lines perpendicular at } P \text{ to } AP, BP, CP \text{ intersect the respective sidelines at three collinear points. The orthocorrespondent of } P \text{ is the trilinear pole } P^\perp \text{ of the line containing these three intersections.}\]
For the second locus problem, we let \( Q \) be given, and ask for the locus of \( P \) such that the triangles \( A_gB_gC_g \) and \( A_sB_sC_s \) are orthologic to \( ABC \). The computations are similar, and again we find a line as the locus:

\[
S_A(g - h)x + S_B(h - f)y + S_C(f - g)z = 0.
\]

This is the line through \( H \), and the two anti-orthocorrespondents of \( Q \). See [1, Figure 2]. It is the anti-orthocorrespondent of the line \( GQ \).

Given \( P \), for both \( A_gB_gC_g \) and \( \tilde{A}_g\tilde{B}_g\tilde{C}_g \) to be orthologic to \( ABC \), the point \( Q \) has to be the intersection of the line \( GP^h \) ((3) above) and

\[
S_A(v - w)x + S_B(w - u)y + S_C(u - v)z = 0,
\]

the anti-orthocorrespondent of \( GP \). This is the point

\[
\tau(P) = (S_A(e^2 - b^2)u^2 + (S_{AC} - S_{BB})uv - (S_{AB} - S_{CC})uw + a^2(e^2 - b^2)vw : \cdots : \cdots).
\]

The point \( \tau(P) \) is not well defined if all three coordinates of \( \tau(P) \) are equal to zero, which is the case exactly when \( P \) is either \( K \), the orthocenter \( H \), or the centroid \( G \). The pre-images of these points are lines: \( GH \) (the Euler line), \( GK \), and \( HK \) for \( K \), \( G \) and \( H \) respectively. Outside these lines the mapping \( P \mapsto \tau(P) \) is an involution. Note that \( P \) and \( \tau(P) \) are collinear with the symmedian point \( K \).

The fixed points of \( \tau \) are the points of the Kiepert hyperbola

\[
(b^2 - c^2)yz + (c^2 - a^2)xz + (a^2 - b^2)xy = 0.
\]

More precisely, the line joining \( \tau(P) \) to \( H \) meets \( GP \) on the Kiepert hyperbola. Therefore we may characterize \( \tau(P) \) as the intersection of the line \( PK \) with the polar of \( P \) in the Kiepert hyperbola.\(^4\)

In the table below we give the first coordinates of some well known triangle centers and their images under \( \tau \). The indexing of triangle centers follows [3].

<table>
<thead>
<tr>
<th>( P )</th>
<th>first coordinate</th>
<th>( \tau(P) )</th>
<th>first coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>( a )</td>
<td>( X_9 )</td>
<td>( a(s - a) )</td>
</tr>
<tr>
<td>( X_7 )</td>
<td>( (s - b)(s - c) )</td>
<td>( X_{948} )</td>
<td>( (s - b)(s - c)F )</td>
</tr>
<tr>
<td>( X_8 )</td>
<td>( s - a )</td>
<td></td>
<td>( a^2 + (b + c)^2 )</td>
</tr>
<tr>
<td>( X_{19} )</td>
<td></td>
<td></td>
<td>( aG )</td>
</tr>
<tr>
<td>( X_{34} )</td>
<td></td>
<td>( X_{72} )</td>
<td>( a(s - b)(s - c)(a^2 + (b + c)^2) )</td>
</tr>
<tr>
<td>( X_{37} )</td>
<td></td>
<td></td>
<td>( a(b + c)S_A )</td>
</tr>
<tr>
<td>( X_{42} )</td>
<td>( a^2(b + c) )</td>
<td>( X_{71} )</td>
<td>( a^2(b + c)S_A )</td>
</tr>
<tr>
<td>( X_{57} )</td>
<td>( a/(s - a) )</td>
<td>( X_{223} )</td>
<td>( a(s - b)(s - c)F )</td>
</tr>
<tr>
<td>( X_{58} )</td>
<td></td>
<td>( X_{572} )</td>
<td>( a^2G )</td>
</tr>
</tbody>
</table>

\(^4\)This is also called the Hirst inverse of \( P \) with respect to \( K \). See the glossary of [3].
Here,
\[ F = a^3 + a^2(b + c) - a(b + c)^2 - (b + c)(b - c)^2, \]
\[ G = a^3 + a^2(b + c) + a(b + c)^2 + (b + c)(b - c)^2, \]

We may also wonder, given \( P \) outside the circumcircle, for which \( Q \) are the Parasix\((P, Q)\) triangles \( A_9B_9C_9 \) and \( A_8B_8C_8 \) orthologic to the circumcevian triangle of \( P \). The \( A \)-vertex of the circumcevian triangle of \( P \) has coordinates
\[ \left( -a^2yz : (b^2z + c^2y)y : (b^2z + c^2y)z \right). \]

Hence we find that the lines from the vertices of the circumcevian triangle of \( P \) perpendicular to the corresponding sides of \( A_9B_9C_9 \) concur if and only if
\[ (uyz + vxz + wxy)L = 0, \tag{4} \]
where
\[ L = \sum_{\text{cyclic}} (c^2v^2 + 2S_Avw + b^2w^2)((c^2S_cv - b^2S_Bw)u^2 + a^2((e^2v^2 - b^2w^2)u + (S_Bv - S_Cw)v)^2)x. \]

The first factor in (4) represents the circumconic with perspector \( P \), and when \( Q \) is on this conic, Parasix\((P, Q)\) is degenerate, see §6 below. The second factor \( L \) yields the locus we are looking for, a line passing through \( P_L \).\(^5\)

A point \( X \) lies on the line \( L = 0 \) if and only if \( P \) lies on a bicircular circumquintic through the in- and excenters\(^6\). For the special case \( X = G \) this quintic decomposes into \( L_\infty \) (with multiplicity 2) and the McCay cubic.\(^7\) In other words, for any \( P \) on the McCay cubic, the circumcevian triangle of \( P \) is orthologic to the Parasix\((P, Q)\) triangles if and only if \( Q \) lies on the line \( GP_L \).

4. **Concyclic Parasix\((P, Q)\)-hexagons**

We may ask, given \( P \), for which \( Q \) the parasix configuration yields a cyclic hexagon. This is equivalent to the circumcenter of \( A_9B_9C_9 \) being equal to the midpoint of segment \( PQ \). Now the midpoint of \( PQ \) lies on the perpendicular bisector of \( B_9C_9 \) if and only if \( Q \) lies on the line
\[ -(w(S_Au - S_Bv - S_Cw) + c^2uv)y + (v(S_Au - S_Bv + S_Cw)v + b^2wu)z = 0, \]
which is indeed the cevian line \( AP_L \). Remarkably, we find the same cevian line as locus for \( Q \) satisfying the condition that \( B_9C_9 \perp AP \).

**Proposition 6.** The following statements are equivalent.

1. Parasix\((P, Q)\) yields a cyclic hexagon.

\(^5\)The line \( L = 0 \) is not defined when \( P \) is an in/excenter. This means that, for any \( Q \) triangles \( A_9B_9C_9 \) and \( A_8B_8C_8 \) in Parasix\((P, Q)\) are orthologic to the circumcevian triangle of \( P \). This is not surprising since \( P \) is the orthocenter of its own circumcevian triangle. For \( P = X_3, L = 0 \) is the line \( GK \), while for \( P = X_{13}, X_{14} \), it is the parallel at \( P \) to the Euler line.

\(^6\)This quintic has equation \( Q_Ax + Q_yy + Q_zz = 0 \) where \( Q_A \) represents the union of the circle center \( A \), radius 0 and the Van Rees focal which is the isogonal pivotal cubic with pivot the infinite point of \( AH \) and singular focus \( A \).

\(^7\)The McCay cubic is the isogonal cubic with pivot \( O \) given by the equation \( \sum_{\text{cyclic}} a^2S_Ax(c^2y^2 - b^2z^2) = 0. \)
(2) $A_g B_g C_g$ and $A_s B_s C_s$ are homothetic to the antipedal triangle of $P$.
(3) $Q$ is the orthocorrespondent of $P$.

The center of the circle containing the 6 points is the midpoint of $PQ$.
The homothetic centers and the circumcenter of the cyclic hexagon are collinear.
A nice example is the circle around Parasix($H, G$). It is homothetic to the
circumcircle and nine point circle through $H$ with factors $\frac{1}{3}$ and $\frac{2}{3}$ respectively. The
center of the circle divides $OH$ in the ratio $2 : 1$.\footnote{This is also the midpoint of $GH$, the center of the orthocentroidal circle, the point $X_{381}$ in [3].} The antipedal triangle of $H$
is clearly the anticomplementary triangle of $ABC$. The two homothetic centers
divide the same segment in the ratios $5 : 2$ and $3 : 2$ respectively.\footnote{These have homogeneous barycentric coordinates $(3a^4 + 2a^2(b^2 + c^2) - 5(b^2 - c^2)^2 : \cdots : \cdots)$ and $(a^4 - 2a^2(b^2 + c^2) + 3(b^2 - c^2)^2 : \cdots : \cdots)$ respectively. They are not in the current edition of [3].}

As noted in [1], $P = P^\perp$ only for the Fermat-Torricelli points $X_{13}$ and $X_{14}$.
The vertices of parasix($X_{13}, X_{13}$) and Parasix($X_{14}, X_{14}$) form regular hexagons.
See Figure 5.

5. Equilateral triangles

The last example raises the question of finding, for given $P$, the points $Q$ for
which the triangles $A_g B_g C_g$ and $A_s B_s C_s$ are equilateral. We find that the $A$-
median of $A_g B_g C_g$ is also an altitude in this triangle if and only if $Q$ lies on the
conic
\[-2((S_Au + S_Bv - S_Cw)w + c^2uv)xy + 2((S_Au - S_Bv + S_Cw)v + b^2uw)yz
- (c^2u^2 + a^2w^2 + 2S_Buw)y^2 + (b^2u^2 + a^2v^2 + 2S_Cuw)z^2 = 0.\]

We find an analogous conic for the \(B\)-median of \(A_gB_gC_g\) to be an altitude. The two conics intersect in four points: two imaginary points and the points
\[Q_{1,2} = \left( -S_Au + S_Bv + S_Cw u + a^2vw \pm \frac{1}{3} \sqrt{3}Su(u+v+w) : \cdots : \cdots \right).\]

**Proposition 7.** Given \(P\), there are two (real) points \(Q\) for which triangles \(A_yB_yC_y\) and \(A_sB_sC_s\) are equilateral. These two points divide \(PP'\) harmonically.

The points \(Q_{1,2}\) from Proposition 7 can be constructed in the following way, using the fact that \(P, G_s, G_g\) and \(P'\) are collinear.

Start with a point \(G'\) on \(PP'\). We shall construct an equilateral triangle \(A'B'C'\) with vertices on \(AP, BP\) and \(CP\) respectively and centroid at \(G\). This triangle must be homothetic to one of the equilateral triangles \(A_yB_yC_y\) of Proposition 7 through \(P\).
Consider the rotation $\rho$ about $G'$ through $\pm \frac{2\pi}{3}$. The image of $AP$ intersects $BP$ in a point $B'$. Now let $C'$ be the image of $B'$ and $A'$ the image of $C'$. Then $A'B'C'$ is equilateral, $A'$ lies on $AP$, $G'$ is the centroid and $C'$ must lie on $CP$.

The homothety with center $A$ that maps $P$ to $A'$ also maps $BC$ to a line $\ell_a$. Similarly we find $\ell_b$ and $\ell_c$. These lines enclose a triangle $A''B''C''$ homothetic to $ABC$. We of course want to find the case for which $A''B''C''$ degenerates into one point, which is the $Q$ we are looking for. Since all possible equilateral $A'B'C'$ of the same orientation are homothetic through $P$, the triangles $A'B''C''$ are all homothetic to $ABC$ through the same point. So the homothety center of $A'B''C''$ and $ABC$ is the point $Q$ we are looking for.

6. Degenerate parasix triangles

We begin with a simple interesting fact.

**Proposition 8.** Every line through $P$ intersects the circumconic $\mathcal{C}_P$ at two real points.

**Proof.** For the special case of the symmedian point $K$ this is clear, since $K$ is the interior of the circumcircle. Now, there is a homography $\varphi$ fixing $A$, $B$, $C$ and transforming $P = (u : v : w)$ into $K = (a^2 : b^2 : c^2)$. It is given by

$$\varphi(x : y : z) = \left( \frac{a^2}{u}x : \frac{b^2}{v}y : \frac{c^2}{w}z \right),$$

and is a projective transformation mapping $\mathcal{C}_P$ into the circumcircle and any line through $P$ into a line through $K$. If $\ell$ is a line through $P$, then $\varphi(\ell)$ is a line through $K$, intersecting the circumcircle at two real points $q_1$ and $q_2$. The circumcircle and
the circumconic $C_P$ have a fourth real point $Z$ in common, which is the trilinear pole of the line $PK$. For any point $M$ on $C_P$, the points $Z, M, \varphi(M)$ are collinear. The second intersections of the lines $Zq_i$ and $Zq_2$ are common points of $\ell$ and the circumconic $C_P$.

In §2, we have seen that the parasix triangles are degenerate if and only if $P \in \mathcal{L}_Q$ or equivalently, $Q \in C_P$. This means that for each line $\ell_P$ through $P$ intersecting the circumconic $C_P$ at $Q_1$ and $Q_2$, the triangles of Parasix$(P, Q_i)$, $i = 1, 2$, are degenerate.

**Theorem 9.** For $i = 1, 2$, the two lines containing the degenerate triangles of the parasix configuration Parasix$(P, Q_i)$ are parallel to a tangent from $P$ to the inscribed conic $C_\ell$ with perspector the trilinear pole of $\ell_P$. The two tangents for $i = 1, 2$ are perpendicular if and only if the line $\ell_P$ contains the orthocorrespondent $P^\perp$.

For example, for $P = K$, the symmedian point, the circumconic $C_P$ is the circumcircle. The orthocorrespondent is the point $K^\perp = (a^2(a^4 - b^4 + 4b^2c^2 - c^4) : \cdots : \cdots)$ on the Euler line. The line $\ell$ joining $K$ to this point has equation

$$
\frac{(b^2 - c^2)(b^2 + c^2 - 2a^2)}{a^2}x + \frac{(c^2 - a^2)(c^2 + a^2 - 2b^2)}{b^2}y + \frac{(a^2 - b^2)(a^2 + b^2 - 2c^2)}{c^2}z = 0.
$$

The inscribed conic $C_\ell$ has center

$$(a^2(b^2 - c^2)(a^4 - b^4 + b^2c^2 - c^4) : \cdots : \cdots).$$

The tangents from $K$ to the conic $C_\ell$ are the Brocard axis $OK$ and its perpendicular at $K$. The points of tangency are

$$
\begin{pmatrix}
\frac{a^2(2a^2 - b^2 - c^2)}{b^2 - c^2} & \frac{b^2(2b^2 - c^2 - a^2)}{c^2 - a^2} & \frac{c^2(2c^2 - a^2 - b^2)}{a^2 - b^2}
\end{pmatrix}.
$$

on the Brocard axis and

$$
\begin{pmatrix}
\frac{a^2(b^2 - c^2)}{2a^2 - b^2 - c^2} & \frac{b^2(c^2 - a^2)}{2b^2 - c^2 - a^2} & \frac{c^2(a^2 - b^2)}{2c^2 - a^2 - b^2}
\end{pmatrix}
$$

on the perpendicular tangent. See Figure 7. The line $\ell$ intersects the circumcircle at the point

$$X_{110} = \begin{pmatrix}
a^2 & b^2 & c^2 \\
b^2 - c^2 & c^2 - a^2 & a^2 - b^2
\end{pmatrix}
$$

and the Parry point

$$X_{111} = \begin{pmatrix}
a^2 & b^2 & c^2 \\
b^2 + c^2 - 2a^2 & c^2 + a^2 - 2b^2 & a^2 + b^2 - 2c^2
\end{pmatrix}.
$$

The lines containing the degenerate triangles of Parasix$(K, X_{110})$ are parallel to the Brocard axis, while those for Parasix$(K, X_{111})$ are parallel to the tangent from $K$ which is perpendicular to the Brocard axis.

---

10The infinite points of these lines are respectively $X_{511}$ and $X_{512}$. 
Figure 7. Degenerate Parasix($K, X_{110}$) and Parasix($K, X_{111}$)

References


Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
*E-mail address*: bg42@wanadoo.fr

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
*E-mail address*: f.v.lamoen@wxs.nl
A Tetrahedral Arrangement of Triangle Centers

Lawrence S. Evans

Abstract. We present a graphic scheme for indexing 25 collinearities of 17 triangle centers three at a time. The centers are used to label vertices and edges of nested polyhedra. Two new triangle centers are introduced to make this possible.

1. Introduction

Collinearities of triangle centers which are defined in apparently different ways has been of interest to geometers since it was first noticed that the orthocenter, centroid, and circumcenter are collinear, lying on Euler’s line. Kimberling [3] lists a great many collinearities, including many more points on Euler’s line. The object of this note is to present a three-dimensional graphical summary of 25 three-center collinearities involving 17 centers, in which the centers are represented as vertices and edge midpoints of nested polyhedra: a tetrahedron circumscribing an octahedron which then circumscribes a cubo-octahedron. Such a symmetric collection of collinearities may be a useful mnemonic. Probably the reason why this has not been recognized before is that two of the vertices of the tetrahedron represent previously undescribed centers. First we describe two new centers, which Kimberling lists as $X_{1276}$ and $X_{1277}$ in his Encyclopedia of Triangle Centers [3]. Then we describe the tetrahedron and work inward to the cubo-octahedron.

2. Perspectors and the excentral triangle

The excentral triangle, $T_x$, of a triangle $T$ is the triangle whose vertices are the excenters of $T$. Let $T_+$ be the triangle whose vertices are the apices of equilateral triangles erected outward on the sides of $T$. Similarly let $T_-$ be the triangle whose vertices are the apices of equilateral triangles erected inward on the sides of $T$. It happens that $T_x$ is in perspective from $T_+$ from a point $V_+$, a previously undescribed triangle center now listed as $X_{1276}$ in [3], and that $T_x$ is also in perspective from $T_-$ from another new center $V_-$ listed as $X_{1277}$ in [3]. See Figure 1.

For $\varepsilon = \pm 1$, the homogeneous trilinear coordinates of $V_\varepsilon$ are

$$1 - v_a + v_b + v_c : 1 + v_a - v_b + v_c : 1 + v_a + v_b - v_c,$$

where $v_a = -\frac{2}{\sqrt{3}} \sin(A + \varepsilon \cdot 60^\circ)$ etc.

It is well known that $T_x$ and $T$ are in perspective from the incenter $I$. Define $T^*$ as the triangle whose vertices are the reflections of the vertices of $T$ in the opposite sides. Then $T_x$ and $T^*$ are in perspective from a point $W$ listed as $X_{184}$ in [3]. See Figure 2. The five triangles $T$, $T_x$, $T_+$, $T_-$, and $T^*$ are pairwise in perspective, giving 10 perspectors. Denote the perspector of two triangles by enclosing the two triangles in brackets, so, for example $[T_x, T] = I$. 

Here is a list of the 10 perspectors with their names and ETC numbers:

<table>
<thead>
<tr>
<th>[T, T+]</th>
<th>F_+</th>
<th>First Fermat point</th>
<th>X_{13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[T, T_]</td>
<td>F_-</td>
<td>Second Fermat point</td>
<td>X_{14}</td>
</tr>
<tr>
<td>[T, T^*]</td>
<td>H</td>
<td>Orthocenter</td>
<td>X_4</td>
</tr>
<tr>
<td>[T, T_x]</td>
<td>I</td>
<td>Incenter</td>
<td>X_1</td>
</tr>
<tr>
<td>[T_+, T_-]</td>
<td>O</td>
<td>Circumcenter</td>
<td>X_3</td>
</tr>
<tr>
<td>[T_+, T^*]</td>
<td>J_-</td>
<td>Second isodynamic point</td>
<td>X_{16}</td>
</tr>
<tr>
<td>[T_-, T^*]</td>
<td>J_+</td>
<td>First isodynamic point</td>
<td>X_{15}</td>
</tr>
<tr>
<td>[T_x, T^*]</td>
<td>W</td>
<td>First Evans perspector</td>
<td>X_{484}</td>
</tr>
<tr>
<td>[T_x, T_+]</td>
<td>V_+</td>
<td>Second Evans perspector</td>
<td>X_{1276}</td>
</tr>
<tr>
<td>[T_x, T_-]</td>
<td>V_-</td>
<td>Third Evans perspector</td>
<td>X_{1277}</td>
</tr>
</tbody>
</table>
3. Collinearities among the ten perspectors

As in [2], we shall write $\mathcal{L}(X, Y, Z, \ldots)$ to denote the line containing $X, Y, Z, \ldots$. The following collinearities may be easily verified:

\[
\begin{align*}
\mathcal{L}(I, O, W), & \quad \mathcal{L}(I, J_-, V_-), & \quad \mathcal{L}(I, J_+, V_+), \\
\mathcal{L}(V_+, H, V_-), & \quad \mathcal{L}(W, F_+, V_-), & \quad \mathcal{L}(W, F_-, V_+).
\end{align*}
\]

What is remarkable is that all five triangles are involved in each collinearity, with $T_x$ used twice. For example, rewrite $\mathcal{L}(I, O, W)$ as

\[\mathcal{L}([T, T_x], [T_+, T_-], [T_x, T^*])\]

to see this. The six collinearities have been stated so that the first and third perspectors involve $T_x$, with the perspector of the remaining two triangles listed second. This lends itself to a graphical representation as a tetrahedron with vertices labelled with $I, V_+, V_-, W$, and the edges labelled with the perspectors collinear with the vertices. See Figure 3. When these centers are actually constructed, they may not be in the order listed in these collinearities. For example, $O$ is not necessarily between $I$ and $W$. There is another collinearity which we do not use, however, namely, $\mathcal{L}(O, J_+, J_-)$, which is the Brocard axis. Triangle $T_x$ is not involved in any of the perspectors in this collinearity.

![Figure 3](image)

If we label each edge of the tetrahedron at its midpoint by the middle center listed in each of the collinearities above, then opposite edge midpoints are pairs of isogonal conjugates: $H$ and $O$, $J_+$ and $F_+$, and $J_-$ and $F_-$. Also the lines $\mathcal{L}(O, H)$, $\mathcal{L}(F_+, J_+)$, and $\mathcal{L}(F_-, J_-)$ are parallel to the Euler line, and may be
interpreted as intersecting at the Euler infinity point \(E\), listed as \(X_{30}\) in [3]. This adds three more collinearities to the tetrahedral scheme:

\[
\mathcal{L}(O, E, H), \ \mathcal{L}(F_+, E, J_+), \ \mathcal{L}(F_-, E, J_-).
\]

The five triangles \(T, T_+, T_-, T^*,\) and \(T_x\) are all inscribed in Neuberg’s cubic curve. Now consider a triangle \(T'_x\) in perspective with \(T_x\) and inscribed in the cubic with vertices very close to those of \(T_x\) (the excenters of \(T\)). The lines of perspective of \(T'_x\) and \(T_x\) approach the tangents to Neuberg’s cubic at the vertices of \(T_x\) as \(T'_x\) approaches \(T_x\). These tangents are known to be parallel to the Euler line and may be thought of as converging at the Euler point at infinity, \(E = X_{30}\).

So we can write \(E = [T_x, T_x]\), interpreting this to mean that \(T_x\) is in perspective from itself from \(E\). I propose the term “ipseperspector” for such a point, from the Latin “ipse” for self. Note that the notion of ipseperspector is dependent on the curve circumscribing the triangle \(T\). A well-known example of an ipseperspector for a triangle circumscribed in Neuberg’s cubic is \(X_{74}\), this being the point where the tangents to the curve at the vertices of \(T\) intersect.

4. Further nested polyhedra

We shall encounter other named centers, which are listed here for reference:

<table>
<thead>
<tr>
<th>Center</th>
<th>Description</th>
<th>X-coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G)</td>
<td>Centroid</td>
<td>(X_2)</td>
</tr>
<tr>
<td>(K)</td>
<td>Symmedian (Lemoine) point</td>
<td>(X_6)</td>
</tr>
<tr>
<td>(N_+)</td>
<td>First Napoleon point</td>
<td>(X_{17})</td>
</tr>
<tr>
<td>(N_-)</td>
<td>Second Napoleon point</td>
<td>(X_{18})</td>
</tr>
<tr>
<td>(N_+^*)</td>
<td>Isogonal conjugate of (N_+)</td>
<td>(X_{61})</td>
</tr>
<tr>
<td>(N_-^*)</td>
<td>Isogonal conjugate of (N_-)</td>
<td>(X_{62})</td>
</tr>
</tbody>
</table>

The six midpoints of the edges of the tetrahedron may be considered as the vertices of an inscribed octahedron. This leads to indexing more collinearities in the following way: label the midpoint of each edge of the octahedron by the point where the lines indexed by opposite edges meet. For example, opposite edges of the octahedron \(\mathcal{L}(F_+, J_-)\) and \(\mathcal{L}(F_-, J_+)\) meet at the centroid \(G\). We can then write two 3-point collinearities as \(\mathcal{L}(F_+, G, J_-)\) and \(\mathcal{L}(F_-, G, J_+)\). Now the edges adjacent to both of these edges index the lines \(\mathcal{L}(F_+, F_-)\) and \(\mathcal{L}(J_+, J_-)\), which meet at the symmedian point \(K\). This gives two more 3-point collinearities, \(\mathcal{L}(F_+, K, F_-)\) and \(\mathcal{L}(J_+, K, J_-)\). Note that \(G\) and \(K\) are isogonal conjugates. This pattern persists with the other pairs of opposite edges of the octahedron.

The intersections of other lines represented as opposite edges intersect at the Napoleon points and their isogonal conjugates. When we consider the four vertices \(O, F_-, H,\) and \(J_+\) of the octahedron, four more 3-point collinearities are indexed in the same manner: \(\mathcal{L}(O, N_+^*, J_-)\), \(\mathcal{L}(H, N_-^*, F_-)\), \(\mathcal{L}(O, N_-, F_-)\), and \(\mathcal{L}(H, N_-, J_-)\). Similarly, from vertices \(O, F_+, H,\) and \(J_+\), four more 3-point collinearities arise in the same indexing process: \(\mathcal{L}(O, N_+^*, J_+)\), \(\mathcal{L}(H, N_-^*, F_+)\), \(\mathcal{L}(O, N_-, F_+)\), and \(\mathcal{L}(H, N_+, J_+)\). So each of the twelve edges of the octahedron indexes a different 3-point collinearity.
Let us carry this indexing scheme further. Now consider the midpoints of the edges of the octahedron to be the vertices of a polyhedron inscribed in the octahedron. This third nested polyhedron is a cubo-octahedron: it has eight triangular faces, each of which is coplanar with a face of the octahedron, and six square faces. Yet again more 3-point collinearities are indexed, but this time by the triangular faces of the cubo-octahedron. It happens that the three vertices of each triangular face of the cubo-octahedron, which inherit their labels as edges of the octahedron, are collinear in the plane of the basic triangle $T$. Opposite edges of the octahedron have the same point labelling their midpoints, so opposite triangular faces of the cubo-octahedron are labelled by the same three centers. This means that there are four instead of eight collinearities indexed by the triangular faces: $\mathcal{L}(G, N_+, N_-^*), \mathcal{L}(G, N_-, N_+^*), \mathcal{L}(K, N_+, N_-), \text{ and } \mathcal{L}(K, N_-^*, N_+^*)$. See Figure 4.

So we have 6 collinearities indexed by edges of the tetrahedron, 3 more by its diagonals, 12 by the inscribed octahedron, and 4 more by the further inscribed cubo-octahedron, for a total of 25.
5. Concluding remarks

In a sense, the location of each center entering into this graphical scheme places it in equal importance to the other centers in similar locations. So the four centers $I, U, V$, and $W$, which arose as perspectors with the excentral triangle are on one level. On the next level we may place the six centers $O, H, J_+, J_-, F_+$, and $F_-$ which index the edges of the tetrahedron and the vertices of the inscribed octahedron. It is interesting that these six centers are the first to appear in the construction given by the author [1], and that the subsequent centers indexed by the midpoints of the edges of the octahedron arise as intersections of lines they determine. The Euler infinity point, $E$, is the only point at the third level of construction. Centers $I, V_+, V_-, W, O, H, F_+, J_+, F_-, J_-$, and $E$ all lie on Neuberg’s cubic curve. The Euler line appears as the collinearity $L(O, E, H)$, with no indication that $G$ lies on the line. The Brocard axis appears four times as $L(J_+, K, J_-), L(K, N^*_+, N^*_+), L(O, N^*_+, J_+), \text{ and } L(O, N^*_+, J_-)$, but the better-known collinearity $L(O, J_+, J_-)$ does not.

References


Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: 75342.3052@compuserve.com
The Apollonius Circle and Related Triangle Centers

Milorad R. Stevanović

Abstract. We give a simple construction of the Apollonius circle without directly invoking the excircles. This follows from a computation of the coordinates of the centers of similitude of the Apollonius circle with some basic circles associated with a triangle. We also find a circle orthogonal to the five circles, circumcircle, nine-point circle, excentral circle, radical circle of the excircles, and the Apollonius circle.

1. The Apollonius circle of a triangle

The Apollonius circle of a triangle is the circle tangent internally to each of the three excircles. Yiu [5] has given a construction of the Apollonius circle as the inversive image of the nine-point circle in the radical circle of the excircles, and the coordinates of its center $Q$. It is known that this radical circle has center the Spieker center $S$ and radius $\rho = \frac{1}{2} \sqrt{r^2 + s^2}$. See, for example, [6, Theorem 4]. Ehrmann [1] found that this center can be constructed as the intersection of the Brocard axis and the line joining $S$ to the nine-point center $N$. See Figure 1. A proof of this fact was given in [2], where Grinberg and Yiu showed that the Apollonius circle is a

Figure 1

The author thanks the editor for his helps in the preparation of this paper.
Tucker circle. In this note we first verify these results by expressing the coordinates of \( Q \) in terms of \( R, r, \) and \( s \), (the circumradius, inradius, and semiperimeter) of the triangle. By computing some homothetic centers of circles associated with the Apollonius circle, we find a simple construction of the Apollonius circle without directly invoking the excircles. See Figure 4.

For triangle centers we shall adopt the notation of Kimberling’s *Encyclopedia of Triangle Centers* \[3\], except for the most basic ones:

- \( G \) centroid
- \( O \) circumcenter
- \( I \) incenter
- \( H \) orthocenter
- \( N \) nine-point center
- \( K \) symmedian point
- \( S \) Spieker center
- \( I' \) reflection of \( I \) in \( O \)

We shall work with barycentric coordinates, absolute and homogeneous. It is known that if the Apollonius circle touches the three excircles respectively at \( A', B', C' \), then the lines \( AA', BB', CC' \) concur in the point \(^{1}\)

\[
X_{181} = \left( \frac{a^2(b + c)^2}{s - a} : \frac{b^2(c + a)^2}{s - b} : \frac{c^2(a + b)^2}{s - c} \right)
\]

We shall make use of the following simple lemma.

**Lemma 1.** Under inversion with respect to a circle, center \( P \), radius \( \rho \), the image of the circle center \( P' \), radius \( \rho' \), is the circle, radius \( \frac{\rho^2}{d^2 - \rho^2} \cdot \rho' \) and center \( Q \) which divides the segment \( PP' \) in the ratio

\[
PQ : QP' = \rho^2 : d^2 - \rho^2 - \rho'^2,
\]

\(^{1}\)The trilinear coordinates of \( X_{181} \) were given by Peter Yff in 1992.
where \(d\) is the distance between \(P\) and \(P'\). Thus,

\[
Q = \frac{(d^2 - \rho^2 - \rho'^2)P + \rho^2 \cdot P'}{d^2 - \rho'^2}.
\]

**Theorem 2.** The Apollonius circle has center

\[
Q = \frac{1}{4Rr} \left( (r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I \right)
\]

and radius \(\frac{r^2 + s^2}{4r}\).

**Proof.** It is well known that the distance between \(O\) and \(I\) is given by

\[
OI^2 = R^2 - 2Rr.
\]

Since \(S\) and \(N\) divide the segments \(IG\) and \(OG\) in the ratio \(3:-1\),

\[
SN^2 = \frac{R^2 - 2Rr}{4}.
\]

Applying Lemma 1 with

\[
P = S = \frac{1}{2}(3G - I) = \frac{1}{2}(2O + H - I), \quad P' = N = \frac{1}{2}(O + H),
\]

\[
\rho^2 = \frac{1}{4}(r^2 + s^2), \quad \rho'^2 = \frac{1}{4}R^2,
\]

\[
d^2 = SN^2 = \frac{1}{4}(R^2 - 2Rr),
\]

we have

\[
Q = \frac{1}{4Rr} \left( (r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I \right).
\]

The radius of the Apollonius circle is \(\frac{r^2 + s^2}{4r}\). \(\square\)

The point \(Q\) appears in Kimberling’s *Encyclopedia of Triangle Centers* [3] as

\[
X_{970} = (a^2(a^3(b + c)^2 + a^2(b + c)(b^2 + c^2) - a(b^4 + 2b^3c + 2bc^3 + c^4)

- (b + c)(b^4 + c^4)) : \ldots : \ldots).
\]

We verify that it also lies on the Brocard axis.

**Proposition 3.**

\[
\overrightarrow{OQ} = -\frac{s^2 - r^2 - 4Rr}{4Rr} \cdot \overrightarrow{OK}.
\]

**Proof.** The oriented areas of the triangles \(KHI, OKI,\) and \(OHK\) are as follows.

\[
\triangle(KHI) = \frac{(a - b)(b - c)(c - a)f}{16(a^2 + b^2 + c^2) \cdot \Delta},
\]

\[
\triangle(OKI) = \frac{abc(a - b)(b - c)(c - a)}{8(a^2 + b^2 + c^2) \cdot \Delta},
\]

\[
\triangle(OHK) = \frac{-(a - b)(b - c)(c - a)(a + b)(b + c)(c + a)}{8(a^2 + b^2 + c^2) \cdot \Delta},
\]
where \( \triangle \) is the area of triangle \( ABC \) and

\[
f = a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) + 2abc
= 8rs(2R + r).
\]

Since \( abc = 4Rrs \) and \((a + b)(b + c)(c + a) = 2s(r^2 + 2Rr + s^2)\), it follows that, with respect to \( OHI \), the symmedian point \( K \) has homogeneous barycentric coordinates

\[
f : 2abc : -2(a + b)(b + c)(c + a)
= 8rs(2R + r) : 8Rrs : -4s(r^2 + 2Rr + s^2)
= 2r(2R + r) : 2Rr : -(r^2 + 2Rr + s^2).
\]

Therefore,

\[
K = \frac{1}{4Rr + r^2 - s^2} \left(2r(2R + r)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right),
\]

and

\[
\overrightarrow{OK} = \frac{1}{4Rr + r^2 - s^2} \left((r^2 + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right)
= -\frac{4Rr}{s^2 - r^2 - 4Rr} \cdot \overrightarrow{OQ}.
\]

\[\square\]

2. Centers of similitude

We compute the coordinates of the centers of similitude of the Apollonius circle with several basic circles. Figure 3 below shows the Apollonius circle with the circumcircle, incircle, nine-point circle, excentral circle, and the radical circle (of the excircles). Recall that the excentral circle is the circle through the excenters of the triangle. It has center \( I' \) and radius \( 2R \).

**Lemma 4.** Two circles with centers \( P, P' \), and radii \( \rho, \rho' \) respectively have internal center of similitude \( \frac{\rho' \cdot P + \rho \cdot P'}{\rho' + \rho} \) and external center of similitude \( \frac{\rho' \cdot P - \rho \cdot P'}{\rho' - \rho} \).

**Proposition 5.** The homogeneous barycentric coordinates (with respect to triangle \( ABC \)) of the centers of similitude of the Apollonius circle with the various circles are as follows.
The Apollonius circle and related triangle centers

Figure 3

<table>
<thead>
<tr>
<th>Type</th>
<th>Internal</th>
<th>External</th>
<th>Incircle</th>
<th>Nine-point Circle</th>
<th>Excentral Circle</th>
<th>Excentral Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumcircle</td>
<td>( X_{573} )</td>
<td>( X_{386} )</td>
<td>( a^2 (a^2(b + c) - abc - (b^3 + c^3)) )</td>
<td>( a^2(a(b + c) + b^2 + bc + c^2) )</td>
<td>( a(a(b + c) - bc) )</td>
<td></td>
</tr>
<tr>
<td>( X_{1682} )</td>
<td>( s-a )</td>
<td>( a^2(s - a)(a(b + c) + b^2 + c^2)^2 )</td>
<td>( \frac{a^2(b+c)^2}{s-a} )</td>
<td>( \frac{1}{s-a} - (\frac{a(b+c)^2 - bc(b+c)}{b+c}) )</td>
<td>( a \cdot F )</td>
<td></td>
</tr>
<tr>
<td>( X_{181} )</td>
<td>( X_{2051} )</td>
<td>( b + c : c + a : a + b )</td>
<td>( \frac{1}{s-a} - (\frac{a(b+c)^2 - bc(b+c)}{b+c}) )</td>
<td>( a(a(b + c) - bc) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_{1695} )</td>
<td>( X_{43} )</td>
<td>( a \cdot F )</td>
<td>( \frac{1}{s-a} - (\frac{a(b+c)^2 - bc(b+c)}{b+c}) )</td>
<td>( a(a(b + c) - bc) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where

\[
F = a^5(b + c) + a^4(4b^2 + 7bc + 4c^2) + 2a^3(b + c)(b^2 + c^2) - 2a^2(2b^3 + 3b^2c + 3bc^2 + 2c^3) - a(b + c)(3b^4 + 2b^3c^2 + 3c^4) - bc(b^2 - c^2)^2.
\]
Proof. The homogenous barycentric coordinates (with respect to triangle $OHI$) of the centers of similitude of the Apollonius circle with the various circles are as follows.

<table>
<thead>
<tr>
<th>Circle</th>
<th>X_{573}</th>
<th>X_{386}</th>
<th>X_{1682}</th>
<th>X_{181}</th>
<th>X_{1815}</th>
<th>X_{2051}</th>
<th>X_{181}</th>
<th>X_{1815}</th>
<th>X_{43}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumcircle</td>
<td>2(r^2 + 2Rr + s^2) : 2Rr : -(r^2 + 2Rr + s^2)</td>
<td>4Rr : 2Rr : -(r^2 + 2Rr + s^2)</td>
<td>2r^2 + 4Rr + s^2 : -2Rr^2 : r^3 + Rr^2 - (R - r)s^2</td>
<td>r^2 + 4Rr + s^2 : -2Rr^2 : r^3 + 3Rr^2 + (R + r)s^2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Incircle</td>
<td>-r(r^2 + 4Rr + s^2) : -2Rr^2 : r^3 + Rr^2 - (R - r)s^2</td>
<td>-r(r^2 + 4Rr + s^2) : -2Rr^2 : r^3 + 3Rr^2 + (R + r)s^2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nine-point circle</td>
<td>2 : 1 : -1</td>
<td>-4Rr : r^2 - 2Rr + s^2 : r^2 + 2Rr + s^2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Excentral circle</td>
<td>4(r^2 + 2Rr + s^2) : 4Rr : -(3r^2 + 4Rr + 3s^2)</td>
<td>8Rr : 4Rr : -(r^2 + 4Rr + s^2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using the relations

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

and the following coordinates of $O$, $H$, $I$ (with equal coordinate sums),

$$O = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)),$$
$$H = (c^2 + a^2 - b^2)(a^2 + b^2 - c^2), (a^2 + b^2 - c^2)(b^2 + c^2 - a^2),$$
$$I = (b + c - a)(c + a - b)(a + b - c)(a, b, c),$$

these can be converted into those given in the proposition. □

Remarks. 1. $X_{386} = OK \cap IG$.
2. $X_{573} = OK \cap HI' = OK \cap X_{55}X_{181}$.
3. $X_{43} = IG \cap X_{57}X_{181}$.

From the observation that the Apollonius circle and the nine-point circle have $S$ as internal center of similitude, we have an easy construction of the Apollonius circle without directly invoking the excircles.

Construct the center $Q$ of Apollonius circle as the intersection of $OK$ and $NS$. Let $D$ be the midpoint of $BC$. Join $ND$ and construct the parallel to $ND$ through $Q$ (the center of the Apollonius circle) to intersect $DS$ at $A'$, a point on the Apollonius circle, which can now be easily constructed. See Figure 4.
**Proposition 6.** The center \( Q \) of the Apollonius circle lies on each of the lines \( X_{21}X_{51}, X_{40}X_{43} \) and \( X_{411}X_{185} \). More precisely,

\[
\begin{align*}
X_{51}X_{21} & : X_{21}Q = 2r : 3R, \\
X_{43}X_{40} & : X_{43}Q = 8Rr : r^2 + s^2, \\
X_{185}X_{411} & : X_{411}Q = 2r : R.
\end{align*}
\]

**Remark.** The Schiffler point \( X_{21} \) is the intersection of the Euler lines of the four triangles \( ABC, IBC, ICA \) and \( IAB \). It divides \( OH \) in the ratio

\[
OX_{21} : X_{21}H = R : 2(R + r).
\]

The harmonic conjugate of \( X_{21} \) in \( OH \) is the triangle center

\[
X_{411} = (a(a^6 - a^5(b + c) - a^4(2b^2 + bc + 2c^2) + 2a^3(b + c)(b^2 - bc + c^2) \\
+ a^2(b^2 + c^2)^2 - a(b - c)^2(b + c)(b^2 + c^2) + bc(b - c)^2(b + c)^2) \\
: \cdots : \cdots).
\]
3. A circle orthogonal to 5 given ones

We write the equations of the circles encountered above in the form
\[ a^2yz + b^2zx + c^2xy + (x + y + z)L_i = 0, \]
where \( L_i, 1 \leq i \leq 5, \) are linear forms given below.

<table>
<thead>
<tr>
<th>( i )</th>
<th>circle</th>
<th>( L_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>circumcircle</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>nine-point circle</td>
<td>(-\frac{1}{2}((b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z))</td>
</tr>
<tr>
<td>3</td>
<td>excentral circle</td>
<td>(bcx + cay + abz)</td>
</tr>
<tr>
<td>4</td>
<td>radical circle</td>
<td>((s - b)(s - c)x + (s - c)(s - a)y + (s - a)(s - b)z)</td>
</tr>
<tr>
<td>5</td>
<td>Apollonius</td>
<td>(s ((s + \frac{bc}{a})x + (s + \frac{ca}{b})y + (s + \frac{ab}{c})z))</td>
</tr>
</tbody>
</table>

Remark. The equations of the Apollonius circle was computed in [2]. The equations of the other circles can be found, for example, in [6].

**Proposition 7.** The four lines \( L_i = 0, \) \( i = 2, 3, 4, 5, \) are concurrent at the point
\[ X_{650} = (a(b - c)(s - a) : b(c - a)(s - b) : c(a - b)(s - c)). \]

It follows that this point is the radical center of the five circles above. From this we obtain a circle orthogonal to the five circles.

**Theorem 8.** The circle
\[ a^2yz + b^2zx + c^2xy + (x + y + z)L = 0, \]
where
\[ L = \frac{bc(b^2 + c^2 - a^2)}{2(c - a)(a - b)}x + \frac{ca(c^2 + a^2 - b^2)}{2(a - b)(b - c)}y + \frac{ab(a^2 + b^2 - c^2)}{2(b - c)(c - a)}z, \]
is orthogonal to the circumcircle, excentral circle, Apollonius circle, nine-point circle, and the radical circle of the excircles. It has center \( X_{650} \) and radius the square root of
\[ \frac{abc \cdot G}{4(a - b)^2(b - c)^2(c - a)^2}, \]
where
\[ G = abc(a^2 + b^2 + c^2) - a^4(b + c - a) - b^4(c + a - b) - c^4(a + b - c) \\
= 16r^2s(r^2 + 5Rr + 4R^2 - s^2). \]

This is an interesting result because among these five circles, only three are coaxal, namely, the Apollonius circle, the radical circle, and the nine-point circle.

Remark. \( X_{650} \) is also the perspector of the triangle formed by the intersections of the corresponding sides of the orthic and intouch triangles. It is the intersection of the trilinear polars of the Gergonne and Nagel points.
4. More centers of similitudes with the Apollonius circle

We record the coordinates of the centers of similitude of the Apollonius circle with the Spieker radical circle. These are

\[
(a^2 - a^3(b + c)^2 - a^2(b + c)(b^2 + c^2) + a(b^4 + 2b^3c + 2bc^3 + c^4) + (b + c)(b^4 + c^4)) \\
\pm abc(b + c)\sqrt{(b + c - a)(c + a - b)(a + b - c)(a^2(b + c) + b^2(c + a) + c^2(a + b) + abc)}
\]

It turns out that the centers of similitude with the Spieker circle (the incircle of the medial triangle) and the Moses circle (the one tangent internally to the nine-point circle at the center of the Kiepert hyperbola) also have rational coordinates in \(a, b, c\):

<table>
<thead>
<tr>
<th>Spieker circle</th>
<th>Moses circle</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>internal</strong></td>
<td><strong>internal</strong></td>
</tr>
<tr>
<td>(a(b + c - a)(a^2(b + c)^2 + a(b + c)(b^2 + c^2) + 2b^2c^2))</td>
<td>(a^2(b + c)^2(a^3 - a(2b^2 - bc + 2c^2) - (b^3 + c^3)))</td>
</tr>
<tr>
<td>(-a(b + c)(b^4 - 2b^3c - 2b^2c^2 - 2bc^3 + c^4) + 2b^2c^2(b + c)^2))</td>
<td>(a^2(a^3(b + c)^2 + 2a^2(b + c)(b^2 + c^2) - abc(b - c)^2))</td>
</tr>
<tr>
<td><strong>external</strong></td>
<td><strong>external</strong></td>
</tr>
<tr>
<td>(a^2(b + c)(b^4 - 2b^3c - 2b^2c^2 - 2bc^3 + c^4) + 2b^2c^2(b + c)^2))</td>
<td>(-(b - c)^2(b + c)(b^2 + bc + c^2)))</td>
</tr>
</tbody>
</table>

References


Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia and Montenegro
E-mail address: milmath@ptt.yu
Two Triangle Centers Associated with the Excircles

Milorad R. Stevanović

Abstract. The triangle formed by the second intersections of the bisectors of a triangle and the respective excircles is perspective to each of the medial and intouch triangles. We identify the perspectors. In the former case, the perspector is closely related to the Yff center of congruence.

1. Introduction

In this note we construct two triangle centers associated with the excircles. Given a triangle $ABC$, let $A'$ be the “second” intersection of the bisector of angle $A$ with the $A$-excircle, which is outside the segment $AI_a$, $I_a$ being the $A$-excenter. Similarly, define $B'$ and $C'$.

![Figure 1](image)

Theorem 1. Triangle $A'B'C'$ is perspective with the medial triangle at the Yff center of congruence of the latter triangle, namely, the point $P$ with homogeneous barycentric coordinates

$$
\left( \sin \frac{B}{2} + \sin \frac{C}{2} : \sin \frac{C}{2} + \sin \frac{A}{2} : \sin \frac{A}{2} + \sin \frac{B}{2} \right)
$$

with respect to $ABC$.

**Theorem 2.** Triangle \( A'B'C' \) is perspective with the intouch triangle at the point \( Q \) with homogeneous barycentric coordinates

\[
\left( \tan \frac{A}{2} \left( \csc \frac{B}{2} + \csc \frac{C}{2} \right) : \tan \frac{B}{2} \left( \csc \frac{C}{2} + \csc \frac{A}{2} \right) : \tan \frac{C}{2} \left( \csc \frac{A}{2} + \csc \frac{B}{2} \right) \right).
\]

**Remark.** These triangle centers now appear as \( X_{2090} \) and \( X_{2091} \) in [2].

2. Notations and preliminaries

We shall make use of the following notations. In a triangle \( ABC \) of sidelengths \( a, b, c \), circumradius \( R \), inradius \( r \), and semiperimeter \( s \), let

\[
s_a = \sin \frac{A}{2}, \quad s_b = \sin \frac{B}{2}, \quad s_c = \sin \frac{C}{2},
\]

\[
c_a = \cos \frac{A}{2}, \quad c_b = \cos \frac{B}{2}, \quad c_c = \cos \frac{C}{2}.
\]

The following formulae can be found, for example, in [1].

\[
r = 4Rs_a s_b s_c, \quad s = 4Rc_a c_b c_c;
\]

\[
s - a = 4Rc_a s_b s_c, \quad s - b = 4Rs_a c_b s_c, \quad s - c = 4Rs_a s_b c_c.
\]

2.1. The medial triangle. The medial triangle \( A_1B_1C_1 \) has vertices the midpoints of the sides \( BC, CA, AB \) of triangle \( ABC \). From

\[
A_1 = \frac{B + C}{2}, \quad B_1 = \frac{C + A}{2}, \quad C_1 = \frac{A + B}{2},
\]

we have

\[
A = B_1 + C_1 - A_1, \quad B = C_1 + A_1 - B_1, \quad C = A_1 + B_1 - C_1. \quad (1)
\]

**Lemma 3.** The barycentric coordinates of the excenters with respect to the medial triangle are

\[
I_a = \frac{s \cdot A_1 - (s - c)B_1 - (s - b)C_1}{s - a},
\]

\[
I_b = \frac{-(s - c)A_1 + s \cdot B_1 - (s - a)C_1}{s - b},
\]

\[
I_c = \frac{-(s - b)A_1 - (s - a)B_1 + s \cdot C_1}{s - c}.
\]

**Proof.** It is enough to compute the coordinates of the excenter \( I_a \):

\[
I_a = \frac{-a \cdot A + b \cdot B + c \cdot C}{b + c - a}
\]

\[
= \frac{-a(B_1 + C_1 - A_1) + b(C_1 + A_1 - B_1) + c(A_1 + B_1 - C_1)}{b + c - a}
\]

\[
= \frac{(a + b + c)A_1 - (a + b - c)B_1 - (c + a - b)C_1}{b + c - a}
\]

\[
= \frac{s \cdot A_1 - (s - c)B_1 - (s - b)C_1}{s - a}.
\]
2.2. The intouch triangle. The vertices of the intouch triangle are the points of tangency of the incircle with the sides. These are

\[
X = \frac{(s-c)B + (s-b)C}{a}, \quad Y = \frac{(s-c)A + (s-a)C}{b}, \quad Z = \frac{(s-b)A + (s-a)B}{c}.
\]

Equivalently,

\[
A = \frac{-a(s-a)X + b(s-b)Y + c(s-c)Z}{2(s-b)(s-c)}, \quad B = \frac{a(s-a)X - b(s-b)Y + c(s-c)Z}{2(s-c)(s-a)}, \quad C = \frac{a(s-a)X + b(s-b)Y - c(s-c)Z}{2(s-a)(s-b)}.
\]

**Lemma 4.** The barycentric coordinates of the excenters with respect to the intouch triangle are

\[
I_a = \frac{a(bc - (s-a)^2)X - b(s-b)^2Y - c(s-c)^2Z}{2(s-a)(s-b)(s-c)},
I_b = \frac{-a(s-a)^2X + b(ca - (s-b)^2)Y - c(s-c)^2Z}{2(s-a)(s-b)(s-c)},
I_c = \frac{-a(s-a)^2X - b(s-b)^2Y + c(ab - (s-c)^2)Z}{2(s-a)(s-b)(s-c)}.
\]

3. Proof of Theorem 1

We compute the barycentric coordinates of \( A' \) with respect to the medial triangle. Note that \( A' \) divides \( AI_a \) externally in the ratio \( AA' : A'I_a = 1 + s_a : -s_a \). It follows that

\[
A' = (1 + s_a)I_a - s_a \cdot A
= \frac{1 + s_a}{s - a} (s \cdot A_1 - (s-c)B_1 - (s-b)C_1) - s_a (B_1 + C_1 - A_1).
\]

From this, the homogeneous barycentric coordinates of \( A' \) with respect to \( A_1B_1C_1 \) are

\[
(1 + s_a)s + s_a(s - a) : -(1 + s_a)(s - c) - s_a(s - a) \\
: -((1 + s_a)(s - b) - s_a(s - a) \\
= s + s_a(b + c) : -(s - c) + s_a b) : -(s - b) + s_a c) \\
= 4Rc_ac_bc_c + 4Rs_a(s_b + s_c)c_a) : -4R(s_a(s_b + s_c) + s_a s_b) : -4R(s_a s_b + s_a s_c) \\
= -\frac{c_a c_b c_c + s_a (s_b + s_c c_c)}{s_a (c_b + c_c)} : s_b : s_c.
\]
Similarly,

\[ B' = \left( s_a : -\frac{c_a c_b c_c + s_b (s_c c_c + s_a c_a)}{s_b (c_c + c_a)} : s_c \right), \]

\[ C' = \left( s_a : s_b : -\frac{c_a c_b c_c + s_c (s_a c_a + s_b c_b)}{s_c (c_a + c_b)} \right). \]

From these, it is clear that \( A'B'C' \) and the medial triangle are perspective at the point with coordinates \( (s_a : s_b : s_c) \) relative to \( A_1 B_1 C_1 \). This is clearly the Yff center of congruence of the medial triangle. See Figure 2. Its coordinates with respect to \( ABC \) are

\[ (s_b + s_c : s_c + s_a : s_a + s_b). \]

This completes the proof of Theorem 1.
Remark. In triangle $ABC$, let $A''$, $B''$, $C''$ be the feet of the bisectors of angles $BIC$, $CIA$, $AIB$ respectively on sides $BC$, $CA$, $AB$. Triangles $A''B''C''$ and $ABC$ are perspective at the Yff center of congruence $X_{174}$, i.e., if the perpendiculars from $X_{174}$ to the bisectors of the angles of $ABC$ intersect the sides of triangle $ABC$ at $X_b$, $X_c$, $Y_a$, $Y_c$, $Z_a$, $Z_b$ (see Figure 3), then the triangles $X_{174}X_bX_c$, $Y_aX_{174}Y_c$ and $Z_aZ_bX_{174}$ are congruent. See [3].

4. Proof of Theorem 2

Consider the coordinates of $A' = (1 + s_a)I_a - s_a \cdot A$ with respect to the intouch triangle $XYZ$. By Lemma 3, the $Y$-coordinate is

$$\frac{-(1 + s_a)b(s - b)^2 - s_a b(s - a)(s - b)}{2(s - a)(s - b)(s - c)}$$

$$= \frac{-b(s - b)((1 + s_a)(s - b) + s_a(s - a))}{2(s - a)(s - b)(s - c)}$$

$$= \frac{-b(s - b)(s - b + s_a \cdot c)}{2(s - a)(s - b)(s - c)}$$

$$= \frac{-(c_b + c_c)}{2c_ac_bc_c} \cdot \frac{c_b^2}{s_b}$$
Similarly for the Z-coordinate is $\frac{-(c_b+c_c)}{2ca_c} \cdot \frac{c_c^2}{s_c}$. Therefore, $A'B'C'$ is perspective with $XYZ$ at

$$Q = \left( \frac{c_a^2}{s_a}, \frac{c_b^2}{s_b}, \frac{c_c^2}{s_c} \right).$$

Note that the angles of the intouch triangles are $X = \frac{B+C}{2}$, $Y = \frac{C+A}{2}$, and $Z = \frac{A+B}{2}$. This means

$$s_a = \cos \frac{B+C}{2} = \cos X, \quad c_a = \sin \frac{B+C}{2} = \sin X,$$

e tc. It follows that $Q$ has homogeneous barycentric coordinates

$$\left( \frac{\sin^2 X}{\cos X}, \frac{\sin^2 Y}{\cos Y}, \frac{\sin^2 Z}{\cos Z} \right).$$
and is the Clawson point of the intouch triangle \(XYZ\). With respect to triangle \(ABC\), this perspector \(Q\) has coordinates given by

\[
\left( \frac{a(s-a)}{s_a} + \frac{b(s-b)}{s_b} + \frac{c(s-c)}{s_c} \right) Q
\]

\[
= \frac{a(s-a)X}{s_a} + \frac{b(s-b)Y}{s_b} + \frac{c(s-c)Z}{s_c}
\]

\[
= \frac{(s-b)(s-c)(s_b+s_c)}{s_b s_c} A + \frac{(s-c)(s-a)(s_c+s_a)}{s_c s_a} B + \frac{(s-a)(s-b)(s_a+s_b)}{s_a s_b} C
\]

\[
= (4R)^2 s_a^2 c_b c_c (s_b+s_c) A + (4R)^2 s_b^2 c_c c_a (s_c+s_a) B + (4R)^2 s_c^2 c_a c_b (s_a+s_b) C
\]

\[
= (4R)^2 c_a c_b c_c \left( \frac{s_a^2 (s_b+s_c)}{c_a} : \frac{s_b^2 (s_c+s_a)}{c_b} : \frac{s_c^2 (s_a+s_b)}{c_c} \right).
\]

Therefore, the homogeneous barycentric coordinates of \(Q\) with respect to \(ABC\) are

\[
\left( \frac{s_a^2 (s_b+s_c)}{c_a} : \frac{s_b^2 (s_c+s_a)}{c_b} : \frac{s_c^2 (s_a+s_b)}{c_c} \right)
\]

\[
= \left( \tan \frac{A}{2} \left( \csc \frac{B}{2} + \csc \frac{C}{2} \right) : \tan \frac{B}{2} \left( \csc \frac{C}{2} + \csc \frac{A}{2} \right) : \tan \frac{C}{2} \left( \csc \frac{A}{2} + \csc \frac{B}{2} \right) \right).
\]

This completes the proof of Theorem 2.

Inasmuch as \(Q\) is the Clawson point of the intouch triangle, it is interesting to point out that the congruent isosceizers point \(X_{173}\), a point closely related to the Yff center of congruence \(X_{174}\) and with coordinates

\[
(a(-c_a+c_b+c_c) : b(c_a-c_b+c_c) : c(c_a+c_b-c_c)),
\]

is the Clawson point of the excentral triangle \(I_a I_b I_c\) (which is homothetic to the intouch triangle at \(X_{177}\)). This fact was stated in an earlier edition of [2], and can be easily proved by the method of this paper.

References


Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia and Montenegro
E-mail address: milmath@ptt.yu
A 5-step Division of a Segment in the Golden Section

Kurt Hofstetter

Abstract. Using ruler and compass only in five steps, we divide a given segment in the golden section.

Inasmuch as we have given in [1] a construction of the golden section by drawing 5 circular arcs, we present here a very simple division of a given segment in the golden section, in 5 euclidean steps, using ruler and compass only. For two points $P$ and $Q$, we denote by $P(Q)$ the circle with $P$ as center and $PQ$ as radius.

Construction. Given a segment $AB$, construct

(1) $C_1 = A(B)$,
(2) $C_2 = B(A)$, intersecting $C_1$ at $C$ and $D$,
(3) $C_3 = C(A)$, intersecting $C_1$ again at $E$,
(4) the segment $CD$ to intersect $C_3$ at $F$,
(5) $C_4 = E(F)$ to intersect $AB$ at $G$.

The point $G$ divides the segment $AB$ in the golden section.

Proof. Suppose $AB$ has unit length. Then $CD = \sqrt{3}$ and $EG = EF = \sqrt{2}$.
Let $H$ be the orthogonal projection of $E$ on the line $AB$. Since $HA = \frac{1}{2}$, and $HG^2 = EG^2 - EH^2 = 2 - \frac{3}{4} = \frac{5}{4}$, we have $AG = HG - HA = \frac{1}{2}(\sqrt{5} - 1)$. This shows that $G$ divides $AB$ in the golden section.

Remark. The other intersection $G'$ of $C_4$ and the line $AB$ is such that $G'A : AB = \frac{1}{2}(\sqrt{5} + 1) : 1$.

References


Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: pendel@sunpendulum.at
Circumcenters of Residual Triangles

Eckart Schmidt

Abstract. This paper is an extension of Mario Dalcín’s work on isotomic inscribed triangles and their residuals [1]. Considering the circumcircles of residual triangles with respect to isotomic inscribed triangles there are two congruent triangles of circumcenters. We show that there is a rotation mapping these triangles to each other. The center and angle of rotation depend on the Miquel points. Furthermore we give an interesting generalization of Dalcín’s definitive example.

1. Introduction

If \(X, Y, Z\) are points on the sides of a triangle \(ABC\), there are three residual triangles \(AZY, BXZ, CYX\). The circumcenters of these triangles form a triangle \(O_aO_bO_c\) similar to the reference triangle \(ABC\) [2]. The circumcircles have a common point \(M\) by Miquel’s theorem. The lines \(MX, MY, MZ\) and the corresponding side lines have the same angle of intersection \(\mu = (AY, YM) = (BZ, ZM) = (CX, XM)\). The angles are directed angles measured between 0 and \(\pi\).

Dalcín considers isotomic inscribed triangles \(XYZ\) and \(X'Y'Z'\). Here, \(X'\), \(Y'\), \(Z'\) are the reflections of \(X, Y, Z\) in the midpoints of the respective sides. The triangle \(XYZ\) may or may not be cevian. If it is the cevian triangle of a point \(P\), then \(X'Y'Z'\) is the cevian triangle of the isotomic conjugate of \(P\). The

The author thanks the editor for his helps in the preparation of this paper.
corresponding Miquel point $M'$ of $X', Y', Z'$ has Miquel angle $\mu' = \pi - \mu$. The circumcircles of the residual triangles $AZY', BX'Z', CY'X'$ give further points of intersection. The intersections $A'$ of the circles $AZY'$ and $AZY'$, $B'$ of $BXZ'$ and $BX'Z'$, and $C'$ of $CYX'$ and $CY'X'$ form a triangle $A'B'C'$ perspective to the reference triangle $ABC$ with the center of perspectivity $Q$. See Figure 2. It can be shown that the points $M, M', A', B', C', Q$ and the circumcenter $O$ of the reference triangle lie on a circle with the diameter $OQ$.

These results can be proved by analytical calculations. We make use of homogeneous barycentric coordinates. Let $X, Y, Z$ divide the sides $BC, CA, AB$ respectively in the ratios

$$BX : XC = x : 1, \quad CY : YA = y : 1, \quad AZ : ZB = z : 1.$$

These points have coordinates

$$X = (0 : 1 : x), \quad Y = (y : 0 : 1), \quad Z = (1 : z : 0);$$
$$X' = (0 : x : 1), \quad Y' = (1 : 0 : y), \quad Z' = (z : 1 : 0).$$
The circumcenter, the Miquel points, and the center of perspectivity are the points

\[ O = \left( a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2) \right), \]
\[ M = \left( a^2x(1 + y)(1 + z) - b^2xy(1 + x)(1 + z) - c^2(1 + x)(1 + y) : \cdots : \cdots \right), \]
\[ M' = \left( a^2x(1 + y)(1 + z) - b^2(1 + x)(1 + z) - c^2xz(1 + x)(1 + y) : \cdots : \cdots \right), \]
\[ Q = \left( \frac{(1 - x)a^2}{1 + x} : \frac{(1 - y)b^2}{1 + y} : \frac{(1 - z)c^2}{1 + z} \right). \]

The Miquel angle \( \mu \) is given by

\[ \cot \mu = \frac{1 - yz}{(1 + y)(1 + z)} \cot A + \frac{1 - zx}{(1 + z)(1 + x)} \cot B + \frac{1 - xy}{(1 + x)(1 + y)} \cot C. \]

For example, let \( X, Y, Z \) divide the sides in the same ratio \( k \), i.e., \( x = y = z = k \), then we have

\[ M = \left( a^2(-c^2 + a^2k - b^2k^2) : b^2(-a^2 + b^2k - c^2k^2) : c^2(-b^2 + c^2k - a^2k^2) \right), \]
\[ M' = \left( a^2(-b^2 + a^2k - c^2k^2) : b^2(-c^2 + b^2k - a^2k^2) : c^2(-a^2 + c^2k - b^2k^2) \right), \]
\[ Q = \left( a^2 : b^2 : c^2 \right) = X_6(\text{Lemoine point}); \]
\[ \cot \mu = \frac{1 - k}{1 + k} \cot \omega, \]

where \( \omega \) is the Brocard angle.

2. Two triangles of circumcenters

Considering the circumcenters of the residual triangles for \( XYZ \) and \( XYZ'Z' \), Dalcín ([1, Theorem 10]) has shown that the triangles \( O_aO_bO_c \) and \( O'_aO'_bO'_c \) are congruent. We show that there is a rotation mapping \( O_aO_bO_c \) to \( O'_aO'_bO'_c \). This rotation also maps the Miquel point \( M \) to the circumcenter \( O \), and \( O \) to the other Miquel point \( M' \). See Figure 3. The center of rotation is therefore the midpoint of \( OQ \). This center of rotation is situated with respect to \( O_aO_bO_c \) and \( O'_aO'_bO'_c \) as the center of perspectivity with respect to the reference triangle \( ABC \). The angle \( \varphi \) of rotation is given by

\[ \varphi = \pi - 2\mu. \]

The similarity ratio of triangles \( O_aO_bO_c \) and \( ABC \) is

\[ \frac{1}{2 \cos \frac{\varphi}{2}} = \frac{1}{2 \sin \mu}, \]

similarly for triangle \( O'_aO'_bO'_c \).
3. Dalcín’s example

If we choose $X$, $Y$, $Z$ as the points of tangency of the incircle with the sides, $XYZ$ is the cevian triangle of the Gergonne point $G_e$ and $X'Y'Z'$ is the cevian triangle of the Nagel point $N_a$. The Miquel point $M$ is the incenter $I$ and the Miquel point $M'$ is the reflection of $I$ in $O$, i.e.,

$$X_{40} = (a(a^3 - b^3 - c^3 + (a - b)(a - c)(b + c)) : \cdots : \cdots).$$

In this case, $O_aO_bO_c$ is homothetic to $ABC$ at $M$, with factor $\frac{1}{2}$. This is also the case when $XYZ$ is the cevian triangle of the Nagel point, with $M = X_{40}$.

Therefore, the circle described in §2, degenerates into a line. The center of perspectivity $Q(a(b - c) : b(c - a) : c(a - b))$ is a point of infinity. The triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are homothetic to the triangle $ABC$ at the Miquel points $M$ and $M'$ with factor $\frac{1}{2}$. There is a parallel translation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$.

The fact that $ABC$ is homothetic to $OaObOc$ with the factor $\frac{1}{2}$ does not only hold for the Gergonne and Nagel points. Here are further examples.
Circumcenters of residual triangles

<table>
<thead>
<tr>
<th>$P$</th>
<th>Homothetic center and Miquel point $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>centroid $G$</td>
<td>circumcenter $O$</td>
</tr>
<tr>
<td>orthocenter $H$</td>
<td>$H$</td>
</tr>
<tr>
<td>$X_{69}$</td>
<td>$X_{20}$</td>
</tr>
<tr>
<td>$X_{189}$</td>
<td>$X_{84}$</td>
</tr>
<tr>
<td>$X_{253}$</td>
<td>$X_{64}$</td>
</tr>
<tr>
<td>$X_{329}$</td>
<td>$X_{1490}$</td>
</tr>
</tbody>
</table>

These points $P(u : v : w)$, whose cevian triangle is also the pedal triangle of the point $M$, lie on the Lucas cubic

$$(b^2 + c^2 - a^2)u(v^2 - w^2) + (c^2 + a^2 - b^2)v(w^2 - u^2) + (a^2 + b^2 - c^2)w(u^2 - v^2) = 0.$$  

The points $M$ lie on the Darboux cubic. Isotomic points $P$ and $P^\Lambda$ on the Lucas cubic have corresponding points $M$ and $M'$ on the Darboux cubic symmetric with respect to the circumcenter. Isogonal points $M$ and $M^*$ on the Darboux cubic have

---

1. The Lucas cubic is invariant under the isotomic conjugation and the isotomic conjugate $X_{69}$ of the orthocenter is the pivot point.

2. The Darboux cubic is invariant under the isogonal conjugation and the pivot point is the De-Longchamps point $X_{20}$, the reflection of the orthocenter in the circumcenter. It is symmetric with respect to the circumcenter.
corresponding points $P$ and $P'$ on the Lucas cubic with $P' = P^{\wedge \wedge \wedge}$. Here, $()^{\ast}$ is the isogonal conjugation with respect to the anticomplementary triangle of $ABC$. The line $PM$ and $MM^{\ast}$ all correspond with the DeLongchamps point $X_{20}$ and so the points $P$, $P^{\wedge \wedge \wedge}$, $M$, $M^{\ast}$ and $X_{20}$ are collinear. For example, for $P = N_a$, the five points $N_a$, $X_{189}$, $X_{40}$, $X_{84}$, $X_{20}$ are collinear.

4. Further results

Dalcín’s example can be extended. The cevian triangle of the Gergonne point $G_e$ is the triangle of tangency of the incircle, the cevian triangle of the Nagel point $N_a$ is the triangle of the inner points of tangency of the excircles. Consider the points of tangency of the excircles with the sidelines:
Circumcenters of residual triangles

<table>
<thead>
<tr>
<th>Excircle</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-excircle</td>
<td>[ B_a = (-a + b - c : 0 : a + b + c) \text{ with } CA ]</td>
</tr>
<tr>
<td></td>
<td>[ C_a = (-a - b + c : a + b + c : 0) \text{ with } AB ]</td>
</tr>
<tr>
<td>B-excircle</td>
<td>[ A_b = (0 : a - b - c : a + b + c) \text{ with } BC ]</td>
</tr>
<tr>
<td></td>
<td>[ C_b = (a + b + c : -a - b + c : 0) \text{ with } AB ]</td>
</tr>
<tr>
<td>C-excircle</td>
<td>[ A_c = (0 : a + b + c : a - b - c) \text{ with } BC ]</td>
</tr>
<tr>
<td></td>
<td>[ B_c = (a + b + c : 0 : -a + b - c) \text{ with } CA ]</td>
</tr>
</tbody>
</table>

The point pairs \((A_b, A_c), (B_c, B_a)\) and \((C_a, C_b)\) are symmetric with respect to the corresponding midpoints of the sides. If \(XYZ = A_bB_cC_a\), then \(X'Y'Z' = A_cB_aC_b\). See Figure 6.

Consider the residual triangles of \(A_bB_cC_a\) and those of \(A_cB_aC_b\), with the circumcenters. The two congruent triangles \(O_aO_bO_c\) and \(O'_aO'_bO'_c\) have a common area

\[
\frac{\Delta}{4} + \frac{(ab + bc + ca)^2}{16\Delta}.
\]

The center of perspectivity is

\[ Q = (a(b + c) : b(c + a) : c(a + b)) = X_{37}. \]

The center of rotation which maps \(O_aO_bO_c\) to \(O'_aO'_bO'_c\) is the midpoint of \(OQ\). The point \(X_{37}\) of a triangle is the complement of the isotomic conjugate of the incenter. The center of rotation is the common point \(X_{37}\) of \(O_aO_bO_c\) and \(O'_aO'_bO'_c\). The angle of rotation is given by

\[
\tan \frac{\varphi}{2} = \frac{ab + bc + ca}{2\Delta} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}.
\]
References


Eckart Schmidt: Hasenberg 27 - D 24223 Raisdorf, Germany
E-mail address: eckart.schmidt@t-online.de
Circumrhombi

Floor van Lamoen

Abstract. We consider rhombi circumscribing a given triangle $ABC$ in the sense that one vertex of the rhombus coincides with a vertex of $ABC$, while the sidelines of the rhombus opposite to this vertex pass through the two remaining vertices of $ABC$ respectively. We construct some new triangle centers associated with these rhombi.

1. Introduction

In this paper we further study the rhombi circumscribing a given reference triangle $ABC$ that the author defined in [4]. These rhombi circumscribe $ABC$ in the sense that each of them shares one vertex with $ABC$, with its two opposite sides passing through the two remaining vertices of $ABC$. These rhombi will depend on a fixed angle $\phi$ and its complement $\overline{\phi} = \frac{\pi}{2} - \phi$. More precisely, for a given $\phi$, the rhombus $R_A(\phi) = AA_cA_aA_b$ will be such that $\angle A_bAA_c = 2\phi$, $B \in A_cA_a$ and $C \in A_bA_a$. Similarly there are rhombi $BB_aB_bB_c$ and $CC_bC_cC_a$.

In [4] it was shown that the vertices of the rhombi opposite to $ABC$ form a triangle $A_aB_bC_c$ perspective to $ABC$, and that their perspector lies on the Kiepert hyperbola. We give another proof of this result (Theorem 3).

We denote by $K(\phi) = A^\phi B^\phi C^\phi$ the Kiepert triangle formed by isosceles triangles built on the sides of $ABC$ with base angles $\phi$. When the isosceles triangles are constructed outwardly, $\phi > 0$. Otherwise, $\phi < 0$. These vertices have homogeneous barycentric coordinates

$$
A^\phi = (-S_B + S_C : S_C + S_\phi : S_B + S_\phi),
B^\phi = (S_C + S_\phi : -(S_C + S_A) : S_A + S_\phi),
C^\phi = (S_B + S_\phi : S_A + S_\phi : -(S_A + S_B)).
$$

From these it is clear that $K(\phi)$ is perspective with $ABC$ at the point

$$
K(\phi) = \left( \frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi} \right).
$$


\footnote{For the notations, see [5].}
2. Circumrhombi to a triangle

**Theorem 1.** Consider \( \triangle ABC \) and \( \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\} \). There are unique rhombi \( R_A(\phi) = AA_cA_aA_b, R_B(\phi) = BB_aB_bB_c \) and \( R_C(\phi) = CC_bC_cC_a \) with

\[
\angle A_bAA_c = \angle B_cBB_a = \angle C_aCC_b = 2\phi,
\]

and \( B \in A_cA_a \) and \( C \in A_bA_a \). Similarly there are rhombi \( C \in B_bB_a, A \in B_cB_b, A \in C_bC_c, B \in C_aC_c \).

**Proof.** It is enough to show the construction of \( R_A = R_A(\phi) \).

Let \( B_r \) be the image of \( B \) after a rotation through \(-2\phi\) about \( A \), and \( C_r \) the image of \( C \) after a rotation through \( 2\phi\) about \( A \). Then let \( A_a = B_rC \cap C_rB \). Points \( A_c \in C_rA_a \) and \( A_b \in B_rA_a \) can be constructed in such a way that \( AA_cA_aA_b \) is a parallelogram. Observe that \( \triangle AC_rB \equiv \triangle ACB \), so that the perpendicular distances from \( A \) to lines \( B_rA_a \) and \( C_rA_a \) are equal. And \( AA_cA_aA_b \) must be a rhombus. See Figure 1.

![Figure 1](image_url)

Note that line \( B_rC = A_aA_b \) is the image of line \( C_rB = A_aA_c \) after rotation through \( 2\phi \) about \( A \), so that the directed angle \( \angle A_cA_aA_b = 2\phi \). It follows that \( AA_cA_aA_b \) is the rhombus desired in the theorem.

It is easy to see that this is the unique rhombus fulfilling these requirements. When we rotate the complete figure of \( \triangle ABC \) and rhombus \( AA_cA_aA_b \) through \(-2\phi\) about \( A \), and let \( B_r \) be the image of \( B \) again, we see immediately that \( B_r \in A_aC \). In the same way we see that the image of \( C \) after rotation through \( 2\phi \) about \( A \) must be on the line \( A_aB \). \( \square \)
Consider $\mathcal{R}_A$ and $\mathcal{R}_B$. We note that $\angle A_A B \equiv \phi \mod \pi$ and also $\angle A_B B \equiv \phi \mod \pi$. This means that $ABA_B$ is cyclic. The center $P$ of its circle should be the apex of the isosceles triangle built on $AB$ such that $\angle APB = 2\phi$, so that $P = C^{\phi}$. This shows that $C^{\phi}$ lies on the perpendicular bisectors of $AA_a$ and $BB_b$, hence $A_b A_c \cap B_c A_c = C^{\phi}$. See Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

**Theorem 2.** The diagonals $A_b A_c$, $B_a B_c$ and $C_a C_b$ of the circumrhombi $\mathcal{R}_A(\phi)$, $\mathcal{R}_B(\phi)$, $\mathcal{R}_C(\phi)$ bound the Kiepert triangle $\mathcal{K}(\phi)$.

3. Radical center of a triad of circles

It is now interesting to further study the circles $A^{\phi}(B)$, $B^{\phi}(C)$ and $C^{\phi}(A)$ with centers at the apices of $\mathcal{K}(\phi)$, passing through the vertices of $ABC$. Since the circle $A^{\phi}(B)$ passes through $B$ and $C$, it is represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - kx(x + y + z) = 0.$$ 

Since it also passes through $A^{\phi/2} = (-S_B + S_C) : S_C - S_{\phi/2} : S_B - S_{\phi/2}$, we find

$$k = \frac{S_{\phi} + 2S_A S_{\phi/2} - S^2}{2S_{\phi/2}} = S_A + S_{\phi},$$

\footnote{Hence, when $\phi$ is negative, the apex is on the same side of $AB$ as the vertex $C$.}
The equations of the three circles are thus
\[
\begin{align*}
    a^2yz + b^2zx + c^2xy - (S_A + S_\phi)x(x + y + z) &= 0, \\
    a^2yz + b^2zx + c^2xy - (S_B + S_\phi)y(x + y + z) &= 0, \\
    a^2yz + b^2zx + c^2xy - (S_C + S_\phi)z(x + y + z) &= 0.
\end{align*}
\]

From this, it is clear that the radical center of the three circles is the point
\[
K(\phi) = \left( \frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi} \right).
\]

The intersections of the circles apart from \(A, B\) and \(C\) are the points
\[
\begin{align*}
    A_a &= \left( \frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi} \right), \\
    B_b &= \left( \frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi} \right), \\
    C_c &= \left( \frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}} \right).
\end{align*}
\]

**Theorem 3.** The triangle \(A_aB_bC_c\) is perspective to \(ABC\) and the perspector is \(K(\phi)\).

**Remark.** For \(\phi = \pm \frac{\pi}{3}\), triangle \(A_aB_bC_c\) degenerates into the Fermat point \(K\left(\pm \frac{\pi}{3}\right)\).
The coordinates of the circumcenter of $A_nB_nC_n$ are too complicated to record here, even in the case of circumsquares. However, we prove the following interesting collinearity.

**Theorem 4.** The circumcenters of triangles $ABC$ and $A_nB_nC_n$ are collinear with $K(\phi)$.

**Proof.** Since $P = K(\phi)$ is the radical center of $A^\phi(B)$, $B^\phi(C)$ and $C^\phi(A)$ we see that

$$
PA \cdot PA = PB \cdot PB_b = PC \cdot PC_c,
$$

which product we will denote by $\Gamma$. When $\Gamma > 0$ then the inversion with center $P$ and radius $\sqrt{\Gamma}$ maps $A$ to $A_n$, $B$ to $B_n$ and $C$ to $C_n$. Consequently the circumcircles of $ABC$ and $A_nB_nC_n$ are inverses of each other, and the centers of these circles are collinear with the center of inversion.

When $\Gamma < 0$ then the inversion with center $P$ and radius $\sqrt{-\Gamma}$ maps $A$, $B$ and $C$ to the reflections of $A_n$, $B_n$ and $C_n$ through $P$. And the collinearity follows in the same way as above.

When $\Gamma = 0$ the theorem is trivial. $\square$

4. **Coordinates of the vertices of the circumrhombi**

Along with the coordinates given in (1), we record those of the remaining vertices of the circumrhombi.

$$
A_n = ((b^2 + S \csc 2\phi)(S_B + S_\phi) : (S_A - S_2\phi)(b^2 + S \csc 2\phi) : -(S_A - S_2\phi)^2),
A = ((c^2 + S \csc 2\phi)(S_C + S_\phi) : -(S_A - S_2\phi)^2 : (S_A - S_2\phi)(c^2 + S \csc 2\phi));
B_n = -(S_B - S_2\phi)^2 : (c^2 + S \csc 2\phi)(S_C + S_\phi) : (S_B - S_2\phi)(c^2 + S \csc 2\phi)),
B = (S_B - S_2\phi)(a^2 + S \csc 2\phi) : (a^2 + S \csc 2\phi)(S_A + S_\phi) : -(S_B - S_2\phi)^2);
C_n = (S_C - S_2\phi)(a^2 + S \csc 2\phi) : -(S_C - S_2\phi)^2 : (a^2 + S \csc 2\phi)(S_A + S_\phi)),
C = -(S_C - S_2\phi)^2 : (S_C - S_2\phi)(b^2 + S \csc 2\phi) : (b^2 + S \csc 2\phi)(S_B + S_\phi)).
$$

5. **The triangle $A'B'C'$**

Let $A' = CC_a \cap BB_a$, $B' = CC_b \cap AA_b$ and $C' = AA_c \cap BB_c$. The coordinates of $A'$, using (2), are

$$
A' = (a^2 + S \csc 2\phi : -(S_C - S_2\phi) : -(S_B - S_2\phi)) = \left(\frac{a^2 + S \csc 2\phi}{(S_B - S_2\phi)(S_C - S_2\phi)} : -\frac{1}{S_B - S_2\phi} : -\frac{1}{S_C - S_2\phi}\right);
$$

Similarly for $B'$ and $C'$. It is clear that $A'B'C'$ is perspective to $ABC$ at $K(-2\phi)$. Note that in absolute barycentric coordinates,
\[ S(csc 2\phi + 2 \cot 2\phi)A' \]
\[ = (a^2 + S csc 2\phi, -(S_C - S_{2\phi}), -(S_B - S_{2\phi})) \]
\[ = (S_B + S_C, -(S_C + S_{2\phi}), -(S_B + S_{2\phi})) + S(csc 2\phi, \cot 2\phi + \tan \phi, \cot 2\phi + \tan \phi) \]
\[ = (S_B + S_C, -(S_C + S_{2\phi}), -(S_B + S_{2\phi})) + S csc 2\phi(1, 1, 1) \]
\[ = S(-2 \tan \phi A' + 3 csc 2\phi G). \]

Now, \[ -2 \tan \phi + 3 csc 2\phi = \frac{4}{1 - 3 \cot^2 \phi}. \] It follows that
\[ A' = h \left( G, \frac{4}{1 - 3 \cot^2 \phi} \right) (A\overline{G}). \]

Similarly for \( B' \) and \( C' \).

**Proposition 5.** Triangles \( A'B'C' \) and \( K(\overline{\phi}) \) are homothetic at \( G \).

**Corollary 6.** \( ABC \) is the Kiepert triangle \( K(-\phi) \) with respect to \( A'B'C' \).

See [5, Proposition 4].
6. The desmic mates

Let $XYZ$ be a triangle perspective with $ABC$ at $P = (u : v : w)$. Its vertices have coordinates

$$X = (x : v : w), \quad Y = (u : y : w), \quad Z = (u : v : z),$$

for some $x, y, z$. The desmic mate of $XYZ$ is the triangle with vertices $X' = BZ \cap CY$, $Y' = CX \cap AZ$, $Z' = AY \cap BX$. These have coordinates

$$X' = (u : y : z), \quad Y' = (x : v : z), \quad Z' = (x : y : w).$$

**Lemma 7.** The triangle $X'Y'Z'$ is perspective to $ABC$ at $(x : y : z)$ and to $XYZ$ at $(u + x : v + y : w + z)$.

See, for example, [1, §4].

The desmic mate of $A_aB_bC_a$ has vertices

$$A'_a = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_2\phi} : \frac{1}{S_C - S_2\phi}\right),$$

$$B'_b = \left(\frac{1}{S_A - S_2\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_2\phi}\right),$$

$$C'_c = \left(\frac{1}{S_A - S_2\phi} : \frac{1}{S_B - S_2\phi} : \frac{1}{S_C + S_\phi}\right).$$
Proof. (1) is clear from the coordinates given in (4). Since $A = (S_A + S \csc 2\phi : (S_A + S_2\phi) : (S_A + S_2\phi))$, $B = (S_B + S \csc 2\phi : (S_B + S_2\phi) : (S_B + S_2\phi))$, and $C = (S_C + S \csc 2\phi : (S_C + S_2\phi) : (S_C + S_2\phi))$, we have

$$P_1(\phi) = \left( \frac{2S_A + S \csc 2\phi}{(S_A + S_2\phi)(S_A + S_2\phi)} : \frac{2S_B + S \csc 2\phi}{(S_B + S_2\phi)(S_B + S_2\phi)} : \frac{2S_C + S \csc 2\phi}{(S_C + S_2\phi)(S_C + S_2\phi)} \right).$$

See Figure 5.

The desmic mate of $A'B'C'$ has vertices

$$A'' = -(S_B - S_2\phi)(S_C - S_2\phi) : (S_B - S_2\phi)(b^2 + S \csc 2\phi) : (S_C - S_2\phi)(c^2 + S \csc 2\phi);$$

$$B'' = ((S_A - S_2\phi)(a^2 + S \csc 2\phi) : -(S_C - S_2\phi)(S_A - S_2\phi) : (S_C - S_2\phi)(c^2 + S \csc 2\phi));$$

$$C'' = ((S_A - S_2\phi)(a^2 + S \csc 2\phi) : (S_B - S_2\phi)(b^2 + S \csc 2\phi) : - (S_A - S_2\phi)(S_B - S_2\phi)).$$

Proposition 9. Triangle $A''B''C''$ is perspective to $ABC$ at $K(-2\phi)$. It is also perspective to $A_bB_cC'$ at

$$P_2(\phi) = ((S_A - S_2\phi)(a^2 + S \csc 2\phi) : \cdots : \cdots),$$

(2) $A'B'C'$ at

$$P_3(\phi) = ((a^2S_A - S_{BC}) - S \csc 2\phi(S_A - S_2\phi \cos 2\phi) : \cdots : \cdots),$$

(3) the dilated triangle\(^3\) at

$$P_4(\phi) = (S_B + S_C - S_A - S_2\phi : \cdots : \cdots).$$

Proof. (1) is clear from the coordinates given in (4). Since

$$(a^2 + S \csc 2\phi)(S_A - S \cot 2\phi) - (S_B - S \cot 2\phi)(S_C - S \cot 2\phi)$$

$$= (a^2S_A - S_{BC}) + S^2 \csc 2\phi \cot A - S \cot 2\phi(a^2 + S \csc 2\phi - (S_B + S_C) + S \cot 2\phi)$$

$$= (a^2S_A - S_{BC}) + S^2 \csc 2\phi \cot A - S^2 \cot 2\phi \cot \phi$$

$$= (a^2S_A - S_{BC}) - S_A S \csc 2\phi + S_2\phi \cot \phi$$

$$= (a^2S_A - S_{BC}) - S \csc 2\phi(S_A - S_2\phi \cos 2\phi),$$

it follows from Lemma 7 that $A''B''C''$ is perspective to $A'B'C'$ at

$$\left( \frac{a^2 + S \csc 2\phi}{(S_B - S_{2\phi})(S_C - S_{2\phi})} - \frac{1}{S_A - S_{2\phi}} : \cdots : \cdots \right)$$

$$= ((a^2S_A - S_{BC}) - S \csc 2\phi(S_A - S_2\phi \cos 2\phi) : \cdots : \cdots).$$

\(^3\)This is also called the anticomplementary triangle, it is formed by the lines through the vertices of $ABC$, parallel to the corresponding opposite sides.
This proves (2). For (3), we rewrite the coordinates for \( A'' \) as
\[
A'' = (S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1) \\
+ (S \csc(2\phi) + S_{2\phi} + S_A) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi}) \\
= (S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1) \\
+ (S_A + S_{\phi}) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi})
\]

From this we see that \( A'' \) is on the line connecting the \( A \)-vertices of the dilated triangle and the cevian triangle of the isotomic conjugate of \( K(-2\phi) \), namely, the point
\[
K^*(-2\phi) = (S_A - S_{2\phi} : S_B - S_{2\phi} : S_C - S_{2\phi}).
\]
This shows that \( A''B''C'' \) is perspective to both triangles, and that the perspector is the cevian quotient \( K^*(-2\phi)/G \), \(^4\) where \( G \) denotes the centroid. It is easy to see that this is the superior of \( K^*(-2\phi) \). Equivalently, it is \( K^*(-2\phi) \) of the dilated triangle, with coordinates
\[
(S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).
\]

We conclude with a table showing the triangle centers associated with the circum-squares, when \( \phi = \pm \frac{\pi}{4} \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( P_k(\pi/4) )</th>
<th>( P_k(-\pi/4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( K(\pi/4) )</td>
<td>( K(-\pi/4) )</td>
</tr>
<tr>
<td>2</td>
<td>circumcenter</td>
<td>circumcenter</td>
</tr>
<tr>
<td>3</td>
<td>de Longchamps point</td>
<td>de Longchamps point</td>
</tr>
<tr>
<td>4</td>
<td>( X_{193} )</td>
<td>( X_{193} )</td>
</tr>
</tbody>
</table>

References


Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands

E-mail address: f.v.lamoen@wxs.nl

---

\(^4\)The cevian quotient \( X/Y \) is the perspector of the cevian triangle of \( X \) and the precevian triangle of \( Y \). This is the \( X \)-Ceva conjugate of \( Y \) in the terminology of [2].
Sawayama and Thébault’s theorem

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of Thébault’s theorem, known earlier to Y. Sawayama.

1. Introduction

In 1938 in a “Problems and Solutions” section of the Monthly [24], the famous French problemist Victor Thébault (1882-1960) proposed a problem about three circles with collinear centers (see Figure 1) to which he added a correct ratio and a relation which finally turned out to be wrong. The date of the first three metric solutions [22] which appeared discreetly in 1973 in the Netherlands was more widely known in 1989 when the Canadian revue *Crux Mathematicorum* [27] published the simplified solution by Veldkamp who was one of the two first authors to prove the theorem in the Netherlands [26, 5, 6]. It was necessary to wait until the end of this same year when the Swiss R. Stark, a teacher of the Kantonsschule of Schaffhausen, published in the Helvetic revue *Elemente der Mathematik* [21] the first synthetic solution of a “more general problem” in which the one of Thébault’s appeared as a particular case. This generalization, which gives a special importance to a rectangle known by J. Neuberg [15], citing [4], has been pointed out in 1983 by the editorial comment of the *Monthly* in an outline publication about the supposed
first metric solution of the English K. B. Taylor [23] which amounted to 24 pages. In 1986, a much shorter proof [25], due to Gerhard Turnwald, appeared. In 2001, R. Shail considered in his analytic approach, a “more complete” problem [19] in which the one of Stark appeared as a particular case. This last generalization was studied again by S. Gueron [11] in a metric and less complete way. In 2003, the *Monthly* published the angular solution by B. J. English, received in 1975 and “lost in the mists of time” [7].

Thanks to JSTOR, the present author has discovered in an ancient edition of the *Monthly* [18] that the problem of Shail was proposed in 1905 by an instructor Y. Sawayama of the central military School of Tokyo, and geometrically resolved by himself, mixing the synthetic and metric approach. On this basis, we elaborate a new, purely synthetic proof of Sawayama-Thébault theorem which includes several theorems that can all be synthetically proved. The initial step of our approach refers to the beginning of the Sawayama’s proof and the end refers to Stark’s proof. Furthermore, our point of view leads easily to the Sawayama-Shail result.

2. A lemma

**Lemma 1.** *Through the vertex* $A$ *of a triangle* $ABC$, *a straight line* $AD$ *is drawn, cutting the side* $BC$ *at* $D$. *Let* $P$ *be the center of the circle* $C_1$ *which touches* $DC$, $DA$ *at* $E$, $F$ *and the circumcircle* $C_2$ *of* $ABC$ *at* $K$. *Then the chord of contact* $EF$ *passes through the incenter* $I$ *of triangle* $ABC$.

\[\text{Figure 2}\]

*Proof.* Let $M$, $N$ be the points of intersection of $KE$, $KF$ with $C_2$, and $J$ the point of intersection of $AM$ and $EF$ (see Figure 3). $KE$ is the internal bisector of $\angle BKC$ [8, Théorème 119]. The point $M$ being the midpoint of the arc $BC$ which does not contain $K$, $AM$ is the $A$-internal bisector of $ABC$ and passes through $I$. 


The circles $C_1$ and $C_2$ being tangent at $K$, $EF$ and $MN$ are parallel.

Figure 3

The circle $C_2$, the basic points $A$ and $K$, the lines $MAJ$ and $NKF$, the parallels $MN$ and $JF$, lead to a converse of Reim’s theorem ([8, Théorème 124]). Therefore, the points $A$, $K$, $F$ and $J$ are concyclic. This can also be seen directly from the fact that angles $FJA$ and $FKA$ are congruent.

Miquel’s pivot theorem [14, 9] applied to the triangle $AFJ$ by considering $F$ on $AF$, $E$ on $FJ$, and $J$ on $AJ$, shows that the circle $C_4$ passing through $E$, $J$ and $K$ is tangent to $AJ$ at $J$. The circle $C_5$ with center $M$, passing through $B$, also passes through $I$ ([2, Livre II, p.46, théorème XXI] and [12, p.185]). This circle being orthogonal to circle $C_1$ is also orthogonal to circle $C_4$ ([10, 1]) as $KEM$ is the radical axis of circles $C_1$ and $C_4$. Therefore, $MB = MJ$, and $J = I$.

Conclusion: the chord of contact $EF$ passes through the incenter $I$. □

Remark. When $D$ is at $B$, this is the theorem of Nixon [16].

3. Sawayama-Thébault theorem

**Theorem 2.** Through the vertex $A$ of a triangle $ABC$, a straight line $AD$ is drawn, cutting the side $BC$ at $D$. $I$ is the center of the incircle of triangle $ABC$. Let $P$ be the center of the circle which touches $DC$, $DA$ at $E$, $F$, and the circumcircle of $ABC$, and let $Q$ be the center of a further circle which touches $DB$, $DA$ in $G$, $H$ and the circumcircle of $ABC$. Then $P$, $I$ and $Q$ are collinear.

---

1From $\angle BKE = \angle MAC = \angle MBE$, we see that the circumcircle of $BKE$ is tangent to $BM$ at $B$. So circle $C_5$ is orthogonal to this circumcircle and consequently also to $C_1$ as $M$ lies on their radical axis.
Proof. According to the hypothesis, $QG \perp BC$, $BC \perp PE$; so $QG \parallel PE$. By Lemma 1, $GH$ and $EF$ pass through $I$. Triangles $DHG$ and $QGH$ being isosceles in $D$ and $Q$ respectively, $DQ$ is

1) the perpendicular bisector of $GH$,
2) the $D$-internal angle bisector of triangle $DHG$.

Mutatis mutandis, $DP$ is

1) the perpendicular bisector of $EF$,
2) the $D$-internal angle bisector of triangle $DEF$.

As the bisectors of two adjacent and supplementary angles are perpendicular, we have $DQ \perp DP$. Therefore, $GH \parallel DP$ and $DQ \parallel EF$. Conclusion: using the converse of Pappus’s theorem ([17, Proposition 139] and [3, p.67]), applied to the hexagon $PEIGQDP$, the points $P$, $I$ and $Q$ are collinear. 

References


Jean-Louis Ayme: 37 rue Ste-Marie, 97400 St.-Denis, La Réunion, France
E-mail address: jeanolouisayme@yahoo.fr
Antiorthocorrespondents of Circumconics

Bernard Gibert

Abstract. This note is a complement to our previous paper [3]. We study how circumconics are transformed under antiorthocorrespondence. This leads us to consider a pencil of pivotal circular cubics which contains in particular the Neuberg cubic of the orthic triangle.

1. Introduction

This paper is a complement to our previous paper [3] on the orthocorrespondence. Recall that in the plane of a given triangle $ABC$, the orthocorrespondent of a point $M$ is the point $M^\perp$ whose trilinear polar intersects the sidelines of $ABC$ at the orthotrajectrices of $M$. If $M = (p : q : r)$ in homogeneous barycentric coordinates, then

$$M^\perp = (p(-pS_A + qS_B + rS_C) + a^2qr : \cdots : \cdots).$$

The antiorthocorrespondents of $M$ consist of the two points $M_1$ and $M_2$, not necessarily real, for which $M_1^\perp = M = M_2^\perp$. We write $M^\perp = \{M_1, M_2\}$, and say that $M_1$ and $M_2$ are orthoassociates. We shall make use of the following basic results.

Lemma 1. Let $M = (p : q : r)$ and $M^\perp = \{M_1, M_2\}$.

1. The line $M_1M_2$ has equation

$$S_A(q - r)x + S_B(r - p)y + S_C(p - q)z = 0.$$

It always passes through the orthocenter $H$, and intersects the line $GM$ at the point

$$((b^2 - c^2)/(q - r) : \cdots : \cdots)$$

on the Kiepert hyperbola.

2. The perpendicular bisector $\ell_M$ of the segment $M_1M_2$ is the trilinear polar of the isotomic conjugate of the anticomplement of $M$, i.e.,

$$(q + r - p)x + (r + p - q)y + (p + q - r)z = 0.$$
We study how circumconics are transformed under antiorthocorrespondence. Let \( P = (u : v : w) \) be a point not lying on the sidelines of \( ABC \). Denote by \( \Gamma_P \) the circumconic with perspector \( P \), namely,
\[
\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.
\]
This has center \(^3\)
\[
G/P = (u(-u + v + w) : v(-v + w + u) : w(u + v - w)),
\]
and is the locus of trilinear poles of lines passing through \( P \).

A point \((x : y : z)\) is the orthocorrespondent of a point on \( \Gamma_P \) if and only if
\[
\sum_{\text{cyclic}} \frac{u}{x(-xS_A + yS_B + zS_C) + a^2yz} = 0. \quad (2)
\]

The antiorthocorrespondent of \( \Gamma_P \) is therefore in general a quartic \( Q_P \). It is easy to check that \( Q_P \) passes through the vertices of the orthic triangle and the pedal triangle of \( P \). It is obviously invariant under orthoassociation, i.e., inversion with respect to the polar circle. See [3, §2]. It is therefore a special case of anallagmatic fourth degree curve.

---

\(^3\)This is the perspector of the medial triangle and anticevian triangle of \( P \).
The equation of $Q_P$ can be rewritten as
\[(u + v + w)C^2 - \left( \sum_{\text{cyclic}} (v + w)S_A x \right) \mathcal{C} - \left( \sum_{\text{cyclic}} uS_B S_C yz \right) \mathcal{L}^2 = 0, \quad (3)\]
with
\[
\mathcal{C} = a^2yz + b^2zx + c^2xy, \quad \mathcal{L} = x + y + z.
\]
From this it is clear that $Q_P$ is a bicircular quartic if and only if $u + v + w \neq 0$; equivalently, $\Gamma_P$ does not contain the centroid $G$. We shall study this case in §3 below, and the case $G \in \Gamma_P$ in §4.

2. The conic $\gamma_P$

A generic point on the conic $\Gamma_P$ is
\[M = M(t) = \left( \frac{u}{(v-w)(u+t)} : \frac{v}{(w-u)(v+t)} : \frac{w}{(u-v)(w+t)} \right).\]
As $M$ varies on the circumconic $\Gamma_P$, the perpendicular bisector $\ell_M$ of $M_1M_2$ envelopes the conic $\gamma_P$:
\[
\sum((u + v + w)^2 - 4vw)x^2 - 2(u + v + w)(v + w - u)yz = 0.
\]

![Figure 2. The conic $\gamma_P$](image)

The point of tangency of $\gamma_P$ and the perpendicular bisector of $M_1M_2$ is
\[T_M = (v(u-v)^2(w+t)^2 + w(u-w)^2(v+t)^2 : \cdots : \cdots).\]
The conic $\gamma_P$ is called the déférente of $\Gamma_P$ in [1]. It has center $\omega_P = (2u + v + w : \cdots : \cdots)$, and is homothetic to the circumconic with perspector $((v + w)^2 : (w + u)^2 : (u + v)^2)$. It is therefore a circle when $P$ is the Nagel point or one of its extravagizations. This circle is the Spieker circle. We shall see in §3.5 below that $Q_P$ is an oval of Descartes.

It is clear that $\gamma_P$ is a parabola if and only if $\omega_P$ and therefore $P$ are at infinity. In this case, $\Gamma_P$ contains the centroid $G$. See §4 below.

3. Antiothocorrespondent of a circumconic not containing the centroid

Throughout this section we assume $P$ a finite point so that the circumconic $\Gamma_P$ does not contain the centroid $G$.

Proposition 2. Let $\ell$ be a line through $G$ intersecting $\Gamma_P$ at two points $M$ and $N$. The antiothocorrespondents of $M$ and $N$ are four collinear points on $Q_P$.

Proof. Let $M_1, M_2$ be the antiothocorrespondents of $M$, and $N_1, N_2$ those of $N$. By Lemma 1, each of the lines $M_1M_2$ and $N_1N_2$ intersects $\ell$ at the same point on the Kiepert hyperbola. Since they both contain $H$, $M_1M_2$ and $N_1N_2$ are the same line.

Corollary 3. Let the medians of $ABC$ meet $\Gamma_P$ again at $A_g, B_g, C_g$. The antiothocorrespondents of these points are the third and fourth intersections of $Q_P$ with the altitudes of $ABC$.

Proof. The antiothocorrespondents of $A$ are $A$ and $H_a$.

In this case, the third and fourth points on $AH$ are symmetric about the second tangent to $\gamma_P$ which is parallel to $BC$. The first tangent is the perpendicular bisector of $AH_a$ with contact $(v + w : v : w)$, the contact with this second tangent is $u(v + w) : uw + (v + w)^2 : uw + (v + w)^2$ while $A_g = (-u : v + w : v + w)$.

For distinct points $P_1$ and $P_2$, the circumconics $\Gamma_{P_1}$ and $\Gamma_{P_2}$ have a “fourth” common point $T$, which is the trilinear pole of the line $P_1P_2$. Let $T^\tau = \{T_1, T_2\}$. The conics $\Gamma_{P_1}$ and $\Gamma_{P_2}$ generate a pencil $\mathcal{F}$ consisting of $\Gamma_P$ for $P$ on the line $P_1P_2$. The antiothocorrespondent of every conic $\Gamma_P \in \mathcal{F}$ contains the following 16 points:

(i) the vertices of $ABC$ and the orthic triangle $H_aH_bH_c$,
(ii) the circular points at infinity with multiplicity 4,\footnote{\protect{It is inscribed in the medial triangle; its anticomplement is the circumconic with center the complement of $P$, with perspector the isotomic conjugate of $P$.}}
(iii) the antiothocorrespondents $T^\tau = \{T_1, T_2\}$.

Proposition 4. Apart from the circular points at infinity and the vertices of $ABC$ and the orthic triangle, the common points of the quartics $Q_{P_1}$ and $Q_{P_2}$ are the antiothocorrespondents of the trilinear pole of the line $P_1P_2$.

\footnote{They are not always real when $ABC$ is obtuse angle.}

\footnote{Think of $Q_{P_1}$ as the union of two circles and $Q_{P_2}$ likewise. These have at most 8 real finite points and the remaining 8 are the circular points at infinity, each counted with multiplicity 4.}
Remarks. 1. \(T_1\) and \(T_2\) lie on the line through \(H\) which is the orthocorrespondent of the line \(GT\). See [3, §2.4]. This line \(T_1T_2\) is the directrix of the inscribed (in \(ABC\)) parabola tangent to the line \(P_1P_2\).

2. The pencil \(\mathcal{F}\) contains three degenerate conics \(BC \cup AT\), \(CA \cup BT\), and \(AB \cup CT\). The antithalcorrespondent of \(BC \cup AT\), for example, degenerates into the circle with diameter \(BC\) and another circle through \(A, H_a, T_1\) and \(T_2\) (see [3, Proposition 2]).

3.1. The points \(S_1\) and \(S_2\). Since \(Q_P\) and the circumcircle have already seven common points, the vertices \(A, B, C\), and the circular points at infinity, each of multiplicity 2, they must have an eighth common point, namely,

\[
S_1 = \left( \frac{a^2}{u} \frac{v}{b^2S_B - w} \frac{w}{c^2S_C} \cdots \cdots \right),
\]

which is the isogonal conjugate of the infinite point of the line

\[
\frac{u}{a^2S_A}x + \frac{v}{b^2S_B}y + \frac{w}{c^2S_C}z = 0.
\]
Similarly, $Q_P$ and the nine-point circle also have a real eighth common point
\[ S_2 = ((S_B(u - v + w) - S_C(u + v - w))(c^2S_Cv - b^2S_Bw) : \cdots : \cdots), \]
which is the inferior of
\[ \left( \frac{a^2}{S_B(u - v + w) - S_C(u + v - w)} : \cdots : \cdots \right) \]
on the circumcircle.

We know that the orthocorrespondent of the circumcircle is the circum-ellipse $\Gamma_O$, with center $K$, the Lemoine point, ([3, §2.6]). If $P \neq O$, this ellipse meets $\Gamma_P$ at $A, B, C$ and a fourth point
\[ S = S(P) = \left( \frac{1}{c^2S_Cv - b^2S_Bw} : \cdots : \cdots \right), \]
which is the trilinear pole of the line $OP$. The point $S$ lies on the circumcircle if and only if $P$ is on the Brocard axis $OK$.

**Proposition 5.** $S^T = \{S_1, S_2\}$.

**Corollary 6.** $S(P) = S(P')$ if and only if $P, P'$ and $O$ are collinear.

**Remark.** When $P = O$ (circumcenter), $\Gamma_P$ is the circum-ellipse with center $K$.

In this case $Q_P$ decomposes into the union of the circumcircle and the nine point circle.

### 3.2. Bitangents.

**Proposition 7.** The points of tangency of the two bitangents to $Q_P$ passing through $H$ are the antiothcorrespondents of the points where the polar line of $G$ in $\Gamma_P$ meets $\Gamma_P$.

**Proof.** Consider a line $\ell_H$ through $H$ which is supposed to be tangent to $Q_P$ at two (orthoassociate) points $M$ and $N$. The orthocorrespondents of $M$ and $N$ must lie on $\Gamma_P$ and on the orthocorrespondent of $\ell_H$ which is a line through $G$. Since $M$ and $N$ are double points, the line through $G$ must be tangent to $\Gamma_P$ and $MN$ is the polar of $G$ in $\Gamma_P$. \[ \square \]

**Remark.** $M$ and $N$ are not necessarily real. If $M^T = \{M_1, M_2\}$ and $N^T = \{N_1, N_2\}$, the perpendicular bisectors of $M_1M_2$ and $N_1N_2$ are the asymptotes of $\gamma_P$.\[ \text{7} \] The four points $M_1, M_2, N_1, N_2$ are concyclic and the circle passing through them is centered at $\omega_P$.

Denote by $H_1, H_2, H_3$ the vertices of the triangle which is self polar in both the polar circle and $\gamma_P$. The orthocenter of this triangle is obviously $H$. For $i = 1, 2, 3$, let $C_i$ be the circle centered at $H_i$ orthogonal to the polar circle and $\Gamma_i$ the circle centered at $\omega_P$ orthogonal to $C_i$. The circle $\Gamma_i$ intersects $Q_P$ at the circular points at infinity (with multiplicity 2) and four other points two by two homologous in the inversion with respect to $C_i$ which are the points of tangency of the (not...
always real) bitangents drawn from $H_i$ to $Q_P$. The orthocorrespondent of $\Gamma_i$ is a conic (see [3, §2.6]) intersecting $\Gamma_P$ at four points whose antiothocorrespondents are eight points, two by two orthoassociate. Four of them lie on $\Gamma_i$ and are the required points of tangency. The remaining four are their orthoassociate and they lie on the circle which is the orthoassociate of $\Gamma_i$. Figure 4 below shows an example of $Q_P$ with three pairs of real bitangents.

**Figure 4.** Bitangents to $Q_P$.

**Proposition 8.** $Q_P$ is tangent at $H_a$, $H_b$, $H_c$ to $BC$, $CA$, $AB$ if and only if $P = H$.

3.3. $Q_P$ as an envelope of circles.

**Theorem 9.** The circle $C_M$ centered at $T_M$ passing through $M_1$ and $M_2$ is bitangent to $Q_P$ at those points and orthogonal to the polar circle.

This is a consequence of the following result from [1, tome 3, p.170]. A bicircular quartic is a special case of “plane cyclic curve”. Such a curve always can be considered in four different ways as the envelope of circles centered on a conic (déférente) cutting orthogonally a fixed circle. Here the fixed circle is the polar circle with center $H$, and since $M_1$ and $M_2$ are anallagmatic (inverse in the polar circle) and collinear with $H$, there is a circle passing through $M_1$, $M_2$, centered on the déférente, which must be bitangent to the quartic.

**Corollary 10.** $Q_P$ is the envelope of circles $C_M$, $M \in \Gamma_P$, centered on $\gamma_P$ and orthogonal to the polar circle.
Construction. It is easy to draw $\gamma_P$ since we know its center $\omega_P$. For $m$ on $\gamma_P$, draw the tangent $t_m$ at $m$ to $\gamma_P$. The perpendicular at $m$ to $Hm$ meets the perpendicular bisector of $AH_a$ at a point which is the center of a circle through $A$ (and $H_a$). This circle intersects $Hm$ at two points which lie on the circle centered at $m$ and orthogonal to the polar circle. This circle intersects the perpendicular at $H$ to $t_m$ at two points of $Q_P$.

Corollary 11. The tangents at $M_1$ and $M_2$ to $Q_P$ are the tangents to the circle $C_M$ at these points.

Figure 5. $Q_P$ as an envelope of circles

3.4. Inversions leaving $Q_P$ invariant.

Theorem 12. $Q_P$ is invariant under three other inversions whose poles are the vertices of the triangle which is self-polar in both the polar circle and $\gamma_P$.

Proof. This is a consequence of [1, tome 3, p.172].

Construction: Consider the transformation $\phi$ which maps any point $M$ of the plane to the intersection $M'$ of the polars of $M$ in both the polar circle and $\gamma_P$. Let $\Sigma_a, \Sigma_b, \Sigma_c$ be the conics which are the images of the altitudes $AH, BH, CH$ under $\phi$. The conic $\Sigma_a$ is entirely defined by the following five points:
(1) the point at infinity of \( BC \).
(2) the point at infinity of the polar of \( H \) in \( \gamma_P \).
(3) the foot on \( BC \) of the polar of \( A \) in \( \gamma_P \).
(4) the intersection of the polar of \( H_a \) in \( \gamma_P \) with the parallel at \( A \) to \( BC \).
(5) the pole of \( AH \) in \( \gamma_P \).

Figure 6. The conics \( \Sigma_a, \Sigma_b, \Sigma_c \)

Similarly, we define the conics \( \Sigma_b, \Sigma_c \). These conics are in the same pencil and meet at four points: one of them is the point at infinity of the polar of \( H \) in \( \gamma_P \) and the three others are the required poles. The circles of inversion are centered at those points and are orthogonal to the polar circle. Their radical axes with the polar circle are the sidelines of the self-polar triangle. □

Another construction is possible: the transformation of the sidelines of triangle \( ABC \) under \( \phi \) gives three other conics \( \sigma_a, \sigma_b, \sigma_c \) but not defining a pencil since the three lines are not now concurrent. \( \sigma_a \) passes through \( A \), the two points where the trilinear polar of \( P^+ \) (anticomplement of \( P \)) meets \( AB \) and \( AC \), the pole of the line \( BC \) in \( \gamma_P \), the intersection of the parallel at \( A \) to \( BC \) with the polar of \( H_a \) in \( \gamma_P \). See Figure 7.

Remark. The Jacobian of \( \sigma_a, \sigma_b, \sigma_c \) is a degenerate cubic consisting of the union of the sidelines of the self-polar triangle.
3.5. Examples. We provide some examples related to common centers of \(ABC\).

<table>
<thead>
<tr>
<th>(P)</th>
<th>(S)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(\Gamma_P)</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H)</td>
<td>(X_{648})</td>
<td>(X_{107})</td>
<td>(X_{125})</td>
<td></td>
<td>see Figure 8</td>
</tr>
<tr>
<td>(K)</td>
<td>(X_{110})</td>
<td>(X_{112})</td>
<td>(X_{115})</td>
<td></td>
<td>circumcircle</td>
</tr>
<tr>
<td>(G)</td>
<td>(X_{648})</td>
<td>(X_{107})</td>
<td>(S_{125})</td>
<td></td>
<td>Steiner circum – ellipse</td>
</tr>
<tr>
<td>(X_{647})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Jerabek hyperbola</td>
</tr>
</tbody>
</table>

Remarks. 1. For \(P = H\), \(Q_P\) is tangent at \(H_a, H_b, H_c\) to the sidelines of \(ABC\). See Figure 8.

2. \(P = X_{647}\), the isogonal conjugate of the tripole of the Euler line: \(\Gamma_P\) is the Jerabek hyperbola.

3. \(Q_P\) has two axes of symmetry if and only if \(P\) is the point such that \(\overrightarrow{OP} = 3\overrightarrow{OH}\) (this is a consequence of [1, tome 3, p.172, §15]. Those axes are the parallels at \(H\) to the asymptotes of the Kiepert hyperbola. See Figure 9.

4. When \(P = X_8\) (Nagel point), \(\gamma_P\) is the incircle of the medial triangle (its center is \(X_{10} =\) Spieker center) and \(\Gamma_P\) the circum-conic centered at \(\Omega_P = ((b + c - a)(b + c - 3a) : \cdots : \cdots)\). Since the déférente is a circle, \(Q_P\) is now an oval.
of Descartes (see [1, tome 1, p.8]) with axis the line $HX_{10}$. We obtain three more ovals of Descartes if $X_8$ is replaced by one of its extraversions. See Figure 10.
4. Antiorthocorrespondent of a circum-conic passing through $G$

We consider the case when the circumconic $\Gamma_P$ contains the centroid $G$; equivalently, $P = (u : v : w)$ is an infinite point. In this case, $\Gamma_P$ has center $(u^2 : v^2 : w^2)$ on the inscribed Steiner ellipse. The trilinear polar of points $Q \neq G$ on $\Gamma_P$ are all parallel, and have infinite point $P$. It is clear from (3) that the curve $Q_P$ decomposes into the union of the line at infinity $L^\infty : x + y + z = 0$ and the cubic $K_P$

$$\sum x(S_B(S_Au - S_Bv)y^2 - S_C(S_Cw - S_Au)z^2) = 0. \quad (7)$$

This is the pivotal isocubic $pK(\Omega_P, H)$, with pivot $H$ and pole

$$\Omega_P = \left(\frac{S_Bv - S_Cw}{S_A}, \frac{S_Cw - S_Au}{S_B}, \frac{S_Au - S_Bv}{S_C}\right).$$

Since the orthocorrespondent of the line at infinity is the centroid $G$, we shall simply say that the antiorthocorrespondent of $\Gamma_P$ is the cubic $K_P$. The orthocenter $H$ is the only finite point whose orthocorrespondent is $G$. We know that $Q_P$ has already the circular points (counted twice) on $L^\infty$. This means that the cubic $K_P$ is also a circular cubic. In fact, equation (7) can be rewritten as

$$(uS_Ax + vS_By + wS_Cz)(a^2yz + b^2zx + c^2xy) + (x + y + z)(uS_BCyz + vS_Czx + wS_Axy) = 0. \quad (8)$$
As $P$ traverses $\mathcal{C}_\infty$, these cubics $K_P$ form a pencil of circular pivotal isocubics since they all contain $A$, $B$, $C$, $H$, $H_a$, $H_b$, $H_c$ and the circular points at infinity. The poles of these isocubics all lie on the orthic axis.

![Figure 11. The circular pivotal cubic $K_P$](image)

4.1. Properties of $K_P$.

1. $K_P$ is invariant under orthoassociation: the line through $H$ and $M$ on $K_P$ meets $K_P$ again at $M'$ simultaneously the $\Omega_P$-isoconjugate and orthoassociate of $M$. $K_P$ is also invariant under the three inversions with poles $A$, $B$, $C$ which swap $H$ and $H_a$, $H_b$, $H_c$ respectively. 8

2. The real asymptote of $K_P$ is the line $\ell_P$

$$\frac{u}{S_B v - S_C w} x + \frac{v}{S_C w - S_A u} y + \frac{w}{S_A u - S_B v} z = 0. \tag{9}$$

It has infinite point

$$P' = (S_B v - S_C w : S_C w - S_A u : S_A u - S_B v),$$

---

8$H$, $H_a$, $H_b$, $H_c$ are often called the centers of anallagmaty of the circular cubic.
and is parallel to the tangents at \(A, B, C,\) and \(H.\) It is indeed the Simson line of the isogonal conjugate of \(P.\) It is therefore tangent to the Steiner deltoid.

(3) The tangents to \(K_P\) at \(H_a, H_b, H_c\) are the reflections of those at \(A, B, C,\) about the perpendicular bisectors of \(AH_a, BH_b, CH_c\) respectively.\(^{10}\) They concur on the cubic at the point

\[
X = \left( \frac{S_Bv - S_Cw}{u}, \frac{b^2S_B}{v} - \frac{c^2S_C}{w}, \cdots : \cdots \right),
\]

which is also the intersection of \(\ell_P\) and the nine point circle. This is the inferior of the isogonal conjugate of \(P'.\) It is also the image of \(P'^*,\) the isogonal conjugate of \(P,\) under the homothety \(h(H, \frac{1}{2}).\)

(4) The antipode \(F\) of \(X\) on the nine point circle is the singular focus of \(K_P:\)

\[
F = (u(b^2v - c^2w) : v(c^2w - a^2u) : w(a^2u - b^2v)).
\]

(5) The orthoassociate \(Y\) of \(X\) is the “last” intersection of \(K_P\) with the circumcircle, apart from the vertices and the circular points at infinity.

(6) The second intersection of the line \(XY\) with the circumcircle is \(Z = P^*.\)

---

\(^9\)The latter is the line \(uS_Ax + vS_By + wS_Cz = 0.\)

\(^{10}\)These are the lines \(S^2ux - (S_Bv - S_Cw)(S_By - S_Cz) = 0 \) etc.
(7) $\mathcal{K}_P$ intersects the sidelines of the orthic triangle at three points lying on the cevian lines of $Y$ in $ABC$.

(8) $\mathcal{K}_P$ is the envelope of circles centered on the parabola $\mathcal{P}_P$ (focus $F$, directrix the parallel at $O$ to the Simson line of $Z$) and orthogonal to the polar circle. See Figure 13.

Figure 13. $\mathcal{K}_P$ and the parabola $\mathcal{P}_P$

(9) $\Gamma_P$ meets the circumcircle again at

$$S = \left( \frac{1}{b^2v-c^2w} : \frac{1}{c^2w-a^2u} : \frac{1}{a^2u-b^2v} \right)$$

and the Steiner circum-ellipse again at

$$R = \left( \frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v} \right).$$

The antiorthocorrespondents of these two points $S$ are four points on $\mathcal{K}_P$. They lie on a same circle orthogonal to the polar circle. See [3, §2.5] and Figure 14.

4.2. $\mathcal{K}_P$ passing through a given point. Since all the cubics form a pencil, there is a unique $\mathcal{K}_P$ passing through a given point $Q$ which is not a base-point of the pencil. The circumconic $\Gamma_P$ clearly contains $G$ and $Q^\perp$, the orthocorrespondent of $Q$. It follows that $P$ is the infinite point of the tripolar of $Q^\perp$.

Here is another construction of $P$. The circumconic through $G$ and $Q^\perp$ intersects the Steiner circum-ellipse at a fourth point $R$. The midpoint $M$ of $GR$ is the center of $\Gamma_P$. The anticevian triangle of $M$ is perspective to the medial triangle at $P$. The lines through their corresponding vertices are parallel to the tangents to
$\mathcal{K}_P$ at $A$, $B$, $C$. The point at infinity of these parallel lines is the point $P$ for which $\mathcal{K}_P$ contains $Q$.

In particular, if $Q$ is a point on the circumcircle, $P$ is simply the isogonal conjugate of the second intersection of the line $HQ$ with the circumcircle.

4.3. Some examples and special cases.

1) The most remarkable circum-conic through $G$ is probably the Kiepert rectangular hyperbola with perspector $P = X_{523}$, point at infinity of the orthic axis. Its antiothcorrespondent is $p_{\mathcal{K}}(X_{1990}, H)$, identified as the orthopivotal cubic $\mathcal{O}(H)$ in [3, §6.2.1]. See Figure 15.

2) With $P = $ isogonal conjugate of $X_{930}$, $\mathcal{K}_P$ is the Neuberg cubic of the orthic triangle. We have $F = X_{137}$, $X = X_{128}$, $Y = $ isogonal conjugate of $X_{639}$, $Z = X_{930}$. The cubic contains $X_5$, $X_{15}$, $X_{16}$, $X_{52}$, $X_{186}$, $X_{1154}$ (at infinity). See Figure 16.

3) $\mathcal{K}_P$ degenerates when $P$ is the point at infinity of one altitude. For example, with the altitude $AH$, $\mathcal{K}_P$ is the union of the sideline $BC$ and the circle through $A$, $H$, $H_b$, $H_c$.

\[^{11}P = (a^2(b^2 - c^2))(4S_A^2 - 3b^2c^2) : \cdots : \cdots \). The point $X_{930}$ is the anticomplement of $X_{137}$ which is $X_{110}$ of the orthic triangle.
Figure 15. $O(H)$ or $K_P$ for $P = X_{523}$

Figure 16. $K_P$ as the Neuberg cubic of the orthic triangle

(4) $K_P$ is a focal cubic if and only if $P$ is the point at infinity of one tangent to the circumcircle at $A, B, C$. For example, with $A, K_P$ is the focal cubic
denoted $K_a$ with singular focus $H_a$ and pole the intersection of the orthic axis with the symmedian $AK$. The tangents at $A, B, C, H$ are parallel to the line $OA$. $\Gamma_P$ is the isogonal conjugate of the line passing through $K$ and the midpoint of $BC$. $P_P$ is the parabola with focus $H_a$ and directrix the line $OA$.

$K_a$ is the locus of point $M$ from which the segments $BH_b, CH_c$ are seen under equal or supplementary angles. It is also the locus of contacts of tangents drawn from $H_a$ to the circles centered on $H_bH_c$ and orthogonal to the circle with diameter $H_bH_c$. See Figure 17.

![Figure 17. The focal cubic $K_a$](image)

4.4. Conclusion. We conclude with the following table showing the repartition of the points we met in the study above in some particular situations. Recall that $P, X, Y$ always lie on $K_P, Y, Z, S$ on the circumcircle, $X, F$ on the nine point circle, $R$ on the Steiner circum-ellipse. When the point is not mentioned in [6], its first barycentric coordinate is given, as far as it is not too complicated. $M^*$ denotes the isogonal conjugate of $M$, and $M^\#$ denotes the isotomic conjugate of $M$. 
Remarks. (1) $\Omega_P = X_{115}$. $\Gamma_P$ is the Kiepert hyperbola. $\mathcal{P}_P$ is the Kiepert parabola of the medial triangle with directrix the Euler line. See Figure 15.

(2) $\Omega_P = X_{1086}$. $\mathcal{P}_P$ is the Yff parabola of the medial triangle. See Figure 14.

(3) $\Omega_P = X_{1084}$. The directrix of $\mathcal{P}_P$ is the Brocard line.

(4) $\Omega_P = X_{1015}$. The directrix of $\mathcal{P}_P$ is the line $OI$.

The points $M_1, \ldots, M_7$ are defined by their first barycentric coordinates as follows.

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
<th>$M_6$</th>
<th>$M_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/((b^2 - c^2)(a^2 S_A + b^2 c^2))$</td>
<td>$a^2 / (S_A((b^2 - c^2)^2 - a^2(b^4 + c^2 - 2a^2)))$</td>
<td>$(b^2 - c^2)^2((b^2 + c^2 - 5a^2)(b^4 + c^2 - a^4 - 4b^2c^2))$</td>
<td>$1/[S_A((b^2 - c^2)(b^4 + c^2 - a^4 - 4b^2c^2))]$</td>
<td>$S_A(b - c)(b^4 + c^4 - a^2b - a^2c + abc)$</td>
<td>$1/[S_A(b - c)(b^2 + c^2 - ab - ac + bc)]$</td>
<td>$1/((b - c)(3b^2 + 3c^2 - a^2 - 2ab - 2ac + 2bc))$</td>
</tr>
</tbody>
</table>

References


Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

E-mail address: bg42@wanadoo.fr
Author Index

Ayme, J.-L.: Sawayama and Thébault’s theorem, 225
Dalcín, M.: Isotomic inscribed triangles and their residuals, 125
Dergiades, N.: Harcourt’s theorem, 117
    Rectangles attached to the sides of a triangle, 145
Ehrmann, J.-P.: Similar pedal and cevian triangles, 101
Emelyanov, L.: A note on the Schiffler point, 113
Emelyanova, T.: A note on the Schiffler point, 113
Evans, L.: Some configurations of triangle centers, 49
    A tetrahedral arrangement of triangle centers, 181
Gibert, B.: Orthocorrespondence and orthopivotal cubics, 1
    The parasix configuration and orthocorrespondence, 169
    Antiorthocorrespondents of Circumconics, 251
Grinberg, D.: On the Kosnita point and the reflection triangle, 105
Hofstetter, K.: A 5-step division of a segment in the golden section, 205
Kimberling, C.: Bicentric pairs of points and related triangle centers, 35
van Lamoen, F. M.: Napoleon triangles and Kiepert perspectors, 65
    Rectangles attached to the sides of a triangle, 145
    The parasix configuration and orthocorrespondence, 169
    Circumrhombi, 225
Myakishev, A.: On the procircumcenter and related points, 29
    On the circumcenters of cevasix configurations, 57
    The M-configuration of a triangle, 135
Reyes, W.: The Lucas circles and the Descartes formula, 95
Salazar, J. C.: Harcourt’s theorem, 117
Schmidt, E.: Circumcenters of residual triangles, 207
Stevanović, M.: Triangle centers associated with the Malfatti circles, 83
    The Apollonius circle and related triangle centers, 187
    Two triangle centers associated with the excircles, 197
Thas, C.: A generalization of the Lemoine point, 161
Woo, P.: On the circumcenters of cevasix configurations, 57
Yiu, P.: On the Fermat lines, 73