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Antiparallels and Concurrent Euler Lines

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Abstract. We study the condition for concurrency of the Euler lines of the three triangles each bounded by two sides of a reference triangle and an antiparallel to the third side. For example, if the antiparallels are concurrent at P and the three Euler lines are concurrent at Q, then the loci of P and Q are respectively the tangent to the Jerabek hyperbola at the Lemoine point, and the line parallel to the Brocard axis through the inverse of the deLongchamps point in the circumcircle. We also obtain an interesting cubic as the locus of the point P for which the three Euler lines are concurrent when the antiparallels are constructed through the vertices of the cevian triangle of P.

1. Thébault's theorem on Euler lines

We begin with the following theorem of Victor Thébault [8] on the concurrency of three Euler lines.

Theorem 1 (Thébault)**.** *Let $A'B'C'$ be the orthic triangle of ABC. The Euler lines of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent at the Jerabek center.*[1]

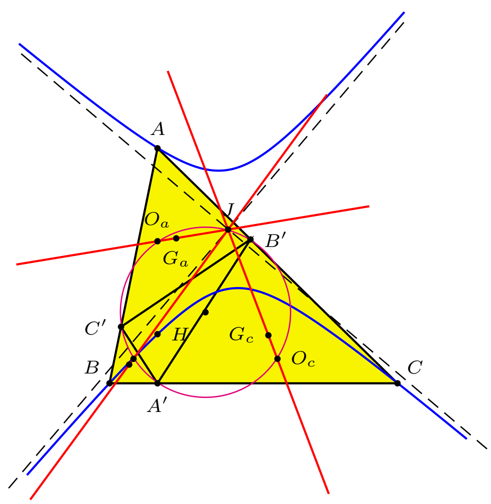

Figure 1. Thébault's theorem on the concurrency of Euler lines

We shall make use of homogeneous barycentric coordinates. With reference to triangle ABC, the vertices of the orthic triangle are the points

$$A' = (0 : S_C : S_B), \qquad B' = (S_C : 0 : S_A), \qquad C' = (S_B : S_A : 0).$$

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[1]Thébault [8] gave an equivalent characterization of this common point. See also [7].

These are the traces of the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$.

The centroid of $AB'C'$ is the point

$$(S_{AA} + 2S_{AB} + 2S_{AC} + 3S_{BC} : S_A(S_C + S_A) : S_A(S_A + S_B)).$$

The circumcenter of $A'BC$, being the midpoint of AH, has coordinates

$$(S_{CA} + S_{AB} + 2S_{BC} : S_{AC} : S_{AB}).$$

It is straightforward to verify that these two points lie on the line

$$S_{AA}(S_B-S_C)(x+y+z) = (S_A+S_B)(S_{AB}+S_{BC}-2S_{CA})y-(S_C+S_A)(S_{BC}+S_{CA}-2S_{AB})z, \quad (1)$$

which is therefore the Euler line of triangle $AB'C'$. Furthermore, the line (1) also contains the point

$$J = (S_A(S_B - S_C)^2 : S_B(S_C - S_A)^2 : S_C(S_A - S_B)^2),$$

which is the center of the Jerabek hyperbola.[2] Similar reasoning gives the equations of the Euler lines of triangles $BC'A'$ and $A'B'C$, and shows that these contain the same point J. This completes the proof of Thébault's theorem.

2. Triangles intercepted by antiparallels

Since the sides of the orthic triangles are antiparallel to the respective sides of triangle ABC, we consider the more general situation when the residuals of the orthic triangle are replaced by triangles intercepted by lines ℓ_1, ℓ_2, ℓ_3 antiparallel to the sidelines of the reference triangle, with the following intercepts on the sidelines

	BC	CA	AB
ℓ_1		B_a	C_a
ℓ_2	A_b		C_b
ℓ_3	A_c	B_c	

These lines are parallel to the sidelines of the orthic triangle $A'B'C'$. We shall assume that they are the images of the lines $B'C'$, $C'A'$, $A'B'$ under the homotheties $\mathsf{h}(A, 1-t_1)$, $\mathsf{h}(B, 1-t_2)$, and $\mathsf{h}(C, 1-t_3)$ respectively. The points B_a, C_a etc. have homogeneous barycentric coordinates

$$\begin{aligned}
B_a &= (t_1S_A + S_C : 0 : (1-t_1)S_A), & C_a &= (t_1S_A + S_B : (1-t_1)S_A : 0),\\
C_b &= ((1-t_2)S_B : t_2S_B + S_A : 0), & A_b &= (0 : t_2S_B + S_C : (1-t_2)S_B),\\
A_c &= (0 : (1-t_3)S_C : t_3S_C + S_B), & B_c &= ((1-t_3)S_C : 0 : t_3S_C + S_A).
\end{aligned}$$

[2]The point J appears as X_{125} in [4].

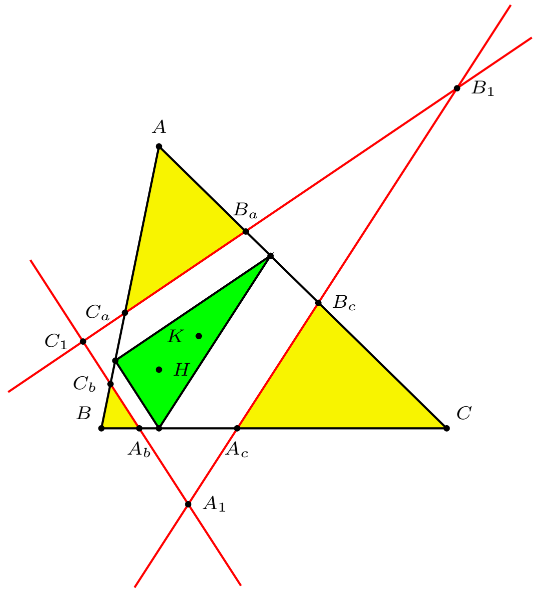

Figure 2. Triangles intercepted by antiparallels

2.1. *The Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$.* Denote by $\mathbf{T}_1$ the triangle AB_aC_a intercepted by ℓ_1; similarly $\mathbf{T}_2$ and $\mathbf{T}_3$. These are oppositely similar to ABC. We shall study the condition of the concurrency of their Euler lines.

Proposition 2. *With reference to triangle ABC, the barycentric equations of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, are*

$$(1-t_1)S_{AA}(S_B - S_C)(x+y+z) = c^2(S_{AB}+S_{BC}-2S_{CA})y - b^2(S_{BC}+S_{CA}-2S_{AB})z,$$
$$(1-t_2)S_{BB}(S_C - S_A)(x+y+z) = a^2(S_{BC}+S_{CA}-2S_{AB})z - c^2(S_{CA}+S_{AB}-2S_{BC})x,$$
$$(1-t_3)S_{CC}(S_A - S_B)(x+y+z) = b^2(S_{CA}+S_{AB}-2S_{BC})x - a^2(S_{AB}+S_{BC}-2S_{CA})y.$$

Proof. It is enough to establish the equation of the Euler line $\mathcal{L}_1$ of $\mathbf{T}_1$. This is the image of the Euler line $\mathcal{L}_1'$ of triangle $AB'C'$ under the homothety $\mathsf{h}(A, 1-t_1)$. A point $(x : y : z)$ on $\mathcal{L}_1$ corresponds to the point $((1-t_1)x - t_1(y+z) : y : z)$ on $\mathcal{L}_1'$. The equation of $\mathcal{L}_1$ can now by obtained from (1). □

From the equations of these Euler lines, we easily obtain the condition for their concurrency.

Theorem 3. *The three Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if*

$$t_1a^2(S_B - S_C)S_{AA} + t_2b^2(S_C - S_A)S_{BB} + t_3c^2(S_A - S_B)S_{CC} = 0. \quad (2)$$

Proof. From the equations of $\mathcal{L}_i$, $i = 1, 2, 3$, given in Proposition 2, it is clear that the condition for concurrency is

$$(1-t_1)a^2(S_B - S_C)S_{AA} + (1-t_2)b^2(S_C - S_A)S_{BB} + (1-t_3)c^2(S_A - S_B)S_{CC} = 0.$$

This simplifies into (2) above. □

2.2. *Antiparallels with given common point of $\mathcal{L}_i$, $i = 1, 2, 3$* . We shall assume triangle ABC scalene, *i.e.*, its angles are unequal and none of them is a right angle. For such triangles, the Euler lines of the residuals of the orthic triangle and the corresponding altitudes intersect at finite points.

Theorem 4. *Given a point Q in the plane of a scalene triangle ABC, there is a unqiue triple of antiparallels ℓ_i, $i = 1, 2, 3$, for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent at Q.*

Proof. Construct the parallel through Q to the Euler line of $AB'C'$ to intersect the line AH at O_a. The circle through A with center O_a intersects AC and AB at B_a and C_a respectively. The line B_aC_a is parallel to $B'C'$. It follows that its Euler line is parallel to that of $AB'C'$. This is the line O_aQ. Similar constructions give the other two antiparallels with corresponding Euler lines passing through Q. □

We make a useful observation here. From the equations of the Euler lines given in Proposition 2 above, the intersection of any two of them have coordinates expressible in linear functions of t_1, t_2, t_3. It follows that if t_1, t_2, t_3 are linear functions of a parameter t, and the three Euler lines are concurrent, then as t varies, the common point traverses a straight line. In particular, $t_1 = t_2 = t_3 = t$, the Euler lines are concurrent by Theorem 3. The locus of the intersection of the Euler lines is a straight line. Since this intersection is the Jerabek center when $t = 0$ (Thébault's theorem), and the orthocenter when $t = -1$,[3] this is the line

$$\mathcal{L}_{\text{c}}: \qquad \sum_{\text{cyclic}} S_{AA}(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

We give a summary of some of the interesting loci of common points of Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, when the lines ℓ_i, $i = 1, 2, 3$, are subjected to some further conditions. In what follows, $\mathbf{T}$ denotes the triangle bounded by the lines ℓ_i, $i = 1, 2, 3$.

Line	Construction	Condition	Reference
$\mathcal{L}_{\text{c}}$	HJ	$\mathbf{T}$ homothetic to orthic triangle at X_{25}	
$\mathcal{L}_{\text{q}}$	Remark below	ℓ_i, $i = 1, 2, 3$, concurrent	§3.2
$\mathcal{L}_{\text{t}}$	KX_{74}	ℓ_i are the antiparallels of a Tucker hexagon	§6
$\mathcal{L}_{\text{f}}$	X_5X_{184}	$\mathcal{L}_i$ intersect on Euler line of $\mathbf{T}$	§7.2
$\mathcal{L}_{\text{r}}$	GX_{110}	$\mathbf{T}$ and ABC perspective	§8.3

Remark. $\mathcal{L}_{\text{q}}$ can be constructed as the line parallel to the Brocard axis through the intersection of the inverse of the deLongchamps point in the circumcircle.

[3]For $t = 1$, this intersection is the point X_{74} on the circumcircle, the isogonal conjugate of the infinite point of the Euler line.

3. Concurrent antiparallels

In this section we consider the case when the antiparallels ℓ_1, ℓ_2, ℓ_3 all pass through a point $P = (u : v : w)$. In this case,

$$\begin{aligned}
B_a &= ((S_C + S_A)u - (S_B - S_C)v : 0 : (S_A + S_B)v + (S_C + S_A)w),\\
C_a &= ((S_A + S_B)u + (S_B - S_C)w : (S_A + S_B)v + (S_C + S_A)w : 0),\\
C_b &= ((S_B + S_C)w + (S_A + S_B)u : (S_A + S_B)v - (S_C - S_A)w : 0),\\
A_b &= (0 : (S_B + S_C)v + (S_C - S_A)u : (S_B + S_C)w + (S_A + S_B)u),\\
A_c &= (0 : (S_C + S_A)u + (S_B + S_C)v : (S_B + S_C)w - (S_A - S_B)u),\\
B_c &= ((S_C + S_A)u + (S_B + S_C)v : 0 : (S_C + S_A)w + (S_A - S_B)v).
\end{aligned}$$

For example, when $P = K$, these are the vertices of the second cosine circle.

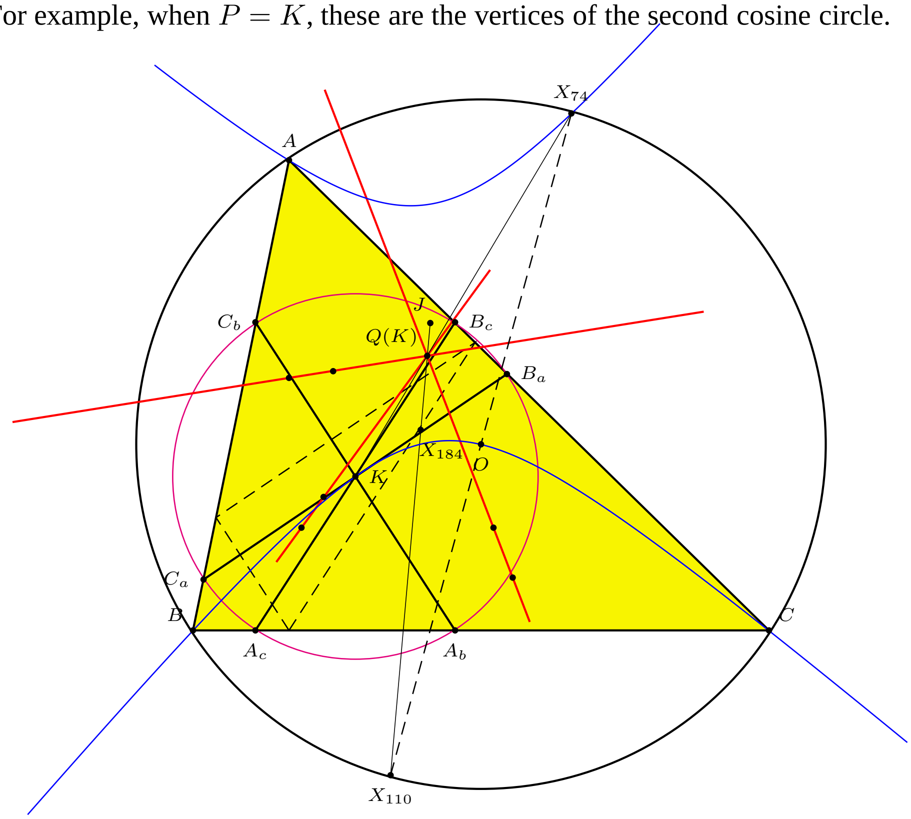

Figure 3. $Q(K)$ and the second Lemoine circle

Proposition 5. *The Euler lines of triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line*

$$\mathcal{L}_\text{p} : \qquad \frac{S_A(S_B - S_C)}{a^2}x + \frac{S_B(S_C - S_A)}{b^2}y + \frac{S_C(S_A - S_B)}{c^2}z = 0.$$

When P traverses $\mathcal{L}_\text{p}$, the intersection Q of the Euler lines traverses the line

$$\mathcal{L}_\text{q} : \qquad \sum_{\text{cyclic}} \frac{(b^2 - c^2)(a^2(S_{AA} + S_{BC}) - 4S_{ABC})}{a^2}x = 0.$$

For a point P on the line $\mathcal{L}_{\rm p}$, we denote by $Q(P)$ the corresponding point on $\mathcal{L}_{\rm q}$.

Proposition 6. *For points P_1, P_2, P_3 on $\mathcal{L}_{\rm p}$, $Q(P_1)$, $Q(P_2)$, $Q(P_3)$ are points on $\mathcal{L}_{\rm q}$ satisfying*

$$Q(P_1)Q(P_2) : Q(P_2)Q(P_3) = P_1P_2 : P_2P_3.$$

3.1. *The line $\mathcal{L}_{\rm p}$.* The line $\mathcal{L}_{\rm p}$ contains K and is the tangent to the Jerabek hyperbola at K. See Figure 4. It also contains, among others, the following points.

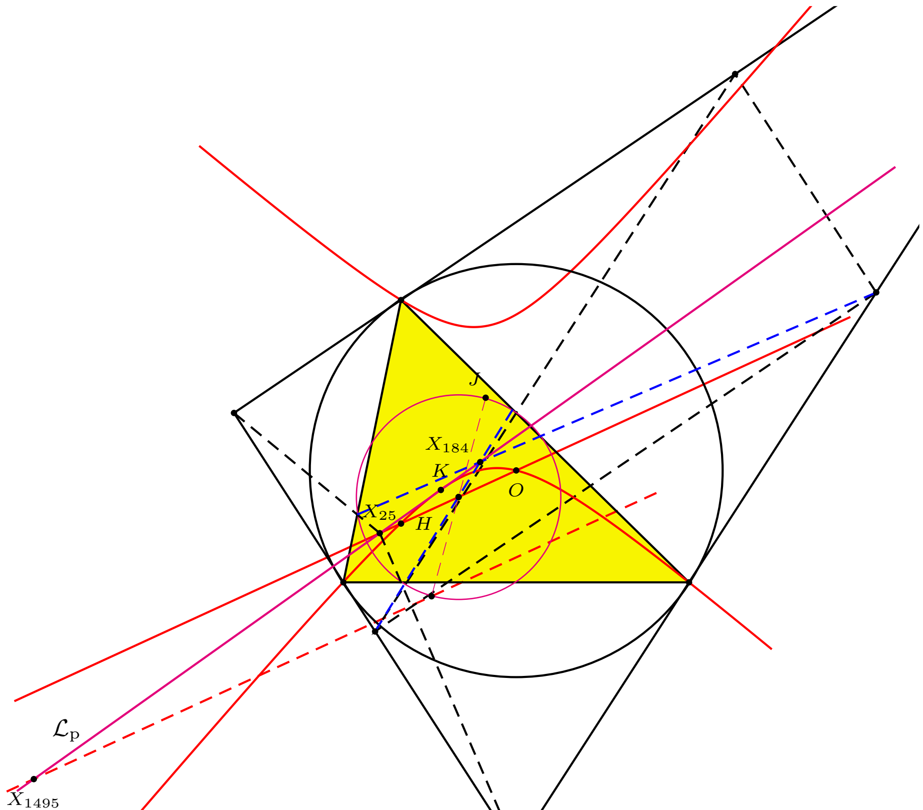

Figure 4. The line $\mathcal{L}_{\rm p}$

(1) $X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$ which is on the Euler line of ABC, and is the homothetic center of the orthic and the tangential triangles,[4]

(2) $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$ which is the homothetic center of the orthic triangle and the medial tangential triangle,[5]

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[4]See also §4.1.

[5]For other interesting properties of X_{184}, see [6], where it is named the procircumcenter of triangle ABC.

(3) $X_{1495} = (a^2(S_{CA} + S_{AB} - 2S_{BC}) : \cdots : \cdots)$ which lies on the parallel to the Euler line through the antipode of the Jerabek center on the nine-point circle. [6]

3.2. *The line* $\mathcal{L}_{\rm q}$. The line $\mathcal{L}_{\rm q}$ is parallel to the Brocard axis. See Figure 5. It contains the following points.

(1) $Q(K) = (a^2S_A(b^2c^2(S_{BB} - S_{BC} + S_{CC}) - 2a^2S_{ABC}) : \cdots : \cdots)$. It can be constructed as the intersection of the lines joining K to X_{74}, and J to X_{110}. See Figure 3 and §6 below. The line $\mathcal{L}_{\rm q}$ can therefore be constructed as the parallel through this point to the Brocard axis.

(2) $Q(X_{1495}) = (a^2S_A(a^2S^2 - 6S_{ABC}) : \cdots : \cdots)$, which is on the line joining O to X_{184} (on $\mathcal{L}_{\rm p}$).

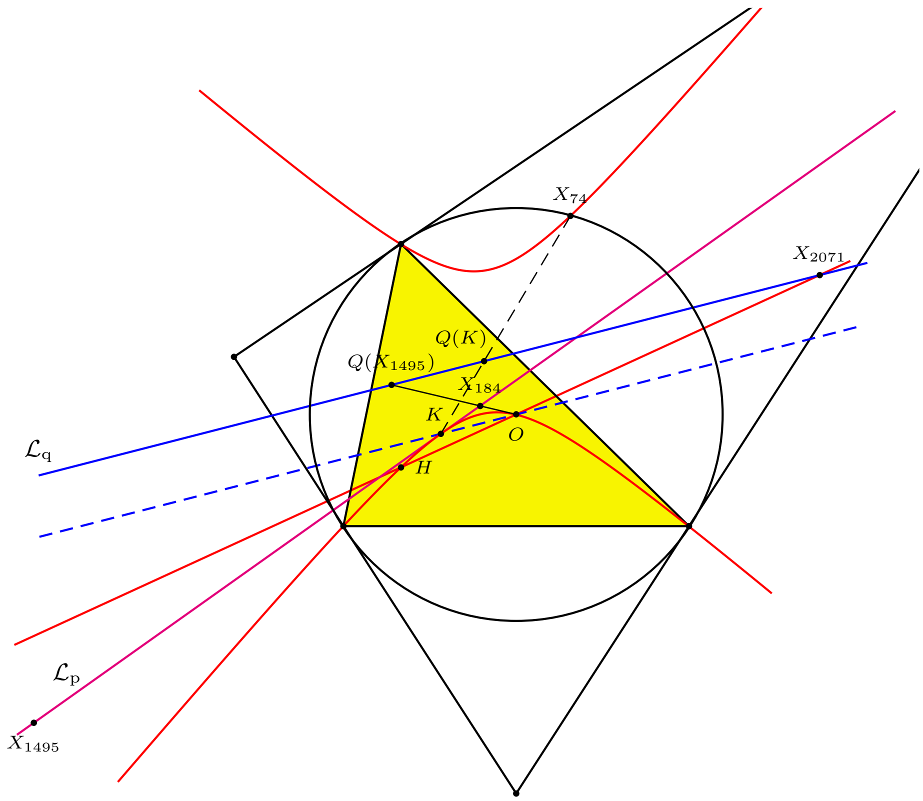

Figure 5. The line $\mathcal{L}_{\rm q}$

The line $\mathcal{L}_{\rm q}$ intersects the Euler line of ABC at the point

$$X_{2071} = (a^2(a^2S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots),$$

[6]This is the point X_{113}.

which is the inverse of the de Longchamps point in the circumcircle. This corresponds to the antiparallels through

$$P_{2071} = (a^4((a^2 S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots)$$

on the line $\mathcal{L}_\text{p}$. This point can be constructed by a simple application of Theorem 4 or Proposition 6. (See also Remark 2 following Theorem 12).

3.3. *The intersection of $\mathcal{L}_\text{p}$ and $\mathcal{L}_\text{q}$.* The lines $\mathcal{L}_\text{p}$ and $\mathcal{L}_\text{q}$ intersect at the point

$$M = (a^2 S_A(S_{AB} + S_{AC} + S_{BB} - 4S_{BC} + S_{CC}) : \cdots : \cdots).$$

(1) $Q(M)$ is the point on $\mathcal{L}_\text{q}$ with coordinates

$$(a^2 S_A(S_{AA}(S_{BB}+S_{CC})+a^2 S_A(S_{BB}-3S_{BC}+S_{CC})+S_{BC}(S_B-S_C)^2) : \cdots : \cdots).$$

(2) The point P on $\mathcal{L}_\text{p}$ for which $Q(P) = M$ has coordinates

$$(a^2(a^2(2S_{AA} - S_{BC}) + 2S_A(S_{BB} - 3S_{BC} + S_{CC})) : \cdots : \cdots).$$

4. The triangle $\mathbf{T}$ bounded by the antiparallels

We assume the line ℓ_i, $i = 1, 2, 3$, nonconcurrent so that they bound a nondegenerate triangle $\mathbf{T} = A_1B_1C_1$. Since these lines have equations

$$\begin{aligned}
-t_1 S_A(x + y + z) &= -S_A x + S_B y + S_C z,\\
-t_2 S_B(x + y + z) &= S_A x - S_B y + S_C z,\\
-t_3 S_C(x + y + z) &= S_A x + S_B y - S_C z,
\end{aligned}$$

the vertices of $\mathbf{T}$ are the points

$$\begin{aligned}
A_1 =&(-a^2(t_2 S_B + t_3 S_C) : 2S_{CA} + t_2 b^2 S_B + t_3 S_C(S_C - S_A)\\
&: 2S_{AB} + t_2 S_B(S_B - S_A) + t_3 c^2 S_C),\\
B_1 =&(2S_{BC} + t_3 S_C(S_C - S_B) + t_1 a^2 S_A : -b^2(t_3 S_C + t_1 S_A)\\
&: 2S_{AB} + t_3 c^2 S_C + t_1 S_A(S_A - S_B))\\
C_1 =&(2S_{BC} + t_1 a^2 S_A + t_2 S_B(S_B - S_C) : 2S_{CA} + t_1 S_A(S_A - S_C) + t_2 b^2 S_B\\
&: -c^2(t_1 S_A + t_2 S_B)).
\end{aligned}$$

4.1. *Homothety with the orthic triangle .* The triangle $\mathbf{T} = A_1B_1C_1$ is homothetic to the orthic triangle $A'B'C'$. The center of homothety is the point

$$P(\mathbf{T}) = \left(\frac{t_2 S_B + t_3 S_C}{S_A} : \frac{t_3 S_C + t_1 S_A}{S_B} : \frac{t_1 S_A + t_2 S_B}{S_C}\right), \tag{3}$$

and the ratio of homothety is

$$1 + \frac{t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC}}{2S_{ABC}}.$$

Proposition 7. *If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, the homothetic center $P(\mathbf{T})$ of $\mathbf{T}$ and the orthic triangle lies on the line $\mathcal{L}_\text{p}$.*

Proof. If we write $P(\mathbf{T}) = (x : y : z)$. From (3), we obtain

$$t_1 = \frac{-xS_A + yS_B + zS_C}{2S_A}, \quad t_2 = \frac{-yS_B + zS_C + xS_A}{2S_B}, \quad t_3 = \frac{-zS_C + xS_A + yS_B}{2S_C}.$$

Substitution in (2) yields the equation of the line $\mathcal{L}_\mathrm{p}$. □

For example, if $t_1 = t_2 = t_3 = t$, $P(\mathbf{T}) = X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$. [7] If the ratio of homothety is 0, triangle $\mathbf{T}$ degenerates into the point X_{25} on $\mathcal{L}_\mathrm{p}$. The intersection of $\mathcal{L}_\mathrm{c}$ and $\mathcal{L}_\mathrm{q}$ is the point

$$\begin{aligned} Q(X_{25}) =&(a^2S_A(b^4S_B^4 + c^4S_C^4 + a^2S_{AAA}(S_B - S_C)^2 \\ &- S_{ABC}(4a^2S_{BC} + 3S_A(S_B - S_C)^2)) : \cdots : \cdots). \end{aligned}$$

Remark. The line $\mathcal{L}_\mathrm{p}$ is also the locus of the centroid of $\mathbf{T}$ for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, concur.

4.2. *Common point of $\mathcal{L}_i$, $i = 1, 2, 3$, on the Brocard axis.* We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Brocard axis. A typical point on the Brocard axis, dividing the segment OK in the ratio $t : 1-t$, has coordinates

$$(a^2(S_A(S_A + S_B + S_C) + (S_{BC} - S_{AA})t) : \cdots : \cdots).$$

This point lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned} t_1 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{AA}) + b^2c^2(S_B + S_C - 2S_A)t}{2S_{AA}(S_A + S_B + S_C)}, \\ t_2 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{BB}) + c^2a^2(S_C + S_A - 2S_B)t}{2S_{BB}(S_A + S_B + S_C)}, \\ t_3 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{CC}) + a^2b^2(S_A + S_B - 2S_C)t}{2S_{CC}(S_A + S_B + S_C)}. \end{aligned}$$

The corresponding triangle $\mathbf{T}$ is homothetic to the orthic triangle at the point

$$(a^2(-(S_A + S_B + S_C) \cdot a^2S_A + t(-(2S_A + S_B + S_C)S_{BC} + b^2S_{CA} + c^2S_{AB}) : \cdots : \cdots),$$

which divides the segment $X_{184}K$ in the ratio $2t : 1-2t$. The ratio of homothety is $-\frac{a^2b^2c^2}{4S_{ABC}}$. These triangles are all directly congruent to the medial tangential triangle of ABC. We summarize this in the following proposition.

Proposition 8. *Corresponding to the family of triangles directly congruent to the medial tangential triangle, homothetic to orthic triangle at points on the line $\mathcal{L}_\mathrm{p}$, the common points of the Euler lines of $\mathcal{L}_i$, $i = 1, 2, 3$, all lie on the Brocard axis.*

[7] See also §3.1(1). The tangential triangle is $\mathbf{T}$ with $t = 1$.

5. Perspectivity of $\mathbf{T}$ with ABC

Proposition 9. *The triangles* $\mathbf{T}$ *and* ABC *are perspective if and only if*

$$\sum_{\text{cyclic}}(S_B - S_C)(t_1 S_{AA} - t_2 t_3 S_{BC}) = 0. \tag{4}$$

Proof. From the coordinates of the vertices of $\mathbf{T}$, it is straightforward to check that $\mathbf{T}$ and ABC are perspective if and only if

$$t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC} + 2S_{ABC} = 0$$

or (4) holds. Since the area of triangle $\mathbf{T}$ is

$$\frac{(t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC} + 2S_{ABC})^2}{a^2 b^2 c^2 S_{ABC}}$$

times that of triangle ABC, we assume $t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC} + 2S_{ABC} \neq 0$ and (4) is the necessary and sufficient condition for perspectivity. □

Theorem 10. *If the triangle* $\mathbf{T}$ *is nondegenerate and is perspective to* ABC*, then the perspector lies on the Jerabek hyperbola of* ABC*.*

Proof. If triangles $A_1B_1C_1$ and ABC are perspective at $P = (x : y : z)$, then

$$A_1 = (u + x : y : z), \quad B_1 = (x : v + y : z), \quad C_1 = (x : y : w + z)$$

for some u, v, w. Since the line B_1C_1 is parallel to $B'C'$, which has infinite point $(S_B - S_C : -(S_C + S_A) : S_A + S_B)$, we have

$$\begin{vmatrix} S_B - S_C & -(S_C + S_A) & S_A + S_B \\ x & y + v & z \\ x & y & z + w \end{vmatrix} = 0,$$

and similarly for the other two lines. These can be rearranged as

$$\begin{aligned}
\frac{(S_C + S_A)x - (S_B - S_C)y}{v} - \frac{(S_B - S_C)z + (S_A + S_B)x}{w} &= S_B - S_C, \\
\frac{(S_A + S_B)y - (S_C - S_A)z}{w} - \frac{(S_C - S_A)x + (S_B + S_C)y}{u} &= S_C - S_A, \\
\frac{(S_B + S_C)z - (S_A - S_B)x}{u} - \frac{(S_A - S_B)y + (S_C + S_A)z}{v} &= S_A - S_B.
\end{aligned}$$

Multiplying these equations respectively by

$$S_A(S_B + S_C)yz, \qquad S_B(S_C + S_A)zx, \qquad S_C(S_A + S_B)xy$$

and adding up, we obtain

$$\left(1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right)\sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz = 0.$$

Since the area of triangle $\mathbf{T}$ is

$$uvw\left(1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right)$$

times that of triangle ABC, we must have $1+\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\neq 0$. It follows that

$$\sum_{\text{cyclic}} S_A(S_{BB}-S_{CC})yz=0.$$

This means that P lies on the Jerabek hyperbola. □

We shall identify the locus of the common points of Euler lines in §8.3 below. In the meantime, we give a construction for the point Q from the perspector on the Jerabek hypebola.

Construction. Given a point P on the Jerabek hyperbola, construct parallels to $A'B'$ and $A'C'$ through an arbitrary point A_1' on the line AP. Let M_1 be the intersection of the Euler lines of the triangles formed by these antiparallels and the sidelines of ABC. With another point A_1'' obtain a point M_2 by the same construction. Similarly, working with two points B_1' and B_1'' on BP, we construct another line M_3M_4. The intersection of M_1M_2 and M_3M_4 is the common point Q of the Euler lines corresponding the antiparallels that bound a triangle perspective to ABC at P.

6. The Tucker hexagons and the line $\mathcal{L}_{\rm t}$

It is well known that if the antiparallels, together with the sidelines of triangle ABC, bound a Tucker hexagon, the vertices lie on a circle whose center is on the Brocard axis. If this center divides the segment OK in the ratio $t:1-t$, the antiparallels pass through the points dividing the symmedians in the same ratio. The vertices of the Tucker hexagon are

$$\begin{array}{ll} B_a=(S_C+(1-t)c^2:0:tc^2), & C_a=(S_B+(1-t)b^2:tb^2:0),\\ C_b=(ta^2:S_A+(1-t)a^2:0), & A_b=(0:S_C+(1-t)c^2:tc^2),\\ A_c=(0:tb^2:S_B+(1-t)b^2), & B_c=(ta^2:0:S_A+(1-t)a^2). \end{array}$$

In this case,

$$1-t_1=\frac{t\cdot b^2c^2}{S_A(S_A+S_B+S_C)},\quad 1-t_2=\frac{t\cdot c^2a^2}{S_B(S_A+S_B+S_C)},\quad 1-t_3=\frac{t\cdot a^2b^2}{S_C(S_A+S_B+S_C)}.$$

It is clear that the Euler lines $\mathcal{L}_i$, $i=1,2,3$, are concurrent. As t varies, this common point traverses a straight line $\mathcal{L}_{\rm t}$. We show that this is the line joining K to $Q(K)$.

(1) For $t=1$, this Tucker circle is the second Lemoine circle with center K, the triangle **T** degenerates into the point K. The common point of the Euler lines is therefore the point $Q(K)$. See §3.2 and Figure 3.

(2) For $t=\frac{3}{2}$, the vertices of the Tucker hexagon are

$$\begin{array}{ll} B_a=(a^2+b^2-2c^2:0:3c^2), & C_a=(c^2+a^2-2b^2:3b^2:0),\\ C_b=(3a^2:b^2+c^2-2a^2:0), & A_b=(0:a^2+b^2-2c^2:3c^2),\\ A_c=(0:3b^2:c^2+a^2-2b^2), & B_c=(3a^2:0:b^2+c^2-2a^2). \end{array}$$

The triangles $\mathbf{T}_i$, $i = 1, 2, 3$, have a common centroid K, which is therefore the common point of their Euler lines. The corresponding Tucker center is the point X_{576} (which divides OK in the ratio $3 : -1$).

From these, we obtain the equation of the line

$$\mathcal{L}_{\mathrm{t}} : \qquad \sum_{\text{cyclic}} b^2c^2S_A(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

Remarks. (1) The triangle $\mathbf{T}$ is perspective to ABC at K. See, for example, [5].

(2) The line $\mathcal{L}_{\mathrm{t}}$ also contains X_{74} which we may regard as corresponding to $t = 0$.

For more about Tucker hexagons, see §8.2.

7. Concurrency of four or more Euler lines

7.1. *Common point of* $\mathcal{L}_i$, $i = 1, 2, 3$, *on the Euler line of* ABC. We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Euler line of ABC. A typical point on the Euler line axis divides the segment OH in the ratio $t : 1 - t$, has coordinates

$$(a^2S_A - (S_{CA} + S_{AB} - 2S_{BC})t) : \cdots : \cdots).$$

This lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned} t_1 =& \frac{-(S^2 - S_{AA}) + (S^2 - 3S_{AA})t}{2S_{AA}}, \\ t_2 =& \frac{-(S^2 - S_{BB}) + (S^2 - 3S_{BB})t}{2S_{BB}}, \\ t_3 =& \frac{-(S^2 - S_{CC}) + (S^2 - 3S_{CC})t}{2S_{CC}}. \end{aligned}$$

Independently of t, the corresponding triangle $\mathbf{T}$ is always homothetic to the medial tangential triangle at the point P_{2071} on the line $\mathcal{L}_{\mathrm{p}}$ for which $Q(P_{2071}) = X_{2071}$, the intersection of $\mathcal{L}_{\mathrm{q}}$ with the Euler line. See the end of §3.2 above. The ratio of homothety is $1 + t - \frac{8S_{ABC}}{a^2b^2c^2}t$. We summarize this in the following proposition.

Proposition 11. *Let P_{2071} be the point on $\mathcal{L}_{\mathrm{p}}$ such that $Q(P_{2071}) = X_{2071}$. The Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, corresponding to the sidelines of triangles homothetic at P_{2071} to the medial tangential triangle intersect on the Euler line of ABC.*

7.2. *The line* $\mathcal{L}_{\mathrm{f}}$. The Euler line of triangle $\mathbf{T}$ is the line

$$\begin{aligned} &(x + y + z) \sum_{\text{cyclic}} t_1a^2S_{AA}(S_B - S_C)(S^2 + S_{BC})(S^2 - S_{AA}) \\ =&2S_{ABC} \sum_{\text{cyclic}} (S^2 + S_{CA})(S^2 + S_{AB})x. \end{aligned} \qquad (5)$$

Theorem 12. *The Euler lines of the four triangles* $\mathbf{T}$ *and* $\mathbf{T}_i$, $i = 1, 2, 3$, *are concurrent if and only if*

$$t_1 = -\frac{16S^2 \cdot S_{ABC} + t(a^2b^4c^4 - 4S_{ABC}(3S^2 - S_{AA}))}{4S_{AA}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_2 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^2c^4 - 4S_{ABC}(3S^2 - S_{BB}))}{4S_{BB}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_3 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^4c^2 - 4S_{ABC}(3S^2 - S_{CC}))}{4S_{CC}(a^2b^2c^2 + 4S_{ABC})},$$

with $t \neq \frac{-24a^2b^2c^2S_{ABC}}{(a^2b^2c^2-8S_{ABC})(3(S_A+S_B+S_C)S^2+S_{ABC})}$. *The locus of the common point of the four Euler lines is the line* $\mathcal{L}_{\rm f}$ *joining the nine-point center of* ABC *to* X_{184}, *with the intersection with* $\mathcal{L}_{\rm q}$ *deleted.*

Proof. The equation of the Euler line $\mathcal{L}_i$, $i = 1, 2, 3$, can be rewritten as

$$\begin{aligned}
&t_1S_A(S_B - S_C)(x + y + z) + S_{AA}(S_B - S_C)x \\
&+(S_{AB}(S_B - S_C) - (S_{AA} - S_{BB})S_C)y + (S_{AC}(S_B - S_C) + (S_{AA} - S_{CC})S_B)z = 0, \quad (6)\\
&t_2S_A(S_B - S_C)(x + y + z) + S_{BB}(S_C - S_A)y \\
&+(S_{BA}(S_C - S_A) + (S_{BB} - S_{AA})S_C)x + (S_{BC}(S_C - S_A) - (S_{BB} - S_{CC})S_A)z = 0, \quad (7)\\
&t_3S_C(S_A - S_B)(x + y + z) + S_{CC}(S_A - S_B)z \\
&+(S_{CA}(S_A - S_B) - (S_{CC} - S_{AA})S_B)x + (S_{CB}(S_A - S_B) + (S_{CC} - S_{BB})S_A)y = 0. \quad (8)
\end{aligned}$$

Multiplying (4), (5), (6) respectively by

$$a^2S_A(S^2+S_{BC})(S^2-S_{AA}), \quad b^2S_B(S^2+S_{CA})(S^2-S_{BB}), \quad c^2S_C(S^2+S_{AB})(S^2-S_{CC}),$$

and adding, we obtain by Theorem 10 the equation of the line

$$\mathcal{L}_{\rm f}: \qquad \sum_{\text{cyclic}} (S_B - S_C)(S^2(2S_{AA} - S_{BC}) + S_{ABC} \cdot S_A)x = 0$$

which contains the common point of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, if it also lies on the Euler line $\mathcal{L}$ of $\mathbf{T}$. The line $\mathcal{L}_{\rm f}$ contains the nine-point center X_5 and $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$. Let Q_t be the point which divides the segment $X_{184}X_5$ in the ratio $t : 1 - t$. It has coordinates

$$\begin{aligned}
&((1 - t)4S^2 \cdot a^4S_A + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + S_{AB} + 2S_{BC}) \\
&: (1 - t)4S^2 \cdot b^4S_B + t(a^2b^2c^2 + 4S_{ABC})(2S_{CA} + S_{AB} + S_{BC}) \\
&: (1 - t)4S^2 \cdot c^4S_C + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + 2S_{AB} + S_{BC})).
\end{aligned}$$

The point Q_t lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, respectively if we choose t_1, t_2, t_3 given above.

If Q lies on $\mathcal{L}_\text{q}$, then $Q_t = Q(P)$ for some point P on $\mathcal{L}_\text{p}$. [8] In this case, the triangle $\mathbf{T}$ degenerates into the point $P \neq Q$ and its Euler line is not defined. It should be excluded from $\mathcal{L}_\text{f}$. The corresponding value of t is as given in the statement above. □

Here are some interesting points on $\mathcal{L}_\text{f}$.

(1) For $t = 0$, $\mathbf{T}$ is perspective with ABC at X_{74}, and the common point of the four Euler lines is X_{184}. The antiparallels are drawn through the intercepts of the trilinear polars of $X_{186} = \left(\frac{a^2}{S_A(S^2-3S_{AA})} : \cdots : \cdots\right)$, the inversive image of the orthocenter in the circumcircle.

(2) For $t = 1$, this common point is the nine-point of triangle ABC. The triangle $\mathbf{T}$ is homothetic to the orthic triangle at X_{51} and to the medial tangential triangle at the point P_{2071} in §3.2.

(3) $t = -\frac{a^2b^2c^2}{4S_{ABC}}$ gives X_{156}, the nine-point center of the tangential triangle.

In these two cases, we have the concurrency of five Euler lines.

(4) The line $\mathcal{L}_\text{f}$ intersects the Brocard axis at X_{569}. This corresponds to $t = \frac{2a^2b^2c^2}{3a^2b^2c^2+4S_{ABC}}$.

Proposition 13. *The triangle* $\mathbf{T}$ *is perspective with* ABC *and its Euler line contains the common point of the Euler lines of* $\mathbf{T}_i$, $i = 1, 2, 3$ *precisely in the following three cases.*

(1) $t = 0$, *with perspector* X_{74} *and common point of Euler line* X_{184}.

(2) $t = \frac{-12a^2b^2c^2S_{ABC}}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}$, *with perspector* K.

Remarks. (1) In the first case,

$$t_1 = \frac{k}{S_{AA}}, \qquad t_2 = \frac{k}{S_{BB}}, \qquad t_3 = \frac{k}{S_{CC}}$$

for $k = -\frac{4S^2 \cdot S_{ABC}}{a^2b^2c^2+4S_{ABC}}$. The antiparallels pass through the intercepts of the trilinear polar of X_{186}, the inversive image of H in the circumcircle.

(2) In the second case, the antiparallels bound a Tucker hexagon. The center of the Tucker circle divides OK in the ratio $t : 1-t$, where

$$t = \frac{S^2(S_A+S_B+S_C)(a^2b^2c^2-16S_{ABC})}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}.$$

It follows that the common point of the Euler lines is the intersection of the lines $\mathcal{L}_\text{f} = X_5X_{184}$ and $\mathcal{L}_\text{t}$.

8. Common points of $\mathcal{L}_i$, $i = 1, 2, 3$, when T is perspective

If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, then, according to (2) we may put

$$t_1 = \frac{k(\lambda+S_A)}{a^2S_{AA}}, \qquad t_2 = \frac{k(\lambda+S_B)}{b^2S_{BB}}, \qquad t_3 = \frac{k(\lambda+S_C)}{c^2S_{CC}}$$

[8] This point is the intersection of $\mathcal{L}_\text{p}$ with the line joining the Jerabek center J to X_{323}, the reflection in X_{110} of the inversive image of the centroid in the circumcircle.

for some λ and k. If, also, the $\mathbf{T}$ is perspective, (4) gives

$$k(k\lambda + S_{ABC})(\lambda + S_A + S_B + S_C)(k(3\lambda + S_A + S_B + S_C) + 2S_{ABC}) = 0.$$

If $k = 0$, $\mathbf{T}$ is the orthic triangle. We consider the remaining three cases below.

8.1. *The case* $k(S_A + S_B + S_C + 3\lambda) + 2S_{ABC} = 0$. In this case,

$$t_1 = -\frac{2S_{ABC} + k(S_B + S_C - 2S_A)}{3a^2S_{AA}},$$
$$t_2 = -\frac{2S_{ABC} + k(S_C + S_A - 2S_B)}{3b^2S_{BB}},$$
$$t_3 = -\frac{2S_{ABC} + k(S_A + S_B - 2S_C)}{3c^2S_{CC}}.$$

The antiparallels are concurrent.

8.2. *The case* $k\lambda + S_{ABC} = 0$. In this case,

$$t_1 = \frac{k - S_{BC}}{a^2S_A}, \qquad t_2 = \frac{k - S_{CA}}{b^2S_B}, \qquad t_3 = \frac{k - S_{AB}}{c^2S_C}.$$

In this case, the perspector is the Lemoine point K. The antiparallels bound a Tucker hexagon. The locus of the common point of Euler lines is the line $\mathcal{L}_t$. Here are some more interesting points on this line.

(1) For $k = 0$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_C + S_A)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B)}.$$

This gives the Tucker hexagon with vertices

$$\begin{array}{ll} B_a = (S_{CC} : 0 : S^2), & C_a = (S_{BB} : S^2 : 0), \\ C_b = (S^2 : S_{AA} : 0), & A_b = (0 : S_{CC} : S^2), \\ A_c = (0 : S^2 : S_{BB}), & B_c = (S^2 : 0 : S_{AA}). \end{array}$$

These are the pedals of A', B', C' on the sidelines. The Tucker circle is the Taylor circle. The triangle $\mathbf{T}$ is the medial triangle of the orthic triangle. The corresponding Euler lines intersect at X_{974}, which is the intersection of $\mathcal{L}_t = KX_{74}$ with X_5X_{125}. See [2].

(2) For $k = \frac{S_{ABC}}{S_A+S_B+S_C}$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_A + S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_A + S_B + S_C)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B + S_C)}.$$

The Tucker circle is the second Lemoine circle, considered in §6.

(3) The line $\mathcal{L}_t$ intersects the Euler line at

$$X_{378} = \left(\frac{a^2(S^2 + 3S_{AA})}{S_A} : \cdots : \cdots\right).$$

The corresponding Tucker circle has center

$$(S^2(S_B+S_C)(S_C-S_A)(S_A-S_B)+3(S_A+S_B)(S_B+S_C)(S_C+S_A)S_{BC}:\cdots:\cdots)$$

which is the intersection of the Brocard axis and the line joining the orthocenter to X_{110}.

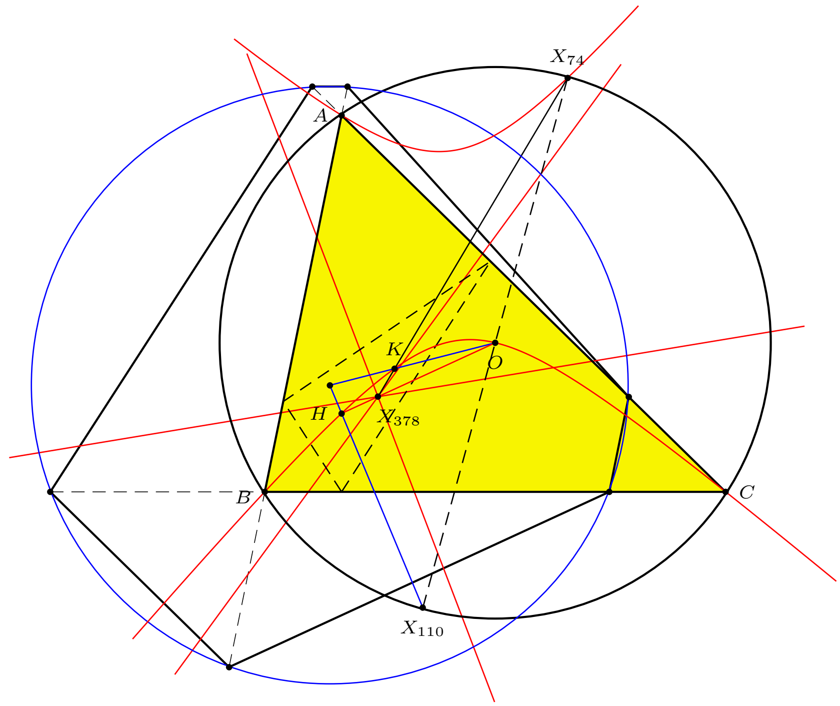

Figure 6. Intersection of 4 Euler lines at X_{378}

8.3. *The case* $\lambda = -(S_A + S_B + S_C)$. In this case, we have

$$t_1 = -\frac{k}{S_{AA}}, \qquad t_2 = -\frac{k}{S_{BB}}, \qquad t_3 = -\frac{k}{S_{CC}}.$$

In this case, the perspector is the point

$$\left(\frac{1}{2S_{ABC}\cdot S_A - k(b^2c^2 - 2S_{BC})}:\cdots:\cdots\right)$$

on the Jerabek hyperbola. If the point on the Jerabek hyperbola is the isogonal conjugate of the point which divides OH in the ratio $t : 1-t$, then

$$k = \frac{4tS^2\cdot S_{ABC}}{a^2b^2c^2(1+t)+4t\cdot S_{ABC}}.$$

The locus of the intersection of the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, is clearly a line. Since this intersection is the Jerabek center for $k = 0$ (Thébault's theorem) and the centroid for $k = \frac{S^2}{3}$, this is the line

$$\mathcal{L}_{\rm r}: \qquad \sum_{\rm cyclic} (S_B - S_C)(S_{BC} - S_{AA})x = 0.$$

This line also contains, among other points, X_{110} and X_{184}. We summarize the general situation in the following theorem.

Theorem 14. *Let P be a point on the Euler line other than the centroid G. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC (at a point on the Jerabek hyperbola). The Euler lines of the triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent (at a point Q on the line $L_{\rm r}$ joining the centroid G to X_{110}).*

Here are some interesting examples with P easily constructed on the Euler line.

P	Perspector	Q
H	H	X_{125}
O	$X_{64} = X_{20}^*$	X_{110}
X_{30}	X_{2071}^*	G
X_{186}	X_{74}	X_{184}
X_{403}	$X_{265} = X186^*$	X_{1899}
X_{23}	$X_{1177} = X_{858}^*$	X_{182}
X_{858}		X_{1352}
X_{1316}		X_{98}

Remarks. (1) X_{186} is the inversive image of H in the circumcircle.
(2) X_{403} is the midpoint between H and X_{186}.
(3) X_{23} is the inversive image of G in the circumcircle.
(4) X_{858} is the inferior of X_{23}.
(5) X_{182} is the midpoint of OK, the center of the Brocard circle.
(6) X_{1352} is the reflection of K in the nine-point center.
(7) X_{1316} is the intersection of the Euler line and the Brocard circle apart from O.

9. Two loci: a line and a cubic

We conclude this paper with a brief discussion on two locus problems.

9.1. *Antiparallels through the vertices of a pedal triangle.* Suppose the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the pedal triangle of a finite point P. Then the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line

$$\sum_{\rm cyclic} S_A(S_B - S_C)(S_{AA} - S_{BC})x = 0.$$

This is the line containing H and the Tarry point X_{98}. For $P = H$, the common point of the Euler line is

$$X_{185} = (a^2 S_A(S_A(S_{BB} + S_{CC}) + a^2 S_{BC}) : \cdots : \cdots).$$

9.2. *Antiparallels through the vertices of a cevian triangle.* If, instead, the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the cevian triangle of P, then the locus of P for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent is the cubic

$$\mathcal{K}: \quad \frac{S_A + S_B + S_C}{S_{ABC}} xyz + \sum_{\text{cyclic}} \frac{x}{S_A(S_B - S_C)} \left(\frac{S_A + S_B}{S_C} y^2 - \frac{S_C + S_A}{S_B} z^2 \right) = 0.$$

This can also be written in the form

$$\begin{aligned} &(\sum_{\text{cyclic}} (S_B + S_C) yz)(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C - S_A)x) \\ =&(\sum_{\text{cyclic}} S_A(S_B - S_C)x)(\sum_{\text{cyclic}} S_A(S_B + S_C)yz). \end{aligned}$$

From this, we obtain the following points on $\mathcal{K}$:

- the orthocenter H (as the intersection of the Euler line and the line $\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C - S_A)x) = 0$),
- the Euler reflection point X_{110} (as the "fourth" intersection of the circumcircle and the circumconic $\sum_{\text{cyclic}} S_A(S_B + S_C)yz = 0$ with center K),
- the intersections of the Euler line with the circumcircle, the points X_{1113} and X_{1114}.

Corresponding to $P = X_{110}$, the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect at the circumcenter O. On the other hand, X_{1113} and X_{1114} are the points

$$(a^2 S_A + \lambda(S_{CA} + S_{AB} - 2S_{BC}) : \cdots : \cdots)$$

for $\lambda = -\frac{abc}{\sqrt{a^2b^2c^2 - 8S_{ABC}}}$ and $\lambda = \frac{abc}{\sqrt{a^2b^2c^2 - 8S_{ABC}}}$ respectively. The antiparallels through the traces of each of these points correspond to

$$t_1 = t_2 = t_3 = \frac{\lambda - 1}{\lambda + 1}.$$

This means that the corresponding intersections of Euler lines lie on the line $\mathcal{L}_c = HJ$ in §2.2.

9.3. *The cubic $\mathcal{K}$.* The infinite points of the cubic $\mathcal{K}$ can be found by rewriting the equation of $\mathcal{K}$ in the form

$$\begin{aligned} &(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C)yz)(\sum_{\text{cyclic}} (S_B + S_C)x) \\ =&(x + y + z)(\sum_{\text{cyclic}} (S_B + S_C)(S_B - S_C)(S_A(S_A + S_B + S_C) - S_{BC})yz) \end{aligned}$$

They are the infinite points of the Jerabek hyperbola and the line $(S_B + S_C)x + (S_C + S_A)y + (S_A + S_B)z = 0$. The latter is $X_{523} = (S_B - S_C : S_C - S_A : S_A - S_B)$. The asymptotes of $\mathcal{K}$ are

- the parallels to the asymptotes of Jerabek hyperbola through the antipode the Jerabek center on the nine-point circle, *i.e.*,

$$X_{113} = ((S_{CA}+S_{AB}-2S_{BC})(b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}) : \cdots : \cdots),$$

- the perpendicular to the Euler line (of ABC) at the circumcenter O, intersecting $\mathcal{K}$ again at

$$Z = \left(\frac{S_{CA}+S_{AB}-2S_{BC}}{b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}} : \cdots : \cdots\right),$$

which also lies on the line joining H to X_{110}. See Figure 7. [9]

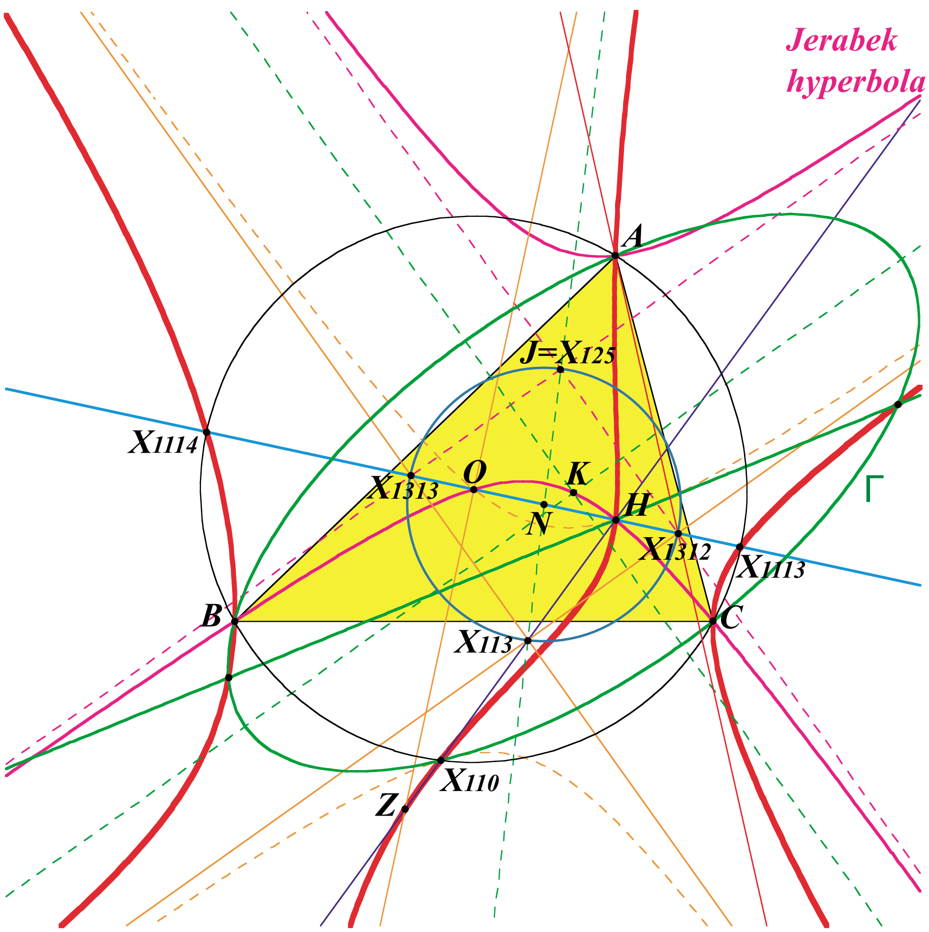

Figure 7. The cubic $\mathcal{K}$

[9] We thank Bernard Gibert for providing the sketch of $\mathcal{K}$ in Figure 7.

Remark. The asymptotes of $\mathcal{K}$ and the Jerabek hyperbola bound a rectangle inscribed in the nine-point circle. Two of the vertices and $J = X_{125}$ and its antipode X_{113}. The other two are the points X_{1312} and X_{1313} on the Euler line.

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