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Extreme Areas of Triangles in Poncelet's Closure Theorem

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Abstract. Among the triangles with the same incircle and circumcircle, we determine the ones with maximum and miniumum areas. These are also the ones with maximum and minimum perimeters and sums of altitudes.

Given two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of radii r and R whose centers are at a distance d apart satisfying Euler's relation

$$R^2 - d^2 = 2Rr, \tag{1}$$

by Poncelet's closure theorem, for every point A_1 on the circle $\mathcal{C}_2$, there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$. In this article we determine those triangles with extreme areas, perimeters, and sum of altitudes.

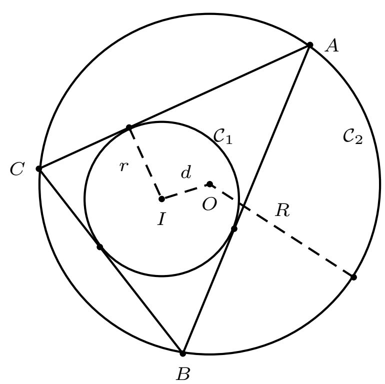

Figure 1a

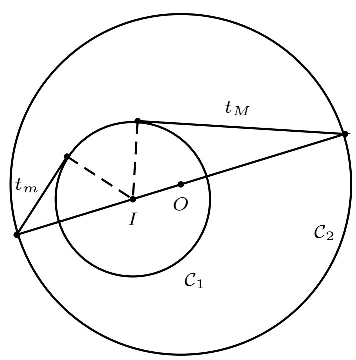

Figure 1b

Denote by t_m and t_M respectively the lengths of the shortest and longest tangents that can be drawn from C_2 to C_1. These are given by

$$t_m = \sqrt{(R-d)^2 - r^2}, \qquad t_M = \sqrt{(R+d)^2 - r^2}. \tag{2}$$

We shall use the following result given in [2, Theorem 2.2]. Let t_1 be any given length satisfying

$$t_m \leq t_1 \leq t_M, \tag{3}$$

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and let t_2 and t_3 be given by

$$t_2 = \frac{2Rrt_1 + \sqrt{D}}{r^2 + t_1^2}, \qquad t_3 = \frac{2Rrt_1 - \sqrt{D}}{r^2 + t_1^2}, \tag{4}$$

where

$$D = 4R^2r^2t_1^2 - r^2(r^2 + t_1^2)(4Rr + r^2 + t_1^2).$$

Then there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$ with side lengths

$$a_i = |A_iA_{i+1}| = t_i + t_{i+1}, \qquad i = 1, 2, 3. \tag{5}$$

Here, the indices are taken modulo 3. It is easy to check that

$$(t_1 + t_2 + t_3)r^2 = t_1t_2t_3,$$
$$t_1t_2 + t_2t_3 + t_3t_1 = 4Rr + r^2,$$

and that these are necessary and sufficient for C_1 and C_2 to be the incircle and circumcircle of triangle $A_1A_2A_3$.

Denote by $J(t_1)$ the area of triangle $A_1A_2A_3$. Thus,

$$J(t_1) = r(t_1 + t_2 + t_3). \tag{6}$$

Note that $D = 0$ when $t_1 = t_m$ or $t_1 = t_M$. In these cases,

$$t_2 = t_3 = \begin{cases} \frac{2Rrt_m}{r^2+t_m^2}, & \text{if } t_1 = t_m, \\ \\ \frac{2Rrt_M}{r^2+t_M^2}, & \text{if } t_1 = t_M. \end{cases}$$

For convenience, we shall write

$$\widehat{t_m} = \frac{2Rrt_m}{r^2 + t_m^2} \quad \text{and} \quad \widehat{t_M} = \frac{2Rrt_M}{r^2 + t_M^2}. \tag{7}$$

Theorem 1. *$J(t_1)$ is maximum when $t_1 = t_M$ and minimum when $t_1 = t_m$. In other words, $J(t_m) \leq J(t_1) \leq J(t_M)$ for $t_m \leq t_1 \leq t_M$.*

Proof. It follows from (6) and (4) that

$$J(t_1) = r\left(t_1 + \frac{4Rrt_1}{r^2 + t_1^2}\right).$$

From $\frac{d}{dt_1}J(t_1) = 0$, we obtain the equation

$$t_1^4 - 2(2Rr - r^2)t_1^2 + 4Rr^3 + r^4 = 0,$$

and

$$t_1^2 = 2Rr - r^2 \pm 2r\sqrt{R^2 - 2Rr} = 2Rr - r^2 \pm 2rd.$$

Since $4R^2r^2 = (R^2 - d^2)^2$, we have

$$\begin{aligned}
&2Rr - r^2 + 2rd - \widehat{t_m}^2 \\
=&2Rr - r^2 + 2rd - \frac{(R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{(R-d)^2(2Rr - r^2 + 2rd) - (R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{((R+d)^2 - (R-d)^2)r^2 + 2r(R+d)(R-d)^2 - (R^2-d^2)^2}{(R-d)^2} \\
=&\frac{4Rdr^2 + 2r(R-d)(2Rr) - (2Rr)^2}{(R-d)^2} \\
=&0.
\end{aligned}$$

Similarly, $2Rr - r^2 - 2rd - \widehat{t_M}^2 = 0$. It follows that $\frac{d}{dt_1}J(t_1) = 0$ for $t_1 = \widehat{t_m}, \widehat{t_M}$. The maximum of J occurs at $t_1 = t_M$ and $\widehat{t_M}$ while the minimum occurs at $t_1 = t_m$ and $\widehat{t_m}$.

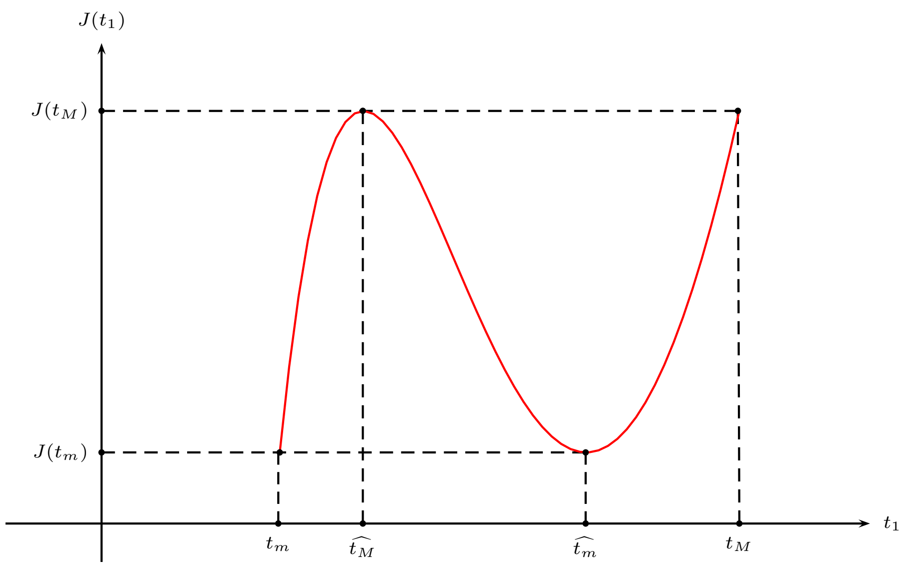

Figure 2

The triangle determined by $\widehat{t_m}$ (respectively $\widehat{t_M}$) is exactly the one determined by t_m (respectively t_M).

□

We conclude with an interesting corollary. Let h_1, h_2, h_3 be the altitudes of the triangle $A_1A_2A_3$. Since

$$2R(h_1 + h_2 + h_3) = a_1a_2 + a_2a_3 + a_3a_1 = (t_1 + t_2 + t_3)^2 + 4Rr + r^2,$$

the following are equivalent:

- the triangle has maximum (respectively minimum) area,
- the triangle has maximum (respectively minimum) perimeter,
- the triangle has maximum (respectively minimum) sum of altitudes.

It follows that these are precisely the two triangles determined by t_M and t_m.

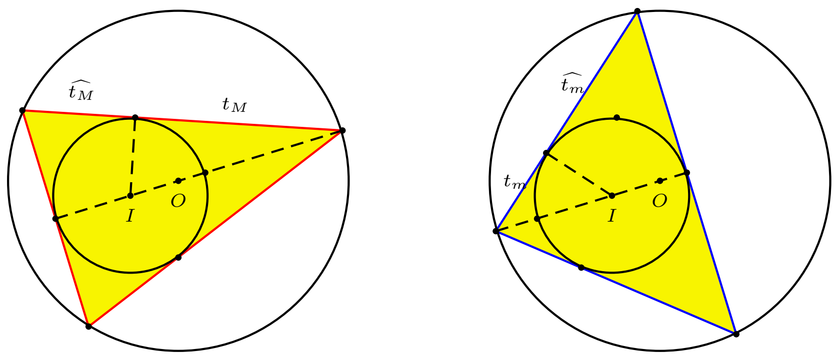

Figure 3a Figure 3b

References

[1] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.

[2] M. Radić, Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem, *Math. Maced.* 1 (2003) 35–58.

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