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The Archimedean Circles of Schoch and Woo

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Abstract. We generalize the Archimedean circles in an arbelos (shoemaker's knife) given by Thomas Schoch and Peter Woo.

1. Introduction

Let three semicircles α, β and γ form an arbelos, where α and β touch externally at the origin O. More specifically, α and β have radii a, $b > 0$ and centers $(a, 0)$ and $(-b, 0)$ respectively, and are erected in the upper half plane $y \geq 0$. The y-axis divides the arbelos into two curvilinear triangles. By a famous theorem of Archimedes, the inscribed circles of these two curvilinear triangles are congruent and have radii $r = \frac{ab}{a+b}$. See Figure 1. These are called the twin circles of Archimedes. Following [2], we call circles congruent to these twin circles Archimedean circles.

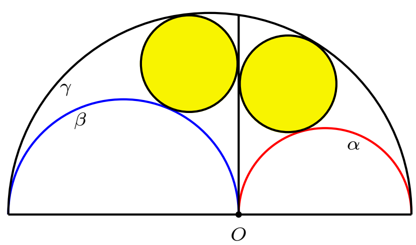

Figure 1

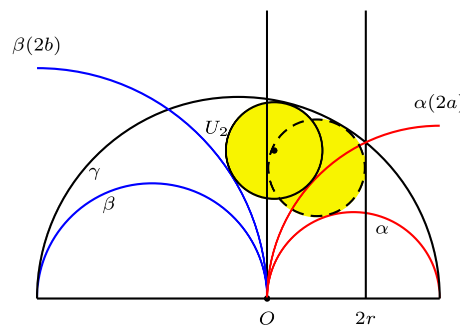

Figure 2

For a real number n, denote by $\alpha(n)$ the semicircle in the upper half-plane with center $(n, 0)$, touching α at O. Similarly, let $\beta(n)$ be the semicircle with center $(-n, 0)$, touching β at O. In particular, $\alpha(a) = \alpha$ and $\beta(b) = \beta$. T. Schoch has found that

(1) the distance from the intersection of $\alpha(2a)$ and γ to the y-axis is $2r$, and
(2) the circle U_2 touching γ internally and each of $\alpha(2a)$, $\beta(2b)$ externally is Archimedean. See Figure 2.

P. Woo considered the Schoch line $\mathcal{L}_s$ through the center of U_2 parallel to the y-axis, and showed that for every nonnegative real number n, the circle U_n with center on $\mathcal{L}_s$ touching $\alpha(na)$ and $\beta(nb)$ externally is also Archimedean. See Figure 3. In this paper we give a generalization of Schoch's circle U_2 and Woo's circles U_n.

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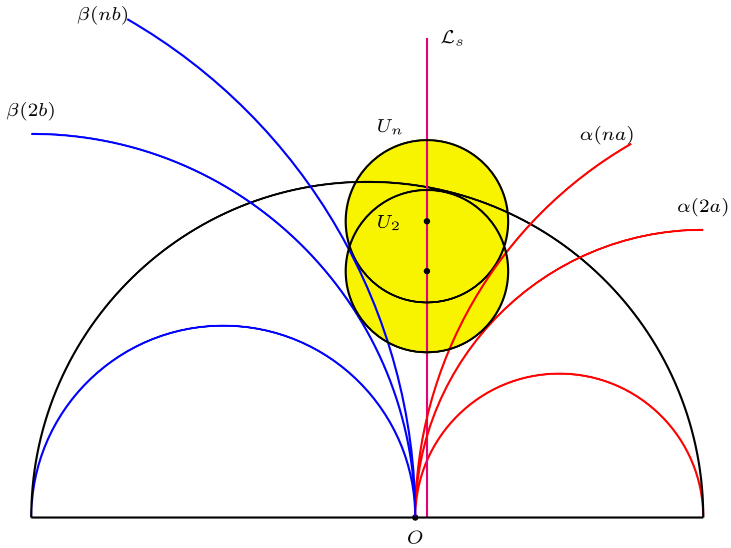

Figure 3

2. A generalization of Schoch's circle U_2

Let a' and b' be real numbers. Consider the semicircles $\alpha(a')$ and $\beta(b')$. Note that $\alpha(a')$ touches α internally or externally according as $a' > 0$ or $a' < 0$; similarly for $\beta(b')$ and β. We assume that the image of $\alpha(a')$ lies on the right side of the image of $\beta(b')$ when these semicircles are inverted in a circle with center O. Denote by $\mathcal{C}(a', b')$ the circle touching γ internally and each of $\alpha(a')$ and $\beta(b')$ at a point different from O.

Theorem 1. *The circle* $\mathcal{C}(a', b')$ *has radius* $\frac{ab(a'+b')}{aa'+bb'+a'b'}$.

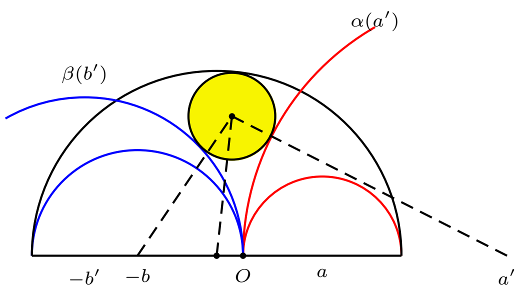

Figure 4a

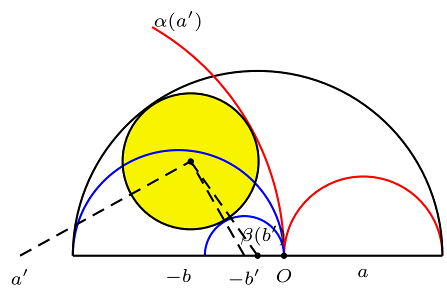

Figure 4b

Proof. Let x be the radius of the circle touching γ internally and also touching $\alpha(a')$ and $\beta(b')$ each at a point different from O. There are two cases in which this circle touches both $\alpha(a')$ and $\beta(b')$ externally (see Figure 4a) or one internally and the other externaly (see Figure 4b). In any case, we have

$$\frac{(a-b+b')^2+(a+b-x)^2-(b'+x)^2}{2(a-b+b')(a+b-x)}$$
$$=-\frac{(a'-(a-b))^2+(a+b-x)^2-(a'+x)^2}{2(a'-(a-b))(a+b-x)},$$

by the law of cosines. Solving the equation, we obtain the radius given above. □

Note that the radius $r=\frac{ab}{a+b}$ of the Archimedean circles can be obtained by letting $a'=a$ and $b'\to\infty$, or $a'\to\infty$ and $b'=b$.

Let $P(a')$ be the external center of similitude of the circles γ and $\alpha(a')$ if $a'>0$, and the internal one if $a'<0$, regarding the two as complete circles. Define $P(b')$ similarly.

Theorem 2. *The two centers of similitude $P(a')$ and $P(b')$ coincide if and only if*

$$\frac{a}{a'}+\frac{b}{b'}=1. \tag{1}$$

Proof. If the two centers of similitude coincide at the point $(t,0)$, then by similarity,

$$a':t-a'=a+b:t-(a-b)=b':t+b'.$$

Eliminating t, we obtain (1). The converse is obvious by the uniqueness of the figure.

□

From Theorems 1 and 2, we obtain the following result.

Theorem 3. *The circle $\mathcal{C}(a',b')$ is an Archimedean circle if and only if $P(a')$ and $P(b')$ coincide.*

When both a' and b' are positive, the two centers of similitude $P(a')$ and $P(b')$ coincide if and only if the three semicircles $\alpha(a')$, $\beta(b')$ and γ share a common external tangent. Hence, in this case, the circle $\mathcal{C}(a',b')$ is Archimedean if and only if $\alpha(a')$, $\beta(b')$ and γ have a common external tangent. Since $\alpha(2a)$ and $\beta(2b)$ satisfy the condition of the theorem, their external common tangent also touches γ. See Figure 5. In fact, it touches γ at its intersection with the y-axis, which is the midpoint of the tangent. The original twin circles of Archimedes are obtained in the limiting case when the external common tangent touches γ at one of the intersections with the x-axis, in which case, one of $\alpha(a')$ and $\beta(b')$ degenerates into the y-axis, and the remaining one coincides with the corresponding α or β of the arbelos.

Corollary 4. *Let m and n be nonzero real numbers. The circle $\mathcal{C}(ma,nb)$ is Archimedean if and only if*

$$\frac{1}{m}+\frac{1}{n}=1.$$

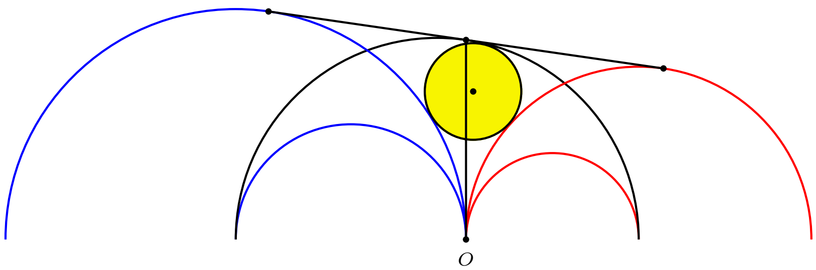

Figure 5

3. Another characterizaton of Woo's circles

The center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right). \tag{2}$$

Denote by $\mathcal{L}$ the half line $x = 2r,\ y \geq 0$. This intersects the circle $\alpha(na)$ at the point

$$\left(2r,\ 2\sqrt{r(na-r)}\right). \tag{3}$$

In what follows we consider β as the complete circle with center $(-b, 0)$ passing through O.

Theorem 5. *If T is a point on the line $\mathcal{L}$, then the circle touching the tangents of β through T with center on the Schoch line $\mathcal{L}_s$ is an Archimedean circle.*

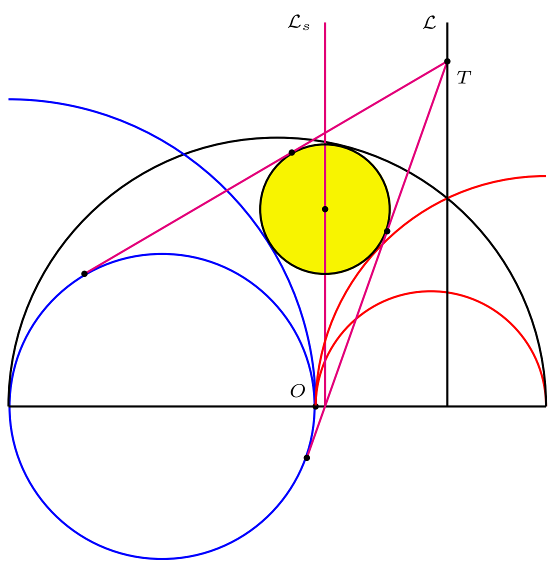

Figure 6

Proof. Let x be the radius of this circle. By similarity (see Figure 6),

$$b + 2r : b = 2r - \frac{b-a}{b+a} r : x.$$

From this, $x = r$. □

The set of Woo circles is a proper subset of the set of circles determined in Theorem 5 above. The external center of similitude of U_n and β has y-coordinate

$$2a\sqrt{n + \frac{r}{a+b}}.$$

When U_n is the circle touching the tangents of β through a point T on $\mathcal{L}$, we shall say that it is determined by T. The y-coordinate of the intersection of α and $\mathcal{L}$ is $2a\sqrt{\frac{r}{a+b}}$. Therefore we obtain the following theorem (see Figure 7).

Theorem 6. *U_0 is determined by the intersection of α and the line $\mathcal{L} : x = 2r$.*

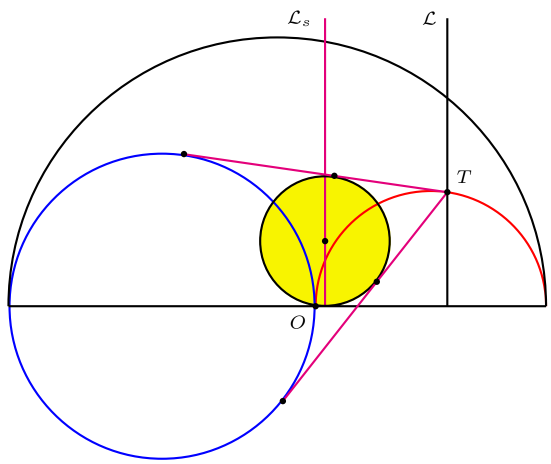

Figure 7

As stated in [2] as the property of the circle labeled as W_{11}, the external tangent of α and β also touches U_0 and the point of tangency at α coincides with the intersection of α and $\mathcal{L}$. Woo's circles are characterized as the circles determined by the points on $\mathcal{L}$ with y-coordinates greater than or equal to $2a\sqrt{\frac{r}{a+b}}$.

4. Woo's circles U_n with $n < 0$

Woo considered the circles U_n for nonnegative numbers n, with U_0 passing through O. We can, however, construct more Archimedean circles passing through points on the y-axis below O using points on $\mathcal{L}$ lying below the intersection with α. The expression (2) suggests the existence of U_n for

$$-\frac{r}{a+b} \leq n < 0. \tag{4}$$

In this section we show that it is possible to define such circles using $\alpha(na)$ and $\beta(nb)$ with negative n satisfying (4).

Theorem 7. *For n satisfying (4), the circle with center on the Schoch line touching $\alpha(na)$ and $\beta(nb)$ internally is an Archimedean circle.*

Proof. Let x be the radius of the circle with center given by (2) and touching $\alpha(na)$ and $\beta(nb)$ internally, where n satisfies (4). Since the centers of $\alpha(na)$ and $\beta(nb)$ are $(na, 0)$ and $(-nb, 0)$ respectively, we have

$$\left(\frac{b-a}{b+a}r - na\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+na)^2,$$

and

$$\left(\frac{b-a}{b+a}r + nb\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+nb)^2.$$

Since both equations give the same solution $x = r$, the proof is complete. □

5. A generalization of U_0

We conclude this paper by adding an infinite set of Archimedean circles passing through O. Let x be the distance from O to the external tangents of α and β. By similarity,

$$b - a : b + a = x - a : a.$$

This implies $x = 2r$. Hence, the circle with center O and radius $2r$ touches the tangents and the lines $x = \pm 2r$. We denote this circle by $\mathcal{E}$. Since U_0 touches the external tangents and passes through O, the circles U_0, $\mathcal{E}$ and the tangent touch at the same point. We easily see from (2) that the distance between the center of U_n and O is $\sqrt{4n+1}r$. Therefore, U_2 also touches $\mathcal{E}$ externally, and the smallest circle touching U_2 and passing through O, which is the Archimedean circle W_{27} in [2] found by Schoch, and U_2 touches $\mathcal{E}$ at the same point. All the Archimedean circles pass through O also touch $\mathcal{E}$. In particular, Bankoff's third circle [1] touches $\mathcal{E}$ at a point on the y-axis.

Theorem 8. *Let $\mathcal{C}_1$ be a circle with center O, passing through a point P on the x-axis, and $\mathcal{C}_2$ a circle with center on the x-axis passing through O. If $\mathcal{C}_2$ and the vertical line through P intersect, then the tangents of $\mathcal{C}_2$ at the intersection also touches $\mathcal{C}_1$.*

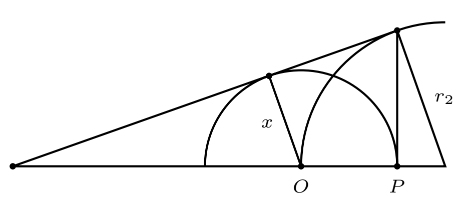

Figure 8a

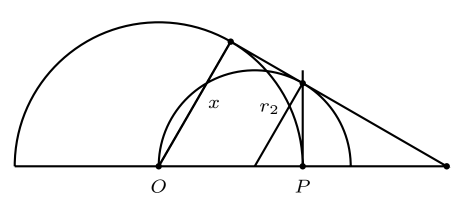

Figure 8b

Proof. Let d be the distance between O and the intersection of the tangent of $\mathcal{C}_2$ and the x-axis, and let x be the distance between the tangent and O. We may assume $r_1 \neq r_2$ for the radii r_1 and r_2 of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. If $r_1 < r_2$, then

$$r_2 - r_1 : r_2 = r_2 + d = x : d.$$

See Figure 8a. If $r_1 > r_2$, then

$$r_1 - r_2 : r_2 = r_2 : d - r_2 = x : d.$$

See Figure 8b. In each case, $x = r_1$. □

Let t_n be the tangent of $\alpha(na)$ at its intersection with the line $\mathcal{L}$. This is well defined if $n \geq \frac{b}{a+b}$. By Theorem 8, t_n also touches $\mathcal{E}$. This implies that the smallest circle touching t_n and passing through O is an Archimedean circle, which we denote by $\mathcal{A}(n)$. Similarlary, another Archimedean circle $\mathcal{A}'(n)$ can be constructed, as the smallest circle through O touching the tangent t'_n of $\beta(nb)$ at its intersection with the line $\mathcal{L}' : x = -2r$. See Figure 9 for $\mathcal{A}(2)$ and $\mathcal{A}'(2)$. Bankoff's circle is $\mathcal{A}\left(\frac{2r}{a}\right) = \mathcal{A}'\left(\frac{2r}{b}\right)$, since it touches $\mathcal{E}$ at $(0, 2r)$. On the other hand, $U_0 = \mathcal{A}(1) = \mathcal{A}'(1)$ by Theorem 6.

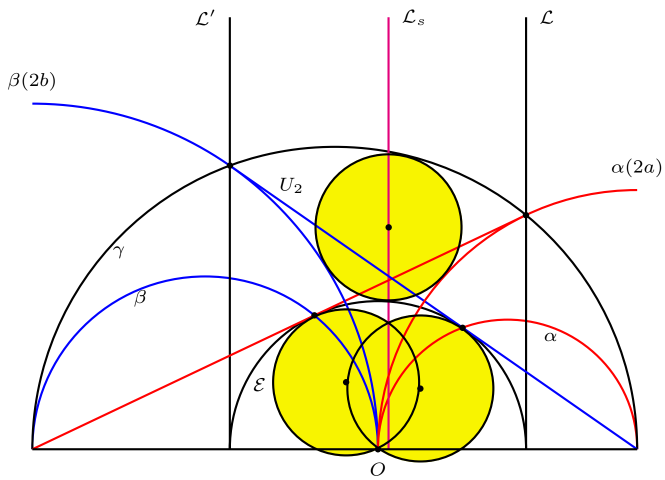

Figure 9

Theorem 9. *Let m and n be positive numbers. The Archimedean circles $\mathcal{A}(m)$ and $\mathcal{A}'(n)$ coincide if and only if m and n satisfy*

$$\frac{1}{ma} + \frac{1}{nb} = \frac{1}{r} = \frac{1}{a} + \frac{1}{b}. \qquad (5)$$

Proof. By (3) the equations of the tangents t_m and t'_n are

$$-(ma + (m-2)b)x + 2\sqrt{b(ma + (m-1)b)}y = 2mab,$$
$$(nb + (n-2)a)x + 2\sqrt{a(nb + (n-1)a)}y = 2nab.$$

These two tangents coincide if and only if (5) holds. □

The line t_2 has equation

$$-ax + \sqrt{b(2a+b)}y = 2ab. \tag{6}$$

It clearly passes through $(-2b, 0)$, the point of tangency of γ and β (see Figure 9). Note that the point

$$\left(-\frac{2r}{a+b}a,\ \frac{2r}{a+b}\sqrt{b(2a+b)}\right)$$

lies on $\mathcal{E}$ and the tangent of $\mathcal{E}$ is also expressed by (6). Hence, t_2 touches $\mathcal{E}$ at this point. The point also lies on β. This means that $\mathcal{A}(2)$ touches t_2 at the intersection of β and t_2. Similarly, $\mathcal{A}'(2)$ touches t_2' at the intersection of α and t_2'. The Archimedean circles $\mathcal{A}(2)$ and $\mathcal{A}'(2)$ intersect at the point

$$\left(\frac{b-a}{b+a}r,\ \frac{r}{a+b}(\sqrt{a(a+2b)} + \sqrt{b(2a+b)})\right)$$

on the Schoch line.

References

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