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Steiner's Theorems on the Complete Quadrilateral

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Abstract. We give a translation of Jacob Steiner's 1828 note on the complete quadrilateral, with complete proofs and annotations in barycentric coordinates.

1. Steiner's note on the complete quadrilateral

In 1828, Jakob Steiner published in Gergonne's *Annales* a very short note [9] listing ten interesting and important theorems on the complete quadrilateral. The purpose of this paper is to provide a translation of the note, to prove these theorems, along with annotations in barycentric coordinates. We begin with a translation of Steiner's note.

Suppose four lines intersect two by two at six points.

(1) These four lines, taken three by three, form four triangles whose circumcircles pass through the same point F.
(2) The centers of the four circles (and the point F) lie on the same circle.
(3) The perpendicular feet from F to the four lines lie on the same line $\mathcal{R}$, and F is the only point with this property.
(4) The orthocenters of the four triangles lie on the same line $\mathcal{R}'$.
(5) The lines $\mathcal{R}$ and $\mathcal{R}'$ are parallel, and the line $\mathcal{R}$ passes through the midpoint of the segment joining F to its perpendicular foot on $\mathcal{R}'$.
(6) The midpoints of the diagonals of the complete quadrilateral formed by the four given lines lie on the same line $\mathcal{R}''$ (Newton).
(7) The line $\mathcal{R}''$ is a common perpendicular to the lines $\mathcal{R}$ and $\mathcal{R}'$.
(8) Each of the four triangles in (1) has an incircle and three excircles. The centers of these 16 circles lie, four by four, on eight new circles.
(9) These eight new circles form two sets of four, each circle of one set being orthogonal to each circle of the other set. The centers of the circles of each set lie on a same line. These two lines are perpendicular.
(10) Finally, these last two lines intersect at the point F mentioned above.

The configuration formed by four lines is called a complete quadrilateral. Figure 1 illustrates the first 7 theorems on the complete quadrilateral bounded by the four lines UVW, UBC, AVC, and ABW. The diagonals of the quadrilateral are the

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segments AU, BV, CW. The four triangles ABC, AVW, BWU, and CUV are called the associated triangles of the complete quadrilateral. We denote by

- H, H_a, H_b, H_c their orthocenters,
- Γ, Γ_a, Γ_b, Γ_c their circumcircles, and
- O, O_a, O_b, O_c the corresponding circumcenters.

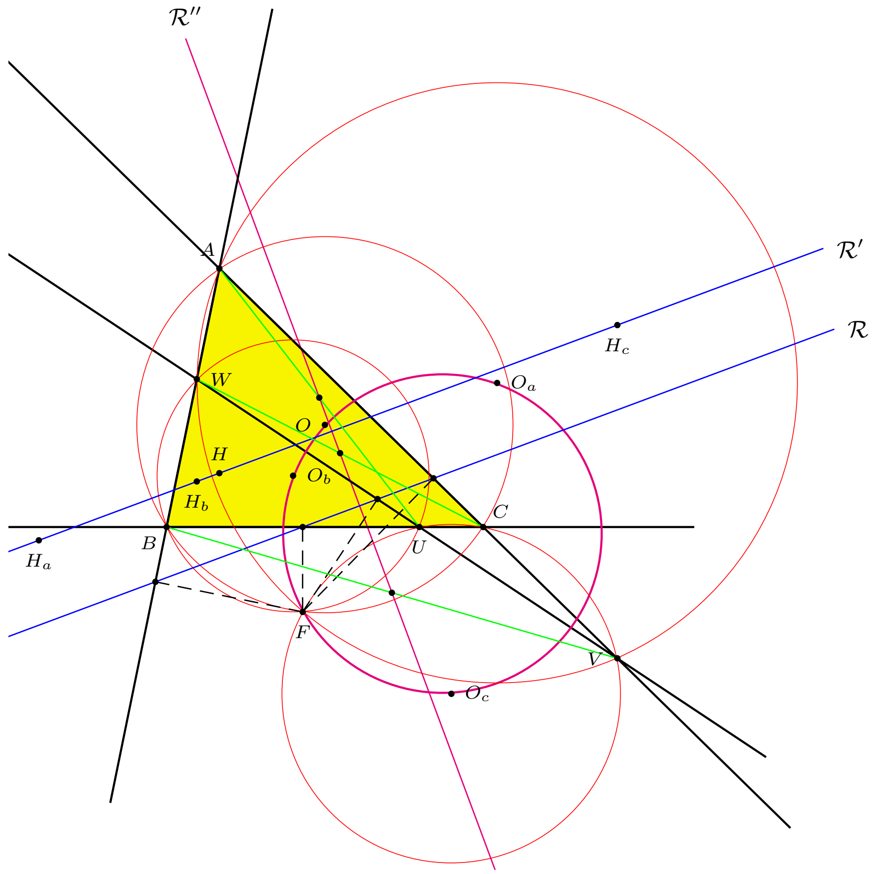

Figure 1.

2. Geometric preliminaries

2.1. *Directed angles.* We shall make use of the notion of *directed* angles. Given two lines ℓ and ℓ', the directed angle (ℓ, ℓ') is the angle through which ℓ must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ'. See [3, §§16–19]. It is defined modulo π.

Lemma 1. (1) $(\ell, \ell'') = (\ell, \ell') + (\ell', \ell'')$.

(2) *Four noncollinear points* P, Q, R, S *are concyclic if and only if* $(PR, PS) = (QR, QS)$.

2.2. *Simson-Wallace lines.* The pedals [1] of a point M on the lines BC, CA, AB are collinear if and only if M lies on the circumcircle Γ of ABC. In this case, the Simson-Wallace line passes through the midpoint of the segment joining M to the orthocenter H of triangle ABC. The point M is the isogonal conjugate (with respect to triangle ABC) of the infinite point of the direction orthogonal to its own Simson-Wallace line.

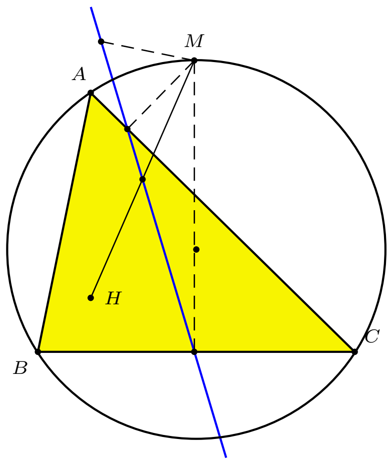

Figure 2

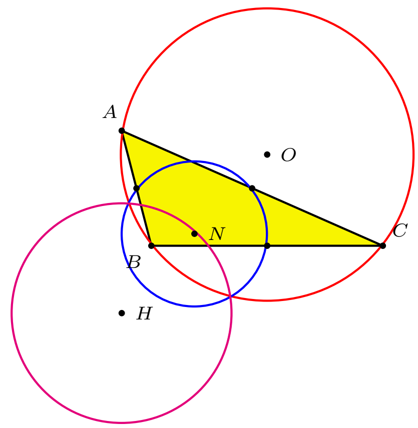

Figure 3

2.3. *The polar circle of a triangle.* There exists one and only one circle with respect to which a given triangle ABC is self polar. The center of this circle is the orthocenter of ABC and the square of its radius is

$$-4R^2 \cos A \cos B \cos C.$$

This *polar* circle is real if and only if ABC is obtuse-angled. It is orthogonal to any circle with diameter a segment joining a vertex of ABC to a point of the opposite sideline. The inversion with respect the polar circle maps a vertex of ABC to its pedal on the opposite side. Consequently, this inversion swaps the circumcircle and the nine-point circle.

2.4. *Center of a direct similitude.* Suppose that a direct similitude with center Ω maps M to M' and N to N', and that the lines MM' and NN' intersect at S. If Ω does not lie on the line MN, then M, N, Ω, S are concyclic; so are M', N', Ω, S. Moreover, if $MN \perp M'N'$, the circles $MN\Omega S$ and $M'N'\Omega S$ are orthogonal.

[1]In this paper we use the word pedal in the sense of orthogonal projection.

3. Steiner's Theorems 1–7

3.1. *Steiner's Theorem 1 and the Miquel point.* Let F be the second common point (apart from A) of the circles Γ and Γ_a. Since

$$(FB, FW) = (FB, FA) + (FA, FW) = (CB, CA) + (VA, VW) = (UB, UW),$$

we have $F \in \Gamma_b$ by Lemma 1(2). Similarly $F \in \Gamma_c$. This proves (1).

We call F the *Miquel point* of the complete quadrilateral.

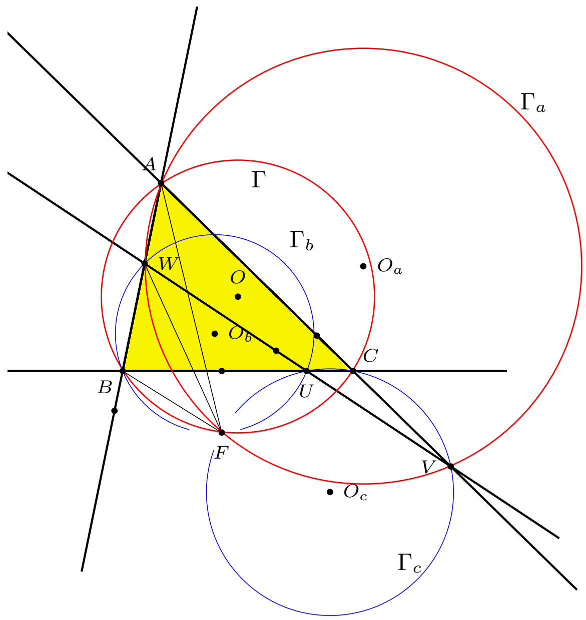

Figure 4.

3.2. *Steiner's Theorem 3 and the pedal line.* The point F has the same Simson-Wallace line with respect to the four triangles of the complete quadrilateral. See Figure 5. Conversely, if the pedals of a point M on the four sidelines of the complete quadrilateral lie on a same line, M must lie on each of the four circumcircles. Hence, $M = F$. This proves (3).

We call the line $\mathcal{R}$ the *pedal line* of the quadrilateral.

3.3. *Steiner's Theorems 4, 5 and the orthocentric line.* As the midpoints of the segments joining F to the four orthocenters lie on $\mathcal{R}$, the four orthocenters lie on a line $\mathcal{R}'$, which is the image of $\mathcal{R}$ under the homothety $\mathsf{h}(F, 2)$. This proves (4) and (5). See Figure 5.

We call the line $\mathcal{R}'$ the *orthocentric line* of the quadrilateral.

Remarks. (1) As U, V, W are the reflections of F with respect to the sidelines of the triangle $O_aO_bO_c$, the orthocenter of this triangle lies on $\mathcal{L}$.

(2) We have $(BC, FU) = (CA, FV) = (AB, FW)$ because, for instance, $(BC, FU) = (UB, UF) = (WB, WF) = (AB, FW)$.

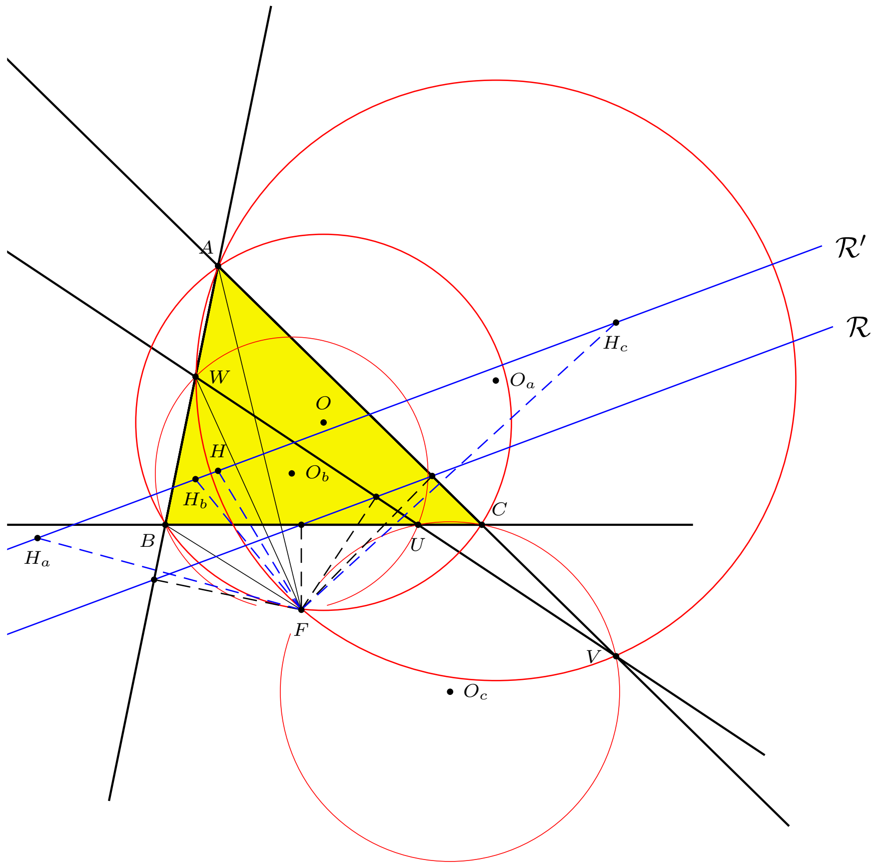

Figure 5.

(3) Let P_a, P_b, P_c be the projections of F upon the lines BC, CA, AB. As P_a, P_b, C, F are concyclic, it follows that F is the center of the direct similitude mapping P_a to U and P_b to V. Moreover, by (2) above, this similitude maps P_c to W.

3.4. *Steiner's Theorem 2 and the Miquel circle.* By Remark (3) above, if F_a, F_b, F_c are the reflections of F with respect to the lines BC, CA, AB, a direct similitude σ with center F maps F_a to U, F_b to V, F_c to W. As A is the circumcenter of FF_bF_c, it follows that $\sigma(A) = O_a$; similarly, $\sigma(B) = O_b$ and $\sigma(C) = O_c$. As A, B, C, F are concyclic, so are O_a, O_b, O_c, F. Hence F and the circumcenters of three associated triangles are concyclic. It follows that O, O_a, O_b, O_c, F lie on the same circle, say, Γ_m. This prove (2).

We call Γ_m the *Miquel circle* of the complete quadrilateral. See Figure 6.

3.5. *The Miquel perspector.* Now, by §2.4, the second common point of Γ and Γ_m lies on the three lines AO_a, BO_b, CO_c. Hence,

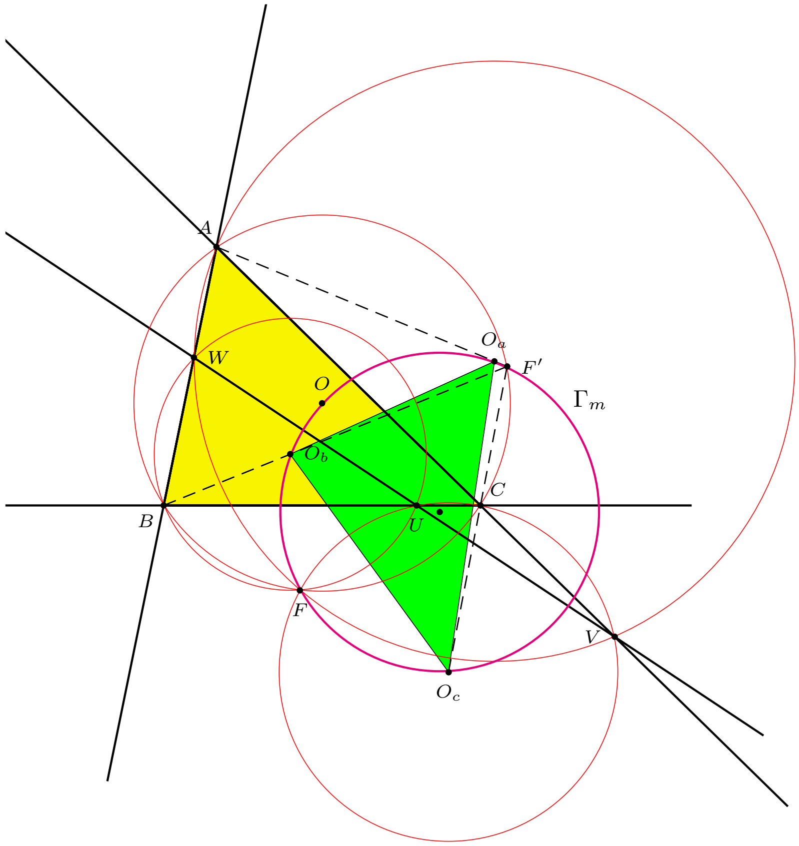

Figure 6.

Proposition 2. *The triangle $O_aO_bO_c$ is directly similar and perspective with ABC. The center of similitude is the Miquel point F and the perspector is the second common point F' of the Miquel circle and the circumcircle Γ of triangle ABC.*

We call F' the *Miquel perspector* of the triangle ABC.

3.6. *Steiner's Theorems 6, 7 and the Newton line.* We call *diagonal triangle* the triangle $A'B'C'$ with sidelines AU, BV, CW.

Lemma 3. *The polar circles of the triangles ABC, AVW, BWU, CUV and the circumcircle of the diagonal triangle are coaxal. The three circles with diameter AU, BV, CW are coaxal. The corresponding pencils of circles are orthogonal.*

Proof. By §2.3, each of the four polar circles is orthogonal to the three circles with diameter AU, BV, CW. More over, as each of the quadruples (A, U, B', C'), (B, V, C', A') and (C, W, A', B') is harmonic, the circle $A'B'C'$ is orthogonal to the three circles with diameter AU, BV and CW. □

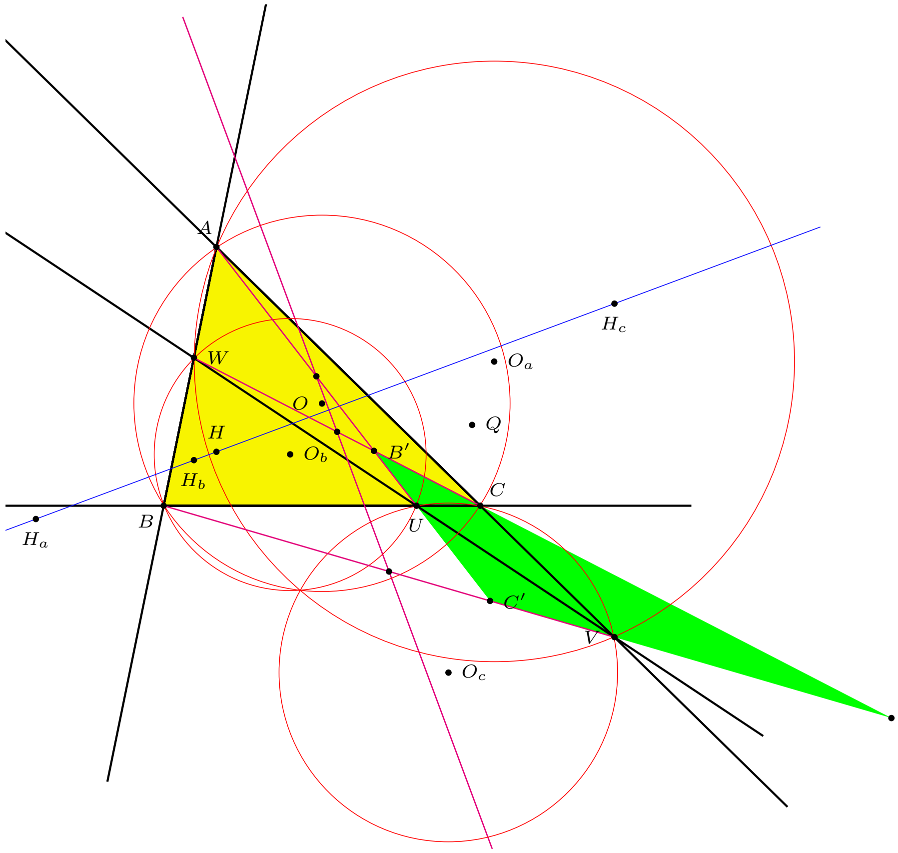

Figure 7.

As the line of centers of the first pencil of circles is the orthocentric line $\mathcal{R}'$, it follows that the midpoints of AU, BV and CW lie on a same line $\mathcal{R}''$ perpendicular to $\mathcal{R}'$. This proves (6) and (7).

4. Some further results

4.1. *The circumcenter of the diagonal triangle.*

Proposition 4. *The circumcenter of the diagonal triangle lies on the orthocentric line.*

This follows from Lemma 3 and §2.3.

We call the line $\mathcal{R}''$ the *Newton line* of the quadrilateral. As the Simson-Wallace line $\mathcal{R}$ of F is perpendicular to $\mathcal{R}''$, we have

Proposition 5. *The Miquel point is the isogonal conjugate of the infinite point of the Newton line with respect to each of the four triangles* ABC, AVW, BWU, CUV.

4.2. *The orthopoles.* Recall that the three lines perpendicular to the sidelines of a triangle and going through the projection of the opposite vertex on a given line go through a same point : the *orthopole* of the line with respect to the triangle.

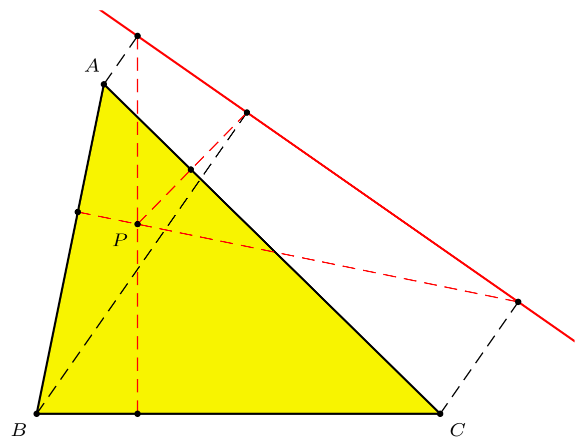

Figure 8

Proposition 6 (Goormaghtigh)**.** *The orthopole of a sideline of the complete quadrilateral with respect to the triangle bounded by the three other sidelines lies on the orthocentric line.*

Proof. See [1, pp.241–242]. □

5. Some barycentric coordinates and equations

5.1. *Notations.* Given a complete quadrilateral, we consider the triangle bounded by three of the four given lines as a reference triangle ABC, and construe the fourth line as the trilinear polar with respect to ABC of a point Q with homogeneous barycentric coordinates $(u : v : w)$, *i.e.*, the line

$$\mathcal{L} : \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

The intercepts of $\mathcal{L}$ with the sidelines of triangle ABC are the points

$$U = (0 : v : -w), \qquad V = (-u : 0 : w), \qquad W = (u : -v : 0).$$

The lines AU, BV, CW bound the diagonal triangle with vertices

$$A' = (-u : v : w), \qquad B' = (u : -v : w), \qquad C' = (u : v : -w).$$

Triangles ABC and $A'B'C'$ are perspective at Q.

We adopt the following notations. If a, b, c stand for the lengths of the sides BC, CA, AB, then

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2).$$

We shall also denote by S *twice* of the signed area of triangle ABC, so that

$$S_A = S \cdot \cot A, \quad S_B = S \cdot \cot B, \quad S_C = S \cdot \cot C,$$

and

$$S_{BC} + S_{CA} + S_{AB} = S^2.$$

Lemma 7. (1) *The infinite point of the line $\mathcal{L}$ is the point*

$$(u(v-w) : v(w-u) : w(u-v)).$$

(2) *Lines perpendicular to $\mathcal{L}$ have infinite point $(\lambda_a : \lambda_b : \lambda_c)$, where*

$$\begin{aligned} \lambda_a &= S_B v(w-u) - S_C w(u-v), \\ \lambda_b &= S_C w(u-v) - S_A u(v-w), \\ \lambda_c &= S_A u(v-w) - S_B v(w-u). \end{aligned}$$

Proof. (1) is trivial. (2) follows from (1) and the fact that two lines with infinite points $(p : q : r)$ and $(p' : q' : r')$ are perpendicular if and only if

$$S_A pp' + S_B qq' + S_C rr' = 0.$$

Consequently, given a line with infinite point $(p : q : r)$, lines perpendicular to it all have the infinite point $(S_B q - S_C r : S_C r - S_A p : S_A p - S_B q)$. □

5.2. *Coordinates and equations.* We give the barycentric coordinates of points and equations of lines and circles in Steiner's theorems.

(1) The Miquel point:

$$F = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}\right).$$

(2) The pedal line:

$$\mathcal{R} : \quad \frac{v-w}{S_C v + S_B w - a^2 u}x + \frac{w-u}{S_A w + S_C u - b^2 v}y + \frac{u-v}{S_B u + S_A v - c^2 w}z = 0.$$

(3) The orthocentric line:

$$\mathcal{R}' : \quad (v-w)S_A x + (w-u)S_B y + (u-v)S_C z = 0.$$

(4) The Newton line:

$$\mathcal{R}'' : \quad (v+w-u)x + (w+u-v)y + (u+v-w)z = 0.$$

(5) The equation of the Miquel circle:

$$a^2yz + b^2zx + c^2xy + \frac{2R^2(x+y+z)}{(v-w)(w-u)(u-v)}\left(\frac{v-w}{a^2}\lambda_a x + \frac{w-u}{b^2}\lambda_b y + \frac{u-v}{c^2}\lambda_c z\right) = 0.$$

(6) The Miquel perspector, being the isogonal conjugate of the infinite point of the direction orthogonal to $\mathcal{L}$, is

$$F' = \left(\frac{a^2}{\lambda_a} : \frac{b^2}{\lambda_b} : \frac{c^2}{\lambda_c}\right).$$

The Simson-Wallace line of F' is parallel to ℓ.

(7) The orthopole of $\mathcal{L}$ with respect to ABC is the point

$$(\lambda_a(-S_BS_Cvw + b^2S_Bwu + c^2S_Cuv) : \cdots : \cdots).$$

5.3. *Some metric formulas* . Here, we adopt more symmetric notations. Let ℓ_i, $i = 1, 2, , 3, 4$, be four given lines.

- For distinct i and j, $A_{i,j} = \ell_i \cap \ell_j$,
- $\mathcal{T}_i$ the triangle bounded by the three lines other than ℓ_i, O_i its circumcenter, R_i its circumradius.
- $F_i = O_jA_{k,l} \cap O_kA_{l,j} \cap O_lA_{j,k}$ its Miquel perspector, *i.e.*, the second intersection (apart from F) of its circumcircle with the Miquel circle; R_m is the radius of the Miquel circle.

Let d be the distance from F to the pedal line $\mathcal{R}$ and $\theta_i = (\mathcal{R}, \ell_i)$. Up to a direct congruence, the complete quadrilateral is characterized by d, θ_1, θ_2, θ_3, and θ_4.

(1) The distance from F to ℓ_i is $\dfrac{d}{|\cos\theta_i|}$.
(2) $|FA_{i,j}| = \dfrac{d}{|\cos\theta_i\cos\theta_j|}$.
(3) $|A_{k,i}A_{k,j}| = d\left|\dfrac{\sin(\theta_j - \theta_i)}{\cos\theta_i\cos\theta_j\cos\theta_k}\right|$.
(4) The directed angle $(FA_{k,i}, FA_{k,j}) = (\ell_i, \ell_j) = \theta_j - \theta_i \mod \pi$.
(5) $R_m = \dfrac{d}{4\,|\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4|} = \dfrac{R_i}{2\,|\cos\theta_i|}$ for $i = 1, 2, 3, 4$.
(6) $|FA_{1,2}| \cdot |FA_{3,4}| = |FA_{1,3}| \cdot |FA_{2,4}| = |FA_{1,4}| \cdot |FA_{2.3}| = 4dR_m$.
(7) $|FF_i| = 2R_i|\sin\theta_i|$.
(8) The *oriented* angle between the vectors $\mathbf{O_iF}$ and $\mathbf{O_iF_i} = -2\theta_i \mod 2\pi$.
(9) The distance from F to $\mathcal{R}''$ is

$$\frac{d}{2}\,|\tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4|\,.$$

6. Steiner's Theorems 8 – 10

At each vertex M of the complete quadrilateral, we associate the pair of angle bisectors m and m'. These lines are perpendicular to each other at M. We denote the intersection of two bisectors m and n by $\mathsf{m} \cap \mathsf{n}$.

- $\mathbf{T}(\mathsf{m}, \mathsf{n}, \mathsf{p})$ denotes the triangle bounded by a bisector at M, one at N, and one at P.
- $\Gamma(\mathsf{m}, \mathsf{n}, \mathsf{p})$ denotes the circumcircle of $\mathbf{T}(\mathsf{m}, \mathsf{n}, \mathsf{p})$.

Consider three bisectors $\mathsf{a}, \mathsf{b}, \mathsf{c}$ intersecting at a point J, the incenter or one of the excenters of ABC. Suppose two bisectors v and w intersect on a. Then so do v' and w'. Now, the line joining $\mathsf{b} \cap \mathsf{w}$ and $\mathsf{c} \cap \mathsf{v}$ is a U-bisector. If we denote this line by u, then u' the line joining $\mathsf{b} \cap \mathsf{w}'$ and $\mathsf{c} \cap \mathsf{v}'$.

The triangles $\mathbf{T}(\mathsf{a}', \mathsf{b}', \mathsf{c}')$, $\mathbf{T}(\mathsf{u}, \mathsf{v}, \mathsf{w})$, and $\mathbf{T}(\mathsf{u}', \mathsf{v}', \mathsf{w}')$ are perspective at J. Hence, by Desargues' theorem, the points $\mathsf{a}' \cap \mathsf{u}$, $\mathsf{b}' \cap \mathsf{v}$, and $\mathsf{c}' \cap \mathsf{w}$ are collinear; so are $\mathsf{a}' \cap \mathsf{u}'$, $\mathsf{b}' \cap \mathsf{v}'$, and $\mathsf{c}' \cap \mathsf{w}'$. Moreover, as the corresponding sidelines of triangles $\mathbf{T}(\mathsf{u}, \mathsf{v}, \mathsf{w})$, and $\mathbf{T}(\mathsf{u}', \mathsf{v}', \mathsf{w}')$ are perpendicular, it follows from §2.4 that their circumcircles $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$, and $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ are orthogonal and pass through J. See Figure 9. [2]

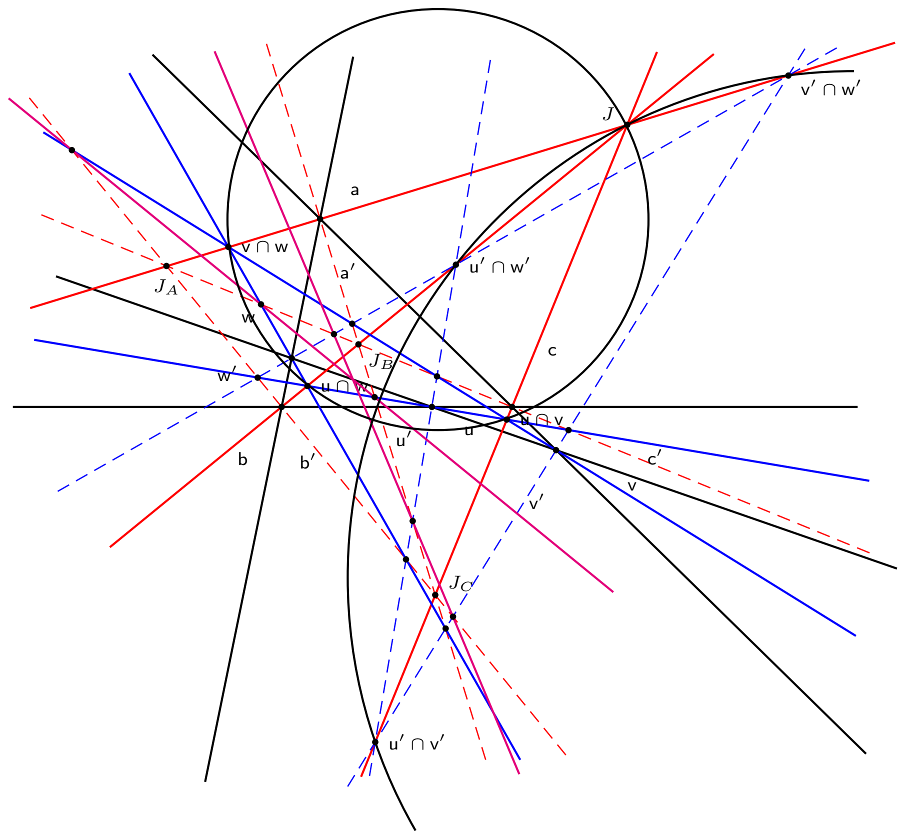

Figure 9

As a intersects the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ at J and $\mathsf{v}'\cap\mathsf{w}'$ and u' intersects the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ at $\mathsf{u}'\cap\mathsf{v}'$ and $\mathsf{u}'\cap\mathsf{w}'$, it follows that the polar line of $\mathsf{a}\cap\mathsf{u}'$ with respect to $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ passes through $\mathsf{b}\cap\mathsf{v}'$ and $\mathsf{c}\cap\mathsf{w}'$. Hence $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}'$, $\mathsf{c}\cap\mathsf{w}'$. Similarly, $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}$, $\mathsf{c}\cap\mathsf{w}$.

By the same reasoning, we obtain the following.

(a) As the triangles $\mathbf{T}(\mathsf{a}',\mathsf{b},\mathsf{c})$, $\mathbf{T}(\mathsf{u},\mathsf{v}',\mathsf{w}')$, and $\mathbf{T}(\mathsf{u}',\mathsf{v},\mathsf{w})$ are perspective at $J_A=\mathsf{a}\cap\mathsf{b}'\cap\mathsf{c}'$, it follows that

- the circles $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}')$ and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ are orthogonal and pass through J_A,
- the points $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}'$, and $\mathsf{c}\cap\mathsf{w}'$ are collinear; so are $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}$, and $\mathsf{c}\cap\mathsf{w}$,

[2] In Figures 9 and 10, at each of the points A, B, C, U, V, W are two bisectors, one shown in solid line and the other in dotted line. The bisectors in solid lines are labeled a, b, c, u, v, w, and those in dotted line labeled a', b', c', u', v', w'. Other points are identified as intersections of two of these bisectors. Thus, for example, $J=\mathsf{a}\cap\mathsf{b}$, and $J_A=\mathsf{b}'\cap\mathsf{c}'$.

- the circle $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}$.

(b) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b}',\mathsf{c})$, $\mathbf{T}(\mathsf{u}',\mathsf{v},\mathsf{w}')$, and $\mathbf{T}(\mathsf{u},\mathsf{v}',\mathsf{w})$ are perspective at $J_B = \mathsf{a}'\cap\mathsf{b}\cap c'$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ are orthogonal and pass through J_B,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, and $\mathsf{c}\cap\mathsf{w}'$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, and $\mathsf{c}\cap\mathsf{w}$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}$.

(c) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b},\mathsf{c}')$, $\mathbf{T}(\mathsf{u}',\mathsf{v}',\mathsf{w})$, and $\mathbf{T}(\mathsf{u},\mathsf{v},\mathsf{w}')$ are perspective at $J_C = \mathsf{a}'\cap\mathsf{b}'\cap c$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ are orthogonal and pass through J_C,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}'$, and $\mathsf{c}'\cap\mathsf{w}$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}$, and $\mathsf{c}'\cap\mathsf{w}'$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}\cap\mathsf{w}$, and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}\cap\mathsf{w}'$.

Therefore, we obtain two new complete quadrilaterals:

(1) $\mathcal{Q}_1$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}),\ (\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}),$$

(2) $\mathcal{Q}_2$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}'),\ (\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}').$$

The polar circles of the triangles associated with $\mathcal{Q}_1$ are

$$\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v},\mathsf{w}').$$

These circles pass through J, J_A, J_B, J_C respectively.

The polar circles of the triangles associated with $\mathcal{Q}_2$ are

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}).$$

These circles pass through J, J_A, J_B, J_C respectively. Moreover, by §2.4, the circles in the first group are orthogonal to those in the second group. For example, as u and u' are perpendicular to each other, the circles $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$ and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ are orthogonal. Now it follows from Lemma 3 applied to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ that

Proposition 8 (Mention [4]). *(1) The following seven circles are members of a pencil Φ:*

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}),$$

and those with diameters

$$(\mathsf{a}\cap\mathsf{u}')(\mathsf{a}'\cap\mathsf{u}),\ (\mathsf{b}\cap\mathsf{v}')(\mathsf{b}'\cap\mathsf{v}),\ (\mathsf{c}\cap\mathsf{w}')(\mathsf{c}'\cap\mathsf{w}).$$

(2) The following seven circles are members of a pencil Φ'*:*

$$\Gamma(\mathsf{u}', \mathsf{v}', \mathsf{w}'),\ \Gamma(\mathsf{u}', \mathsf{v}, \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}', \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}, \mathsf{w}'),$$

and those with diameters

$$(\mathsf{a} \cap \mathsf{u})(\mathsf{a}' \cap \mathsf{u}'),\ (\mathsf{b} \cap \mathsf{v})(\mathsf{b}' \cap \mathsf{v}'),\ (\mathsf{c} \cap \mathsf{w})(\mathsf{c}' \cap \mathsf{w}').$$

(3) The circles in the two pencils Φ *and* Φ' *are orthogonal.*

This clearly gives Steiner's Theorems 8 and 9.

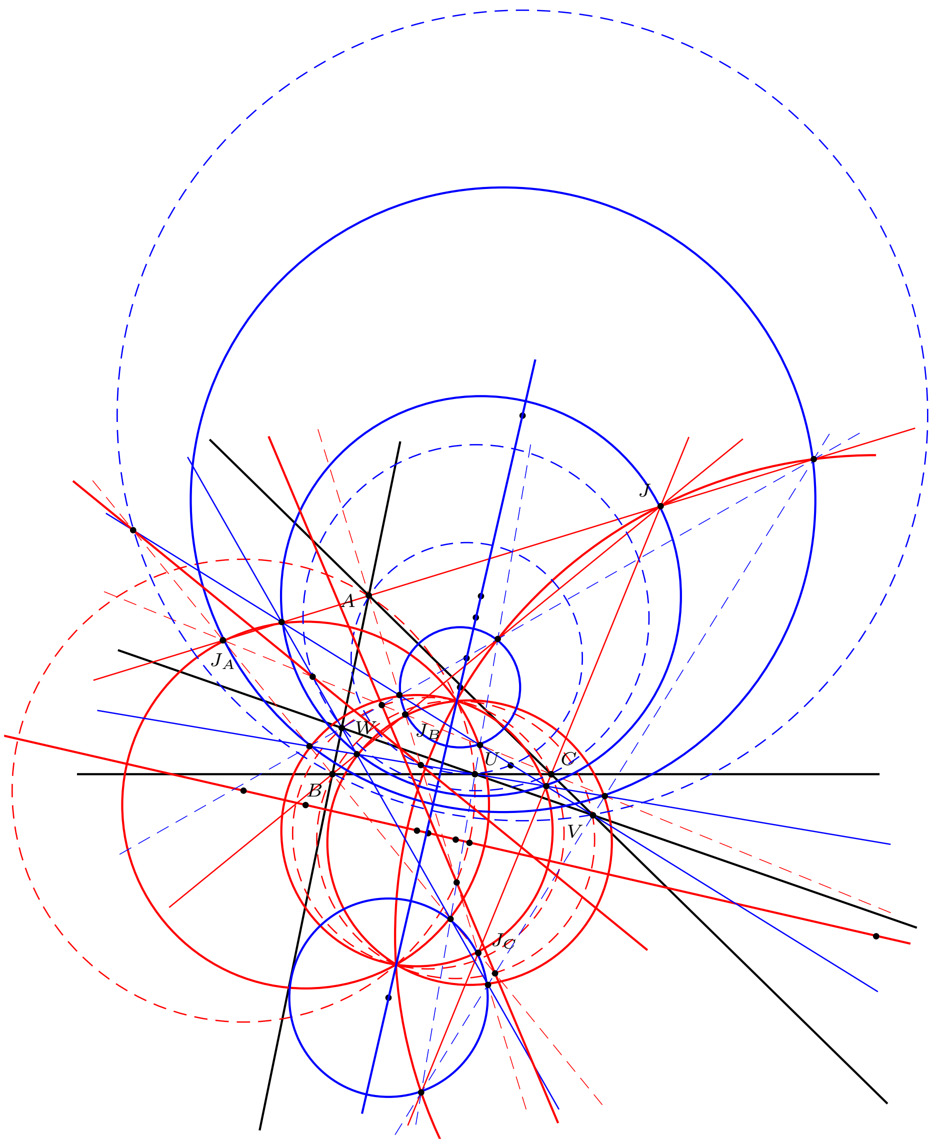

Figure 10

Let P be the midpoint of the segment joining $\mathsf{a}\cap\mathsf{u}'$ and $\mathsf{a}'\cap\mathsf{u}$, and P' the midpoint of the segment joining $\mathsf{a}\cap\mathsf{u}$ and $\mathsf{a}'\cap\mathsf{u}'$. The nine-point circle of the orthocentric system

$$\mathsf{a}\cap\mathsf{u},\quad \mathsf{a}'\cap\mathsf{u}',\quad \mathsf{a}\cap\mathsf{u}',\quad \mathsf{a}'\cap\mathsf{u}$$

is the circle with diameter PP'. This circle passes through A and U. See Figure 11. Furthermore, P and P' are the midpoints of the two arcs AU of this circle. As P is the center of the circle passing through A, U, $\mathsf{a}\cap\mathsf{u}'$ and $\mathsf{a}'\cap\mathsf{u}$, we have

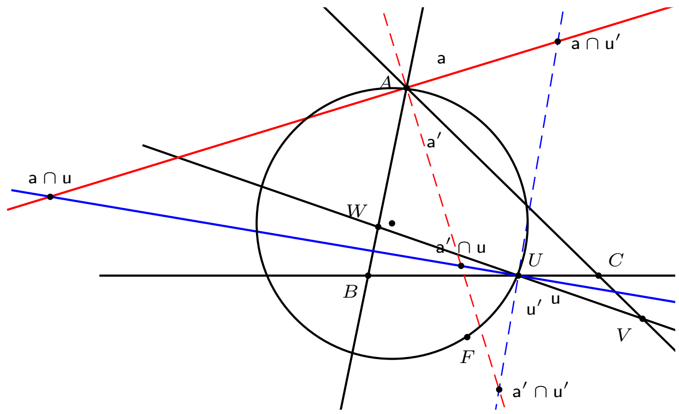

Figure 11.

$$\begin{aligned}(PA,\ PU) =&2((\mathsf{a}\cap\mathsf{u}')A,\ (\mathsf{a}\cap\mathsf{u}')U)\\ =&2((\mathsf{a}\cap\mathsf{u}')A,\ AB)+2(AB,\ UV)+2(UV,\ (\mathsf{a}\cap\mathsf{u}')U)\\ =&(AC,\ AB)+2(AB,\ UV)+(UV, BC)\\ =&(CA,\ CB)+(AB,\ UV)\\ =&(CA,\ CB)+(WB,\ WU)\\ =&(FA,\ FB)+(FB,\ FU)\\ =&(FA,\ FU).\end{aligned}$$

Hence, F lies on the circle with diameter PP', and the lines FP, FP' bisect the angles between the lines FA and FU.

As the central lines of the pencils Φ and Φ' are perpendicular and pass respectively through P and P', their common point lies on the circle FAU. Similarly, this common point must lie on the circles FBV and FCW. Hence, this common point is F. This proves Steiner's Theorem 10 and the following more general result.

Proposition 9 (Clawson)**.** *The central lines of the pencils Φ and Φ' are the common bisectors of the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, and $(FC,\ FW)$.*

Note that, as $(FA,\ FB) = (FV,\ FU) \ = (CA,\ CB)$, it is clear that the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, $(FC,\ FW)$ have a common pair of bisectors $(\mathsf{f}, \mathsf{f}')$. These bisectors are called the *incentric lines* of the complete quadrilateral. With the notations of §5.3, we have

$$2(\mathcal{R},\ \mathsf{f}) = 2(\mathcal{R},\ \mathsf{f}') = \theta_1 + \theta_2 + \theta_3 + \theta_4 \bmod \pi.$$

7. Inscribed conics

7.1. *Centers and foci of inscribed conics.* We give some classical properties of the conics tangent to the four sidelines of the complete quadrilateral.

Proposition 10. *The locus of the centers of the conics inscribed in the complete quadrilateral is the Newton line $\mathcal{R}''$.*

Proposition 11. *The locus of the foci of these conics is a circular focal cubic* (van Rees focal).

This cubic γ passes through A, B, C, U, V, W, F, the circular points at infinity I_∞, J_∞ and the feet of the altitudes of the diagonal triangle.

The real asymptote is the image of the Newton line under the homothety $\mathsf{h}(F, 2)$, and the imaginary asymptotes are the lines FI_∞ and FJ_∞. In other words, F is the singular focus of γ. As F lies on the γ, γ is said to be *focal*. The cubic γ is self isogonal with respect to each of the four triangles ABC, AVW, BWU, CUV. It has barycentric equation

$$\begin{aligned} &ux\left(c^2y^2 + b^2z^2\right) + vy\left(a^2z^2 + c^2x^2\right) + wz\left(b^2x^2 + a^2y^2\right) \\ &+2\left(S_A u + S_B v + S_C w\right) xyz = 0. \end{aligned}$$

If we denote by $\overline{PQRS}$ the van Rees focal of P, Q, R, S, *i.e.*, the locus of M such as $(MP, MQ) = (MR, MS)$, then

$$\gamma = \overline{ABVU} = \overline{BCWV} = \overline{CAUW} = \overline{AVBU} = \overline{BWCV} = \overline{CUAW}.$$

Here is a construction of the cubic γ.

Construction. Consider a variable circle through the pair of isogonal conjugate points on the Newton line. [3] Draw the lines through F tangent to the circle. The locus of the points of tangency is the cubic γ. See Figure 12

7.2. *Orthoptic circles.* Recall that the Monge (or orthoptic) circle of a conic is the locus of M from which the tangents to the conic are perpendicular.

Proposition 12 (Oppermann)**.** *The circles of the pencil generated by the three circles with diameters AU, BV, CW are the Monge circle's of the conics inscribed in the complete quadrilateral.*

Proof. See [5, pp.60–61]. □

[3] These points are not necessarily real.

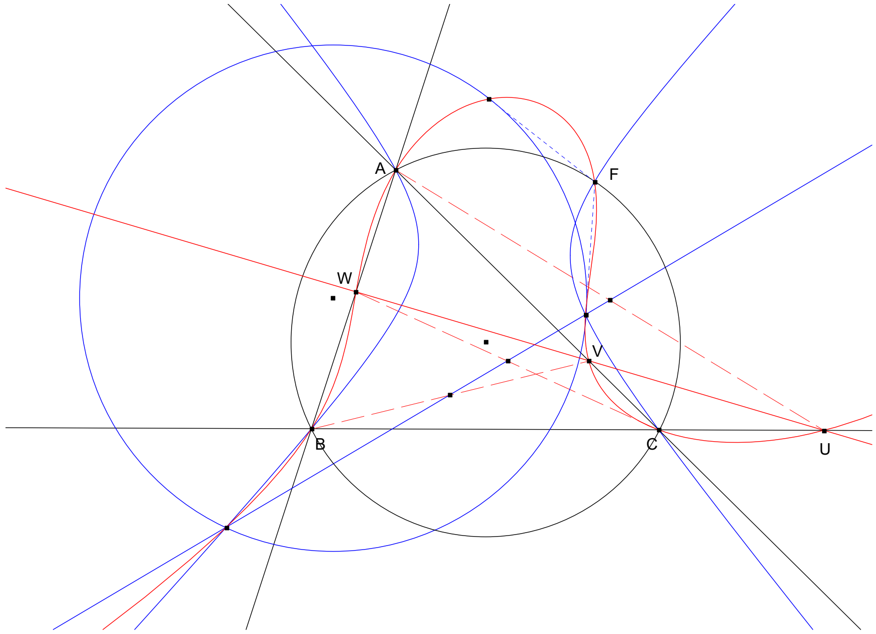

Figure 12.

7.3. *Coordinates and equations.* Recall that the perspector (or Brianchon point) of a conic inscribed in the triangle ABC is the perspector of ABC and the contact triangle. Suppose the perspector is the point $(p : q : r)$.

(1) The center of the conic is the point

$$(p(q+r) : q(r+p) : r(p+q)).$$

(2) The equation of the conic is

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} - 2\frac{xy}{pq} - 2\frac{yz}{qr} - 2\frac{zx}{rp} = 0.$$

(3) The line $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$ is tangent to the conic if and only if $\frac{u}{p} + \frac{v}{q} + \frac{w}{r} = 0$.

(4) The equation of the Monge circle of the conic is

$$\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)(a^2yz + b^2zx + c^2xy) = (x+y+z)\left(\frac{S_A}{p}x + \frac{S_B}{q}y + \frac{S_C}{r}z\right).$$

The locus of the perspectors of the conics inscribed in the complete quadrilateral is the circumconic

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0,$$

i.e., the circumconic with perspector Q.

7.4. *Inscribed parabola.*

Proposition 13. *The only parabola inscribed in the quadrilateral is the parabola with focus F and directrix the orthocentric line $\mathcal{R}'$.*

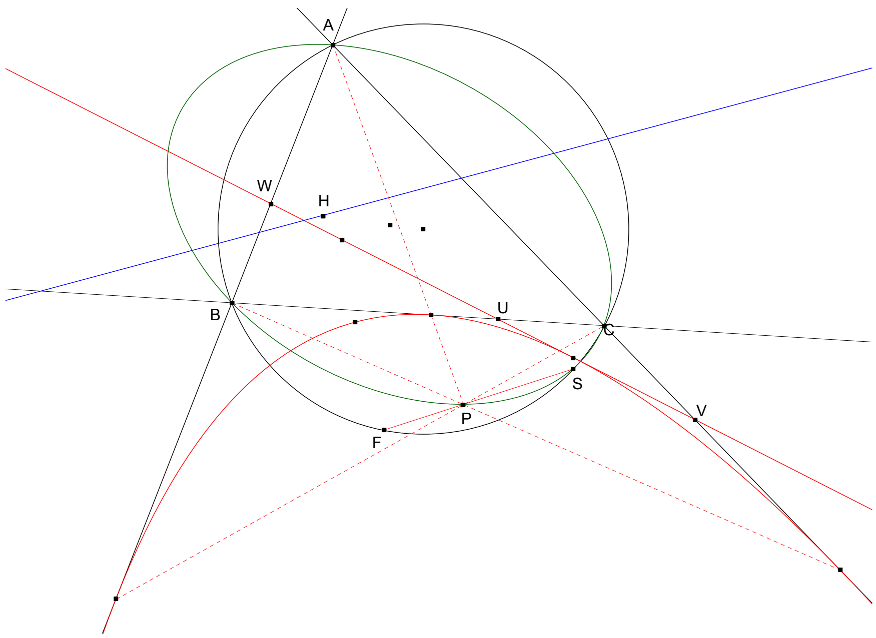

Figure 13

The perspector of the parabola has barycentric coordinates

$$\left(\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right).$$

This point is the isotomic conjugate of the infinite point of the Newton line. It is also the second common point (apart from the Steiner point S of triangle ABC) of the line SF and the Steiner circum-ellipse.

If a line ℓ' tangent to the parabola intersects the lines BC, CA, AB respectively at U', V', W', we have

$$(FU,\ FU') = (FV,\ FV') = (FW,\ FW') = (\ell,\ \ell').$$

If four points P, Q, R, S lie respectively on the sidelines BC, CA, AB, ℓ and verify

$$(FP,\ BC) = (FQ,\ CA) = (FR,\ AB) = (FS,\ \ell),$$

then these four points lie on the same line tangent to the parabola. This is a generalization of the pedal line.

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