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Signed Distances and the Erdős-Mordell Inequality

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Abstract. Using signed distances from the sides of a triangle we prove an inequality from which we get the Erdős-Mordell inequality as a simple consequence.

Let P be an arbitrary point in the plane of triangle ABC. Denote by x_1, x_2, x_3 the distances of P from the vertices A, B, C, and d_1, d_2, d_3 the *signed* distances of P from the sidelines BC, CA, AB respectively. Let a, b, c be the lengths of these sides. We establish an inequality from which the famous Erdős-Mordell inequality easily follows.

Theorem.

$$x_1 + x_2 + x_3 \geq \left(\frac{b}{c} + \frac{c}{b}\right) d_1 + \left(\frac{c}{a} + \frac{a}{c}\right) d_2 + \left(\frac{a}{b} + \frac{b}{a}\right) d_3; \tag{1}$$

equality holds if and only if P is the circumcenter of ABC.

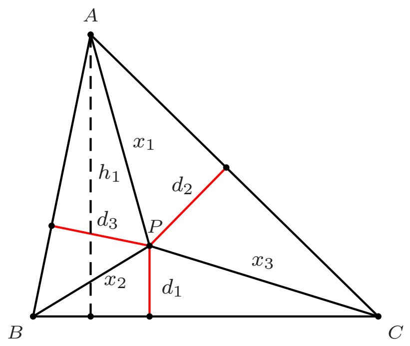

Figure 1

Proof. Let h_1 be the length of the altitude from A to BC, and Δ the area of ABC. Clearly,

$$2\Delta = ah_1 = ad_1 + bd_2 + cd_3.$$

Note that $x_1 + d_1 \geq h_1$. This is true even if $d_1 < 0$, *i.e.*, when P is not an interior point of the triangle. Also, equality holds if and only if P lies on the line containing the A-altitude. We have $ax_1 + ad_1 \geq ah_1 = ad_1 + bd_2 + cd_3$, or

$$ax_1 \geq bd_2 + cd_3. \tag{2}$$

If we apply inequality (2) to triangle $AB'C'$ symmetric to ABC with respect to the A-bisector of ABC we get

$$ax_1 \geq cd_2 + bd_3$$

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or

$$x_1 \geq \frac{c}{a}d_2 + \frac{b}{a}d_3. \tag{3}$$

Equality holds only when P lies on the A-altitude of $AB'C'$, *i.e.*, the line passing through A and the circumcenter of ABC.

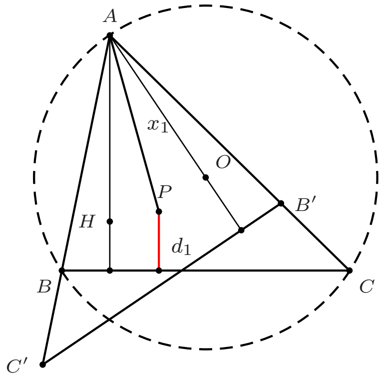

Figure 2

Similarly we get

$$x_2 \geq \frac{a}{b}d_3 + \frac{c}{b}d_1, \tag{4}$$

$$x_3 \geq \frac{b}{c}d_1 + \frac{a}{c}d_2, \tag{5}$$

and by addition of (3), (4), (5), we get the inequality (1). Equality holds only when P is the circumcenter of ABC. □

If P is an internal point of ABC, $d_1,\ d_2,\ d_3 > 0$. Since $\frac{b}{c} + \frac{c}{b} \geq 2$, $\frac{c}{a} + \frac{a}{c} \geq 2$, $\frac{a}{b} + \frac{b}{a} \geq 2$, we have

$$x_1 + x_2 + x_3 \geq 2(d_1 + d_2 + d_3).$$

This is the famous Erdős-Mordell inequality. The equality holds only when $a = b = c$, *i.e.*, ABC is equilateral, and P is the circumcenter of ABC.

There are numerous proofs of the Erdős-Mordell inequality. See, for example, [3] and the bibliography therin. In Mordell's original proof [2], the inequality (1) was established assuming $d_1, d_2, d_3 > 0$. See also [1, §12.13]. Our proof of (1) is more transparent and covers all positions of P.

References

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

[2] P. Erdős and L. J. Mordell, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396; solutions, *ibid.*, 44 (1937) 252.

[3] H.J Lee, Another Proof of the Erdős-Mordell Theorem, *Forum Geom.*, 1 (2001) 7–8.

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