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On the Intercepts of the OI-Line

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Abstract. We prove a new property of the intercepts of the line joining the circumcenter and the incenter on the sidelines of a triangle.

Given a triangle ABC with circumcenter O and incenter I, consider the intouch triangle XYZ. Let X' be the reflection of X in YZ, and similarly define Y' and Z'.

Theorem 1. *The intersections of AX' with BC, BY' with CA, and CZ' with AB are all on the line OI.*

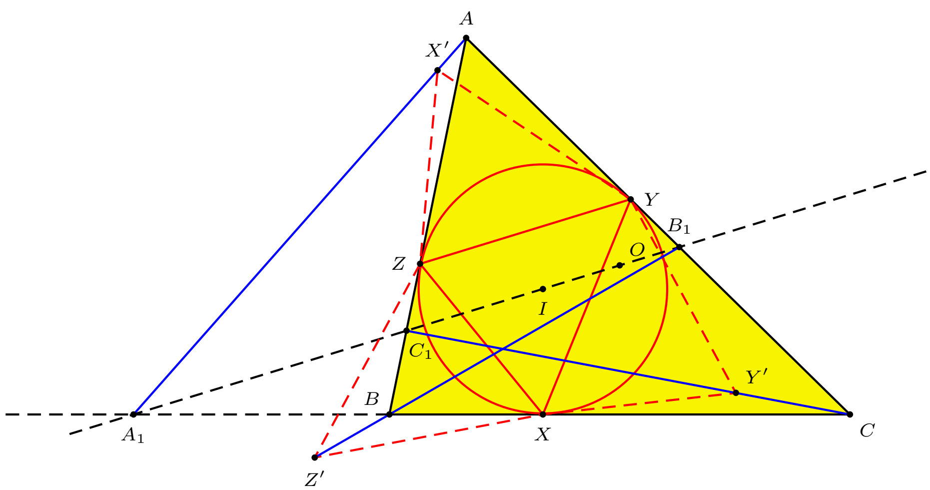

Figure 1.

Lemma 2. *The orthocenter H' of the intouch triangle lies on the line OI.*

Proof. Let $I_1I_2I_3$ be the excentral triangle. The lines YZ and I_2I_3 are parallel because both are perpendicular to AI. Similarly, $ZX//I_3I_1$ and $XY//I_1I_2$. See Figure 2. Hence, the excentral triangle and the intouch triangle are homothetic and their Euler lines are parallel. Now, I and O are the orthocenter and nine-point center of the excentral triangle. On the other hand, I is the circumcenter of the intouch triangle. Therefore, the line OI is their common Euler line, contains the orthocenter H' of XYZ. □

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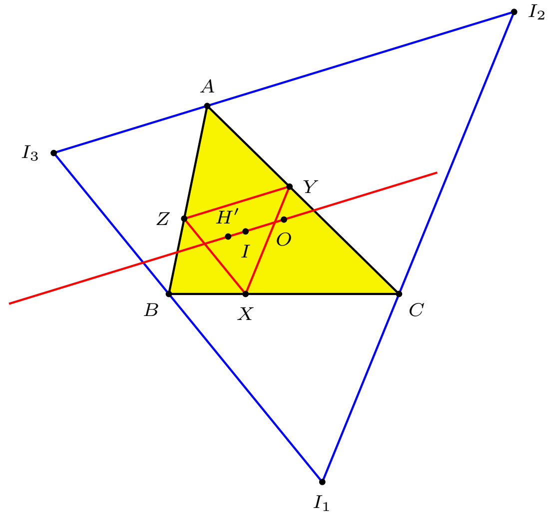

Figure 2.

Proof of Theorem 1. To prove that the intersection point A_1 of OI and AX' lies BC it is sufficient to show that $\frac{X'H'}{H'X} = \frac{AI}{IA_2}$, where A_2 is the foot of the bisector AI. See Figure 3.

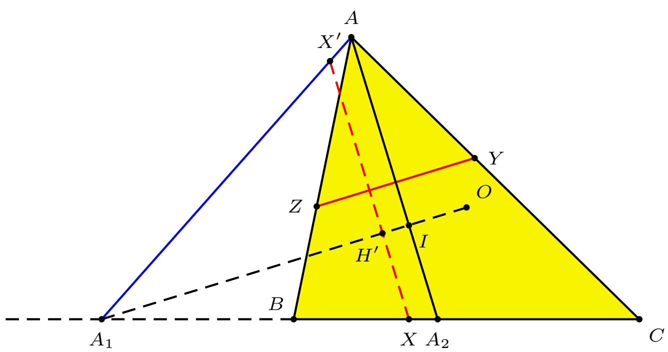

Figure 3.

It is known that

$$\frac{AI}{IA_2} = \frac{CA + AB}{BC} = \frac{\sin B + \sin C}{\sin A}.$$

For any acute triangle, $AH = 2R\cos A$. The angles of the intouch triangle are

$$X = \frac{B + C}{2}, \quad Y = \frac{C + A}{2}, \quad Z = \frac{A + B}{2}.$$

It is clear that triangle XYZ is always acute, and

$$XH' = 2r\cos X = 2r\cos\frac{B + C}{2} = 2r\sin\frac{A}{2},$$

where r is the inradius of triangle ABC.

$$\begin{aligned}\frac{X'H'}{H'X} &= \frac{X'X - H'X}{H'X} = \frac{X'X \cdot YZ}{H'X \cdot YZ} - 1\\ &= \frac{2 \cdot \text{area of } XYZ}{H'X \cdot YZ} - 1\\ &= \frac{2r^2(\sin 2X + \sin 2Y + \sin 2Z)}{2r \sin X \cdot 2r \cos X} - 1\\ &= \frac{\sin 2Y + \sin 2Z}{\sin 2X} = \frac{\sin B + \sin C}{\sin A}.\end{aligned}$$

This completes the proof of Theorem 1.

Similar results hold for the extouch triangle. In part it is in [1]. The following corollaries are clear.

Corollary 3. *The line joining A_1 to the projection of X on YZ passes through the midpoint of the bisector of angle A.*

Proof. In Figure 3, $X'X$ is parallel to the bisector of angle A and its midpoint is the projection of X on YZ. □

Corollary 4. *The OI-line is parallel to BC if and only if the projection of X on YZ lies on the line joining the midpoints of AB and AC.*

Corollary 5. *Let XYZ be the tangential triangle of ABC, X' the reflection of X in BC. If A_1 is the intersection of the Euler line and XX', then AA_1 is tangent to the circumcircle.*

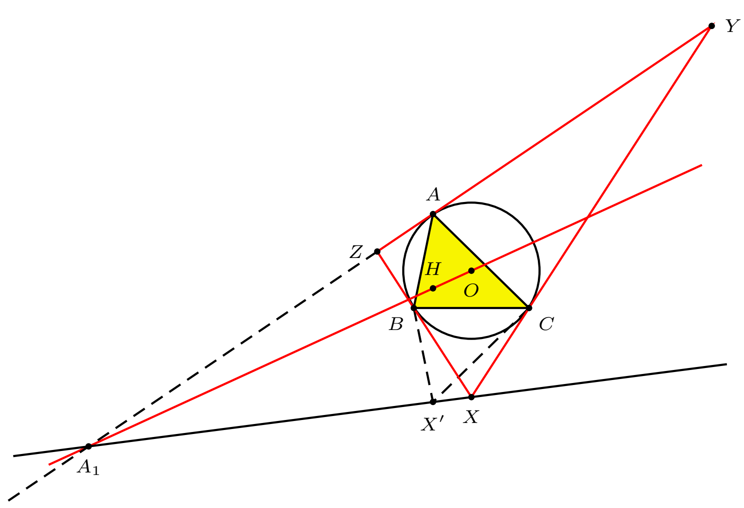

Figure 4.

References

[1] L. Emelyanov and T. Emelyanova, A note on the Schiffler point, *Forum Geom.*, 3 (2003) 113–116.

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