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The Vertex-Midpoint-Centroid Triangles

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Abstract. This paper explores six triangles that have a vertex, a midpoint of a side, and the centroid of the base triangle ABC as vertices. They have many interesting properties and here we study how they monitor the shape of ABC. Our results show that certain geometric properties of these six triangles are equivalent to ABC being either equilateral or isosceles.

Let A', B', C' be midpoints of the sides BC, CA, AB of the triangle ABC and let G be its centroid (*i.e.*, the intersection of medians AA', BB', CC'). Let G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+ be triangles BGA', CGA', CGB', AGB', AGC', BGC' (see Figure 1).

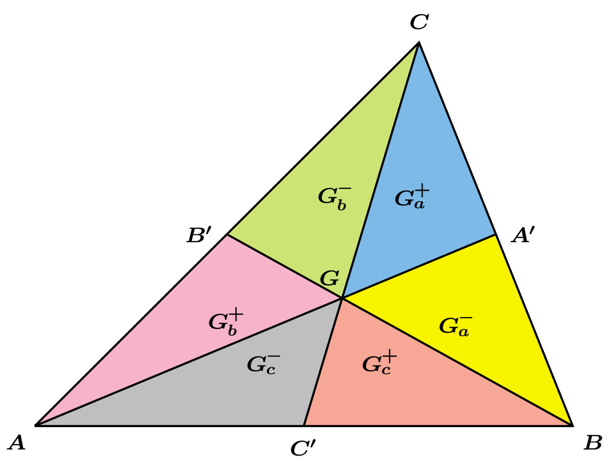

Figure 1. Six vertex–midpoint–centroid triangles of ABC.

This set of six triangles associated to the triangle ABC is a special case of the cevasix configuration (see [5] and [7]) when the chosen point is the centroid G. It has the following peculiar property (see [1]).

Theorem 1. *The triangle ABC is equilateral if and only if any three of the triangles from the set $\sigma_G = \{G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+\}$ have the same either perimeter or inradius.*

In this paper we wish to show several similar results. The idea is to replace perimeter and inradius with other geometric notions (like k-perimeter and Brocard angle) and to use various central points (like the circumcenter and the orthocenter – see [4]) of these six triangles.

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Let a, b, c be lengths of sides of the base triangle ABC. For a real number k, the sum $p_k = p_k(ABC) = a^k + b^k + c^k$ is called the k-*perimeter* of ABC. Of course, the 1-perimeter $p_1(ABC)$ is just the perimeter $p(ABC)$. The above theorem suggests the following problem.

Problem. *Find the set Ω of all real numbers k such that the following is true: The triangle ABC is equilateral if and only if any three of the triangles from σ_G have the same k-perimeter.*

Our first goal is to show that the set Ω contains some values of k besides the value $k = 1$. We start with $k = 2$ and $k = 4$.

Theorem 2. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same either 2-perimeter or 4-perimeter.*

Proof for $k = 2$. We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r(f + g), 0)$, and the vertex C has coordinates $\left(\frac{rg(f^2-1)}{fg-1}, \frac{2rfg}{fg-1}\right)$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (*i.e.*, that angles A and B are acute).

Nice features of this placement are that many important points of the triangle have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{rf\left(g^2+1\right)}{fg-1}, \quad b = \frac{rg\left(f^2+1\right)}{fg-1}, \quad c = r\,(f+g)\,,$$
$$f = \frac{(b+c)^2-a^2}{4\Delta}, \quad g = \frac{(a+c)^2-b^2}{4\Delta}, \quad r = \frac{2\Delta}{a+b+c},$$

where the area Δ is $\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}$.

There are 20 ways in which we can choose 3 triangles from the set σ_G. The following three cases are important because all other cases are similar to one of these.

Case 1: (G_a^-, G_a^+, G_b^-). When we compute the 2-perimeters $p_2(G_a^-)$, $p_2(G_a^+)$, and $p_2(G_b^-)$ and convert to lengths of sides we get

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3}.$$

Both of these differences are by assumption zero. From the first we get $b = c$ and when we substitute this into the second the conclusion is $\frac{(a-c)(a+c)}{6} = 0$. Hence, $b = c = a$ so that ABC is equilateral.

Case 2: (G_a^-, G_a^+, G_b^+). Now we have

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^+) = \frac{(a-b)(a+b)}{2},$$

which makes the conclusion easy.

Case 3: (G_a^-, G_b^-, G_c^-). This time we have

$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3},$$
$$p_2(G_a^-) - p_2(G_c^-) = \frac{a^2}{2} - \frac{b^2}{3} - \frac{c^2}{6}.$$

The only solution of this linear system in a^2 and b^2 is $a^2 = c^2$ and $b^2 = c^2$. Thus the triangle ABC is equilateral because the lengths of sides are positive. □

Recall that the Brocard angle ω of the triangle ABC satisfies the relation

$$\cot\omega = \frac{p_2(ABC)}{4\Delta}.$$

Since all triangles in σ_G have the same area, from Theorem 2 we get the following corollary.

Corollary 3. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same Brocard angle.*

On the other hand, when we put $k = -2$ then for $a = \sqrt{-5 + 3\sqrt{3}}$ and $b = c = 1$ we find that the triangles G_a^-, G_a^+, and G_b^- have the same (-2)-perimeter while ABC is not equilateral. In other words the value -2 is not in Ω.

The following result answers the final question in [1]. It shows that some pairs of triangles from the set σ_G could be used to detect if ABC is isosceles. Let τ denote the set whose elements are pairs (G_a^-, G_a^+) (G_a^-, G_b^+), (G_a^-, G_c^+), (G_a^+, G_b^-), (G_a^+, G_c^-), (G_b^-, G_b^+), (G_b^-, G_c^+), (G_b^+, G_c^-), (G_c^-, G_c^+).

Theorem 4. *The triangle ABC is isosceles if and only if triangles from some element of τ have the same perimeter.*

Proof. This time there are only two representative cases.

Case 1: (G_a^-, G_a^+). By assumption,

$$p(G_a^-) - p(G_a^+) = \frac{\sqrt{2a^2 - b^2 + 2c^2}}{3} - \frac{\sqrt{2a^2 + 2b^2 - c^2}}{3} = 0.$$

When we move the second term to the right then take the square of both sides and move everything back to the left we obtain $\frac{(c-b)(c+b)}{3} = 0$. Hence, $b = c$ and ABC is isosceles.

Case 2: (G_a^-, G_b^+). This time our assumption is

$$p(G_a^-) - p(G_b^+) = \frac{a-b}{2} + \frac{\sqrt{2a^2 - b^2 + 2c^2}}{6} - \frac{\sqrt{2c^2 + 2b^2 - a^2}}{6} = 0.$$

When we move the third term to the right then take the square of both sides and move the right hand side back to the left and bring the only term with the square root to the right we obtain

$$\frac{2a^2 - 3ab + b^2}{6} = \frac{(b-a)\sqrt{2a^2 - b^2 + 2c^2}}{6}.$$

In order to eliminate the square root, we take the square of both sides and move the right hand side to the left to get $\frac{(a-b)^2(a-b-c)(a-b+c)}{18} = 0$. Hence, $a = b$ and the triangle ABC is again isosceles. □

Remark. The above theorem is true also when the perimeter is replaced with the 2-perimeter and the 4-perimeter. It is not true for $k = -2$ but it holds for any $k \neq 0$ when only pairs (G_a^-, G_a^+), (G_b^-, G_b^+), (G_c^-, G_c^+) are considered.

We continue with results that use various central points (see [4], [5, 6]) (like the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian or the Grebe-Lemoine point, and the Longchamps point) of the triangles from the set σ_G and try to detect when ABC is either equilateral or isosceles.

Recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point in which they concur is their homology *center* and the line containing intersections of pairs of lines (BC, YZ), (CA, ZX), and (AB, XY) is their homology *axis*. Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*.

The triangles ABC and XYZ are *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto corresponding sides of ABC are concurrent at a point $[XYZ, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and two centers of paralogy $\langle ABC, XYZ\rangle$ and $\langle XYZ, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [2] and [3]).

Theorem 5. *The centroids $G_{G_a^-}$, $G_{G_a^+}$, $G_{G_b^-}$, $G_{G_b^+}$, $G_{G_c^-}$, $G_{G_c^+}$ of the triangles from σ_G lie on the image of the Steiner ellipse of ABC under the homothety $\mathsf{h}(G, \frac{\sqrt{7}}{6})$. This ellipse is a circle if and only if ABC is equilateral. The triangles $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$ are both homologic and paralogic to triangles $A_bB_bC_b$, $B_bC_bA_b$ and $C_bA_bB_b$ and they share with ABC the centroid and the Brocard angle and both have $\frac{7}{36}$ of the area of ABC. They are directly similar to each other or to ABC if and only if ABC is an equilateral triangle. They are orthologic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is an equilateral triangle.*

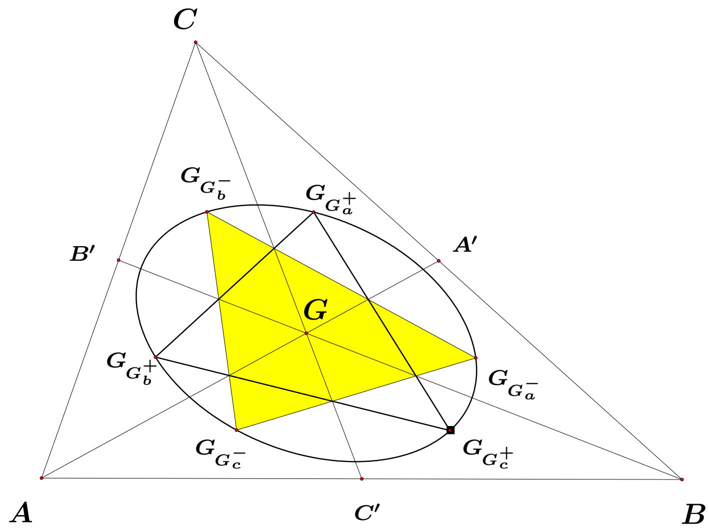

Figure 2. The ellipse containing vertices of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$.

Proof. We look for the conic through five of the centroids and check that the the sixth centroid lies on it. The trilinear coordinates of $G_{G_a^-}$ are $\frac{2}{a} : \frac{11}{b} : \frac{5}{c}$ while those of other centroids are similar. It follows that they all lie on the ellipse with the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0,$$

where

$$\begin{aligned} &a_{11} = 432\Delta^2, \quad a_{12} = 108\Delta(a-b)(a+b), \\ &a_{22} = 27(a^4 + b^4 + 3c^4 - 2a^2b^2), \\ &a_{13} = -216\Delta^2 c, \quad a_{23} = -54\Delta c(a^2 - b^2 + c^2), \quad a_{33} = 116\Delta^2 c^2. \end{aligned}$$

Since $D_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \frac{3c^4}{16\Delta^2} > 0$, and $\frac{A_0}{I_0} = \frac{-7c^4}{72(a^2+b^2+c^2)} < 0$ with $I_0 = a_{11} + a_{22}$, and $A_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$ it follows that this is an ellipse whose center is G. It will be a circle provided either $I_0^2 = 4D_0$ or $a_{11} = a_{22}$ and $a_{12} = 0$. This happens if and only if ABC is equilateral.

The precise identification of this ellipse is now easy. We take a point (p, q) which is on the Steiner ellipse of ABC (with the equation $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$ in trilinear coordinates) and denote its image under $\mathsf{h}(G, \frac{\sqrt{7}}{6})$ by (x, y). By eliminating p and q we check that this image satisfies the above equation (of the common Steiner ellipse of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$).

Since the trilinear coordinates of A_b are $abc : c^3 : b^3$, the line $A_b G_{G_a^-}$ has the equation

$$a(11b^2 - 5c^2)x + b(5a^2 - 2b^2)y + c(11a^2 - 2c^2)z = 0.$$

The lines $B_b G_{G_b^-}$ and $C_b G_{G_c^-}$ have similar equations. The determinant of the coefficients of these three lines is equal to zero so that we conclude that the triangles $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ and $A_b B_b C_b$ are homologic. The other claims about homologies and paralogies are proved in a similar way. We note that $\langle G_{G_a^-} G_{G_b^-} G_{G_c^-}, A_b B_b C_b \rangle$ is on the (above) Steiner ellipse of $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ while $\langle A_b B_b C_b, G_{G_a^-} G_{G_b^-} G_{G_c^-} \rangle$ is on the Steiner ellipse of $A_b B_b C_b$. The other centers behave accordingly.

When we substitute the coordinates of the six centroids into the conditions

$$x_1(v_2 - v_3) + x_2(v_3 - v_1) + x_3(v_1 - v_2) - u_1(y_2 - y_3) - u_2(y_3 - y_1) - u_3(y_1 - y_2) = 0,$$
$$x_1(u_2 - u_3) + x_2(u_3 - u_1) + x_3(u_1 - u_2) - y_1(v_2 - v_3) - y_2(v_3 - v_1) - y_3(v_1 - v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be directly similar and convert to the side lengths, we get

$$\frac{4\Delta(a-b)(a+b+c)}{9c^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)}{9c^2} = 0,$$

where

$$h(u, v, w, x, y, z) = ub^2c^2 + vc^2a^2 + wa^2b^2 - xa^4 - yb^4 - zc^4.$$

The first relation implies $a = b$, which gives $h(1,1,2,1,1,2) = 2c^2(c-b)(c+b)$. Therefore, $b = c$ so that ABC is an equilateral triangle.

Substituting the coordinates of $G_{G_a^-}$, $G_{G_b^-}$, $G_{G_c^-}$, A_b, B_b, C_b into the left hand side of the condition

$$x_1(u_2-u_3)+x_2(u_3-u_1)+x_3(u_1-u_2)+y_1(v_2-v_3)+y_2(v_3-v_1)+y_3(v_1-v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be orthologic, we obtain

$$\frac{-h(1,1,1,1,1,1)}{3p_2(ABC)} = \frac{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2}{6p_2(ABC)}$$

so that the triangles $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ and $A_b B_b C_b$ are orthologic if and only if ABC is equilateral.

The remaining statements are proved similarly or by substitution of coordinates into well-known formulas for the area, the centroid, and the Brocard angle. □

Let m_a, m_b, m_c be lengths of medians of the triangle ABC. The following result is for the most part already proved in [7]. The center of the circle is given in [6] as $X(1153)$.

Theorem 6. *The circumcenters $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ of the triangles from σ_G lie on the circle whose center O_G is a central point with the first*

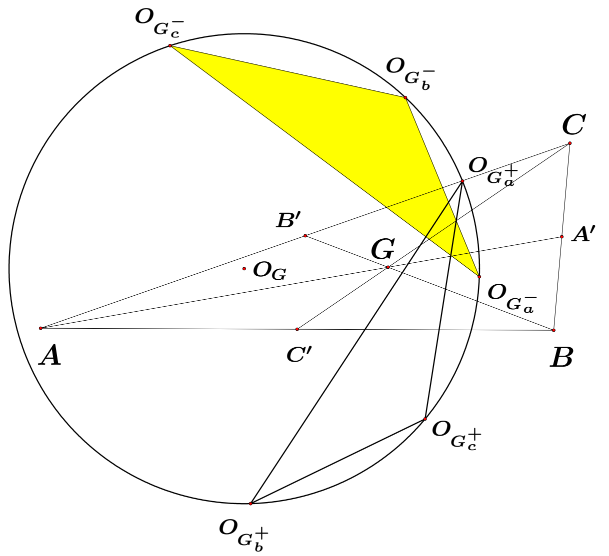

Figure 3. The vertices of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ are on a circle.

trilinear coordinate

$$\frac{10a^4 - 13a^2(b^2+c^2) + 4b^4 + 4c^4 - 10b^2c^2}{a}$$

and whose radius is

$$\frac{m_a m_b m_c \sqrt{2(a^4+b^4+c^4) - 5(b^2c^2+c^2a^2+a^2b^2)}}{72\Delta}.$$

Also, $|O_G G| = \frac{m_a m_b m_c \sqrt{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}}{72\sqrt{2}\Delta}$.

Proof. The proof is conceptually simple but technically involved so that we shall only outline how it could be done on a computer. In order to find points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we use the circumcenter function and evaluate it in vertices of the triangles from σ_G. Applying it again in points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$ we obtain the point O_G. The remaining points $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ are at the same distance from it as the vertex $O_{G_a^-}$ is. The remaining tasks are standard (they involve only the distance function and the conversion to the side lengths). □

The last sentence in Theorem 6 implies the following corollary.

Corollary 7. *The triangle ABC is equilateral if and only if the circumcenters of any three of the triangles in σ_G have the same distance from the centroid G.*

Let P, Q and R denote vertices of similar isosceles triangles BCP, CAQ and ABR.

Theorem 8. *(1) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are congruent. They are orthologic to BCA and CAB, respectively.*

(2) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ *are orthologic to* QRP *and* RPQ *if and only if* ABC *is an equilateral triangle.*

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *are orthologic if and only if the lengths of sides of* ABC *satisfy* $h(7,7,7,4,4,4) = 0$*.*

(4) The line joining the centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *will go through the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the Longchamps point, or the Bevan point of* ABC *(*i.e., $X(2)$*,* $X(3)$*,* $X(4)$*,* $X(5)$*,* $X(20)$*, or* $X(40)$ *in* [6]*) if and only if it is an equilateral triangle.*

(5) The line joining the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *goes through the centroid of* ABC*. It will go through the centroid of its orthic triangle (*i.e., $X(51)$ *in* [6]*) if and only if* ABC *is an equilateral triangle.*

(6) The centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *have the same distance from* $X(2)$*,* $X(3)$*,* $X(4)$*,* $X(5)$*,* $X(6)$*,* $X(20)$*,* $X(39)$*,* $X(40)$*, or* $X(98)$ *if and only if* ABC *is an isosceles triangle.*

Proof. (1) The points $O_{G_a^-}$ and $O_{G_a^+}$ have trilinear coordinates

$$a(5c^2 - a^2 - b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$$a(5b^2 - a^2 - c^2) : \frac{h(6,3,1,1,4,2)}{b} : \frac{2h(3,5,3,2,1,2)}{c},$$

while the trilinears of the points $O_{G_b^-}$, $O_{G_c^-}$, $O_{G_b^+}$, $O_{G_c^+}$ are their cyclic permutations. We can show easily that $|O_{G_b^-}O_{G_c^-}|^2 - |O_{G_c^+}O_{G_a^+}|^2 = 0$, $|O_{G_c^-}O_{G_a^-}|^2 - |O_{G_a^+}O_{G_b^+}|^2 = 0$, and $|O_{G_a^-}O_{G_b^-}|^2 - |O_{G_b^+}O_{G_c^+}|^2 = 0$, so that $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are indeed congruent.

Substituting the coordinates of $O_{G_a^-}$, $O_{G_b^-}$, $O_{G_c^-}$, B, C, A into the left hand side of the above condition for triangles to be orthologic we conclude that it holds. The same is true for the triangles $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ and CAB.

(2) The point P has the trilinear coordinates

$$2ka : \frac{k(a^2 + b^2 - c^2) + 2\Delta}{b} : \frac{k(a^2 - b^2 + c^2) + 2\Delta}{c}$$

for some real number $k \neq 0$. The coordinates of Q and R are analogous. It follows that the triangles $O_{G_c^-}O_{G_a^-}O_{G_b^-}$ and QRP are orthologic provided

$$\frac{h(1,1,1,1,1,1)k}{8\Delta} = 0,$$

i.e., if and only if ABC is equilateral.

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are orthologic provided $\frac{p_2(ABC)h(7,7,7,4,4,4)}{384\Delta^2} = 0$. The triangle with lengths of sides $4, 4, 3\sqrt{2} + \sqrt{10}$ satisfies this condition.

(4) for $X(40)$. The first trilinear coordinates of the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{3a^4-(2b^2+7c^2)a^2+b^4-3b^2c^2+2c^4}{a}$$

and

$$\frac{3a^4-(7b^2+2c^2)a^2+2b^4-3b^2c^2+c^4}{a}.$$

The line joining these centroids will go through $X(40)$ with the first trilinear coordinate $a^3+(b+c)a^2-(b+c)^2a-(b+c)(b-c)^2$ provided

$$\frac{(a^2+b^2+c^2-bc-ca-ab)(3bc+3ca+3ab+a^2+b^2+c^2)}{96\Delta}=0.$$

Since $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\left((b-c)^2+(c-a)^2+(a-b)^2\right)$ it follows that this will happen if and only if ABC is equilateral.

(5) The first trilinear coordinates of the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{2a^6-(b^2+3c^2)a^4+(3b^4-12b^2c^2-7c^4)a^2+2c^2(b^2-c^2)(b^2-2c^2)}{a}$$

and

$$\frac{2a^6-(3b^2+c^2)a^4-(7b^4+12b^2c^2-3c^4)a^2+2b^2(b^2-c^2)(2b^2-c^2)}{a}.$$

The line joining these symmedian points will go through $X(51)$ with the first trilinear coordinate $a\left((b^2+c^2)a^2-(b^2-c^2)^2\right)$ provided

$$\frac{2\Delta h(1,1,1,0,0,0)h(1,1,1,1,1,1)}{9a^2b^2c^2(a^2+b^2+c^2)}=0.$$

Since $h(1,1,1,1,1,1)=\frac{1}{2}\left((b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\right)$ we see that this will happen if and only if ABC is equilateral. The trilinear coordinates $\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$ of the centroid G satisfy the equation of this line.

(6) for $X(40)$. Using the information from the proof of (4), we see that the difference of squares of distances from $X(40)$ to the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ is $\frac{(b-c)(c-a)(a-b)M}{192\Delta^2}$, where

$$M=2(a^3+b^3+c^3)+5(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+18abc$$

is clearly positive. Hence, these distances are equal if and only if ABC is isosceles. □

With points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we can also detect if ABC is isosceles as follows.

Theorem 9. *(1) The relation $b=c$ holds in ABC if and only if $O_{G_a^-}$ is on BG and/or $O_{G_a^+}$ is on CG.*

(2) The relation $c=a$ holds in ABC if and only if $O_{G_b^-}$ is on CG and/or $O_{G_b^+}$ is on AG.

(3) The relation $a = b$ holds in ABC if and only if $O_{G_c^-}$ is on AG and/or $O_{G_a^+}$ is on BG.

Proof. (1) for $O_{G_a^-}$. Since the trilinear coordinates of $O_{G_a^-}$, G and B are

$$a(5c^2 - a^2 - b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$ and $(0 : 1 : 0)$, it follows that these points are collinear if and only if $\frac{m_b^2(b-c)(b+c)}{72\Delta} = 0$. □

For the following result I am grateful to an anonymous referee. It refers to the point T on the Euler line which divides the segment joining the circumcenter with the centroid in ratio k for some real number $k \neq -1$. Notice that for $k = 0, -\frac{3}{4}, -\frac{3}{2}, -3$ the point T will be the circumcenter, the Longchamps point, the orthocenter, and the center of the nine-point circle, respectively.

Theorem 10. *The triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar to each other or to ABC if and only if ABC is equilateral.*

Proof. For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$.

The point $T_{G_a^-}$ has $\frac{p_1}{a} : \frac{p_2}{b} : \frac{p_3}{c}$ as trilinear coordinates, where

$$\begin{aligned}p_1 =& 3a^2(a^2 + b^2 - 5c^2) - 32\Delta^2 k,\\ p_2 =& 12a^4 - 6(5b^2 + 3c^2)a^2 + 6(b^2 - c^2)(2b^2 - c^2) - 176\Delta^2 k,\\ p_3 =& 12a^4 - 6(3b^2 + 5c^2)a^2 + 6(b^2 - c^2)(b^2 - 2c^2) - 176\Delta^2 k.\end{aligned}$$

Applying the method of the proof of Theorem 4 we see that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar if and only if

$$\frac{(a^2 - b^2)M}{288\Delta c^2(k+1)^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)M}{1152S^2c^2(k+1)^2} = 0,$$

where $M = 128\Delta^2 k^2 + 240\Delta^2 k + h(15,15,15,6,6,6)$. The discriminant

$$-48\Delta^2 h(10,10,10,-11,-11,-11)$$

of the trinomial M is negative so that M is always positive. Hence, from the first condition it follows that $a = b$. Then the factor $h(1,1,2,1,1,2)$ in the second condition is $2c^2(c - b)(c + b)$ so that $b = c$ and ABC is equilateral. The converse is easy because for $a = b = c$ the left hand sides of both conditions are equal to zero.

For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC. The two conditions are

$$\begin{aligned}&32\Delta^2(a^2 - b^2)k - a^6 + (4b^2 + 3c^2)a^4\\ &- (5b^4 + 2b^2c^2 + c^4)a^2 - 3b^4c^2 + 2b^2c^4 + 2b^6 + c^6 = 0\end{aligned}$$

and

$$h(2,2,4,2,2,4)k + h(1,2,3,1,2,3) = 0.$$

When $a \neq b$, we can solve the first equation for k and substitute it into the second to obtain $\frac{c^4(a^2+b^2+c^2)h(1,1,1,1,1,1)}{8\Delta^2(a^2-b^2)} = 0$. This implies that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are directly similar if and only if ABC is equilateral because the first condition is $c^2\,(b-c)\,(b+c)\left(c^2+2b^2\right) = 0$ for $a = b$. □

Theorem 11. *(1) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to ABC if and only if $k = -\frac{3}{2}$.*

(2) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to $A_bB_bC_b$ if and only if either ABC is equilateral or $k = -\frac{3}{4}$.

(3) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are paralogic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is equilateral.

(4) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ is orthologic to $B_bC_bA_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$ and to $C_bA_bB_b$ if and only if ABC is equilateral.

(5) $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ is orthologic to $B_bC_bA_b$ if and only if ABC is equilateral and to $C_bA_bB_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$.

Proof. All parts have similar proofs. For example, in the first, we find that the triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are orthologic if and only if $-\frac{(a^2+b^2+c^2)(2k+3)}{12(k+1)} = 0$. □

The orthocenters $H_{G_a^-}$, $H_{G_a^+}$, $H_{G_b^-}$, $H_{G_b^+}$, $H_{G_c^-}$, $H_{G_c^+}$ of the triangles from σ_G also monitor the shape of the triangle ABC.

Theorem 12. *The triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ are orthologic if and only if ABC is an equilateral triangle.*

Proof. Substituting the coordinates of $H_{G_a^-}$, $H_{G_b^-}$, $H_{G_c^-}$, $H_{G_a^+}$, $H_{G_b^+}$, $H_{G_c^+}$ into the condition for triangles to be orthologic (see the proof of Theorem 6), we obtain

$$\frac{(a^2+b^2+c^2)[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2]}{192\Delta^2} = 0.$$

Hence, $a = b = c$ and the triangle ABC is equilateral. □

Remark. Note that the triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ have the same Brocard angle and both have the area equal to one fourth of the area of ABC.

The centers $F_{G_a^-}$, $F_{G_a^+}$, $F_{G_b^-}$, $F_{G_b^+}$, $F_{G_c^-}$, $F_{G_c^+}$ of the nine point circles of the triangles from σ_G allow the following analogous result.

Theorem 13. *The triangles $F_{G_a^-}F_{G_b^-}F_{G_c^-}$ and $F_{G_a^+}F_{G_b^+}F_{G_c^+}$ have the same Brocard angle and area. The triangle ABC is equilateral if and only if this area is $\frac{3}{16}$ of the area of ABC.*

Proof. Recall the formula $\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$ for the area of the triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3). Since

$$\frac{3}{16}|ABC| - |F_{G_a^-}F_{G_b^-}F_{G_c^-}| = \frac{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}{1536\Delta},$$

the second claim is true. The proof of the first are also substitutions of coordinates into well-known formulas. □

The symmedian points $K_{G_a^-}$, $K_{G_a^+}$, $K_{G_b^-}$, $K_{G_b^+}$, $K_{G_c^-}$, $K_{G_c^+}$ of the triangles from σ_G play the similar role.

Theorem 14. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the area equal to $\frac{7}{64}$ of the area of ABC if and only if ABC is an equilateral triangle.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - \frac{7}{64}|ABC|$ is equal to

$$\frac{3\Delta T}{64(5b^2+8c^2-a^2)(5c^2+8a^2-b^2)(5a^2+8b^2-c^2)},$$

where

$$T = 40(a^6+b^6+c^6)+231(b^4c^2+c^4a^2+a^4b^2)-147(b^2c^4+c^2a^4+a^2b^4)-372a^2b^2c^2.$$

We shall argue that T is equal to zero if and only if $a=b=c$. We can assume that $a \le b \le c$, $a=\sqrt{d}$, $b=\sqrt{(1+h)d}$, $c=\sqrt{(1+h+k)d}$ for some positive real numbers d, h and k. In new variables $\frac{T}{d^3}$ is

$$164h^3+(204+57k)h^2+3k(68-9k)h+4k^2(51+10k).$$

The quadratic part has the discriminant $-3k^2(41616+30056k+2797k^2)$. Thus T is always positive except when $h=k=0$ which proves our claim. □

Theorem 15. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the same area if and only if the triangle ABC is isosceles.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - |K_{G_a^+}K_{G_b^+}K_{G_c^+}|$ is equal to

$$\frac{81\Delta(b-c)(b+c)(c-a)(c+a)(a-b)(a+b)T}{2t(-1,8,5)t(-1,5,8)t(8,-1,5)t(5,-1,8)t(8,5,-1)t(5,8,-1)},$$

where $t(u,v,w)=ua^2+vb^2+wc^2$ and

$$T = 10(a^6+b^6+c^6)-105(b^4c^2+c^4a^2+a^4b^2+b^2c^4+c^2a^4+a^2b^4)-156a^2b^2c^2.$$

We shall now argue that T is always negative. Without loss of generality we can assume that $a \le b \le c$ and that

$$a=\sqrt{d}, \quad b=\sqrt{(1+h)d}, \quad c=\sqrt{(1+h+k)d},$$

for some positive real numbers d, h and k. Since $a+b>c$ it follows that

$$k < 1+2\sqrt{h+1} \le h+3$$

because $\sqrt{h+1}=\sqrt{1\cdot(h+1)} \le \frac{1+(h+1)}{2}$. In new variables,

$$-\frac{T}{d^3} = 190h^3+(285k+936)h^2+(1512+936k+75k^2)h-10k^3+180k^2+756k+756.$$

For $k \le h$ it is obvious that the above polynomial is positive since $190h^3-10k^3 > 0$. On the other hand, when $k \in (h, h+3)$, then k can be represented as $(1-w)h + w(h+3)$ for some $w \in (0,1)$. The above polynomial for this k is

$$540h^3+(2052+1215w)h^2+(3888w+405w^2+2268)h-270w^3+1620w^2+2268w+756.$$

But, the free coefficient of this polynomial for w between 0 and 1 is positive. Thus T is always negative which proves our claim. $\square$

The Longchamps points (*i.e.*, the reflections of the orthocenters in the circumcenters) $L_{G_a^-}$, $L_{G_a^+}$, $L_{G_b^-}$, $L_{G_b^+}$, $L_{G_c^-}$, $L_{G_c^+}$ of the triangles from σ_G offer the following result.

Theorem 16. *The triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and $L_{G_a^+}L_{G_b^+}L_{G_c^+}$ have the same areas and Brocard angles. This area is equal to $\frac{3}{4}$ of the area of ABC and/or this Brocard angle is equal to the Brocard angle of ABC if and only if ABC is an equilateral triangle.*

Proof. The common area is $\frac{h(10,10,10,1,1,1)}{112\Delta}$ while the tangent of the common Brocard angle is $\frac{h(10,10,10,1,1,1)}{4\Delta p_2(ABC)h(2,2,2,-7,-7,-7)}$. It follows that the difference

$$\frac{3}{4}|ABC| - |L_{G_a^-}L_{G_b^-}L_{G_c^-}| = \frac{h(1,1,1,1,1,1)}{24\Delta}$$

while the difference of tangents of the Brocard angles of the triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and ABC is $\frac{32\Delta h(1,1,1,1,1,1)}{p_2(ABC)h(2,2,2,-7,-7,-7)}$. From here the conclusions are easy because $h(1,1,1,1,1,1) = \frac{1}{2}\left((b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right)$. $\square$

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