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The Intouch Triangle and the OI-line

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Abstract. We prove some interesting results relating the intouch triangle and the OI line of a triangle. We also give some interesting properties of the triangle center X_{57}, the homothetic center of the intouch and excentral triangles.

1. Introduction

L. Emelyanov [4] has recently given an interesting relation between the OI-line and the triangle of reflections of the intouch triangle. Here, O and I are respectively the circumcenter and incenter of the triangle. Given triangle ABC with intouch triangle XYZ, let X_2, Y_2, Z_2 be the reflections of X, Y, Z in their respective opposite sides YZ, ZX, XY. Then the lines AX_2, BY_2, CZ_2 intersect BC, CA, AB at the intercepts of the OI-line.

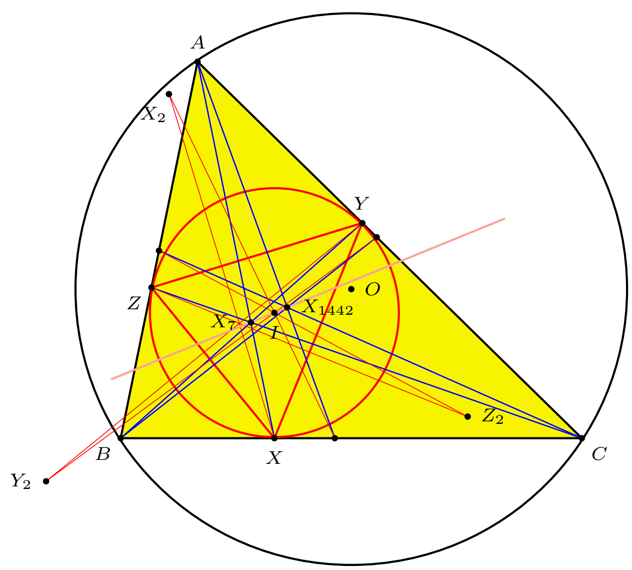

Figure 1.

Emelyanov [3] also noted that the intercepts of the points $IX_2 \cap BC$, $IY_2 \cap CA$, $IZ_2 \cap AB$ form a triangle perspective with ABC. See Figure 1. According to [7], this perspector is the point

$$X_{1442} = \left(\frac{a(b^2+bc+c^2-a^2)}{s-a} : \frac{b(c^2+ca+a^2-b^2)}{s-b} : \frac{c(a^2+ab+b^2-c^2)}{s-c}\right)$$

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on the Soddy line joining the incenter and the Gergonne point.

In this paper we generalize these results. We work with barycentric coordinates with reference to triangle ABC.

2. The triangle center X_{57}

Let a, b, c be the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$ the semiperimeter. The intouch triangle XYZ and the excentral triangle (with the excenters as vertices) are clearly homothetic, since their corresponding sides are perpendicular to the same angle bisector of triangle ABC. These triangles are respectively the cevian triangle of the Gergonne point $\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ and the anticevian triangle of the incenter $(a : b : c)$, their homothetic center has coordinates

$$\begin{aligned}&(a(-a(s-a)+b(s-b)+c(s-c)) : \cdots : \cdots)\\ =&(2a(s-b)(s-c) : \cdots : \cdots)\\ =&\left(\frac{a}{s-a} : \cdots : \cdots\right).\end{aligned}$$

This is the triangle center X_{57} in [6], defined as the isogonal conjugate of the Mittenpunkt $X_9 = (a(s-a) : b(s-b) : c(s-c))$. This is a point on the OI-line since the two triangles in question have circumcenters I and X_{40} (the reflection of I in O), [1]

We give some interesting properties of the triangle X_{57}.

Since ABC is the orthic triangle of the excentral triangle, it is homothetic to the orthic triangle $X_1Y_1Z_1$ of XYZ with the same homothetic center X_{57}. See Figure 2.

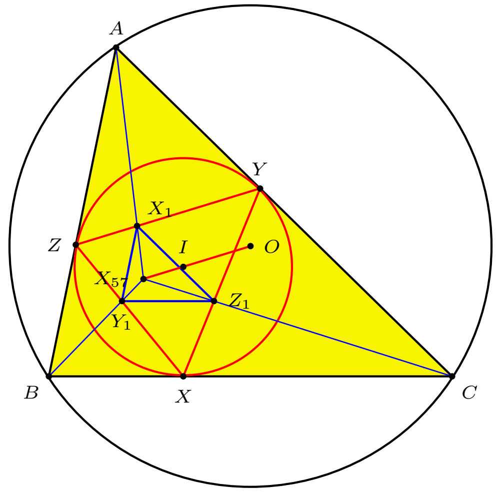

Figure 2.

[1]The circumcircle of ABC is the nine-point circle of the excentral triangle.

Let DEF be the circumcevian triangle of the incenter I, and D', E', F' the antipodes of D, E, F in the circumcircle. In other words, D and D' are the midpoints of the two arcs BC, D' on the arc containing the vertex A; similarly for the other two pairs. Clearly,

$$D = \left(\frac{a^2}{-(b+c)} : \frac{b^2}{b} : \frac{c^2}{c}\right) = (-a^2 : b(b+c) : c(b+c)).$$

Similarly,

$$E = (a(c+a) : -b^2 : c(c+a)) \quad \text{and} \quad F = (a(a+b) : b(a+b) : -c^2).$$

To compute the coordinates of D', E', F', we make use of the following formula.

Lemma 1. *Let $P = (a^2vw : b^2wu : c^2uv)$ be a point on the circumcircle (so that $u + v + w = 0$). For a point $Q = (x : y : z)$ different from P and not lying on the circumcircle, the line PQ intersects the circumcircle again at the point $(a^2vw + tx : b^2wu + ty : c^2uv + tz)$, where*

$$t = \frac{b^2c^2u^2x + c^2a^2v^2y + a^2b^2w^2z}{a^2yz + b^2zx + c^2xy}. \tag{1}$$

Proof. Entering the coordinates

$$(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = (a^2vw + tx : b^2wu + ty : c^2uv + tz)$$

into the equation of the circumcircle

$$a^2\mathbb{YZ} + b^2\mathbb{ZX} + c^2\mathbb{XY} = 0,$$

we obtain

$$\begin{aligned}
&(a^2yz + b^2zx + c^2xy)t^2\\
&+(b^2c^2u(v+w)x + c^2a^2v(w+u)y + a^2b^2w(u+v)z)t\\
&+a^2b^2c^2uvw(u+v+w) = 0.
\end{aligned}$$

Since $u + v + w = 0$, this gives $t = 0$ or the value given in (1) above. □

Let $M = (0 : 1 : 1)$ be the midpoint of BC. Applying Lemma 1 to D and M, we obtain

$$D' = (-a^2 : b(b-c) : c(c-b)).$$

Similarly,

$$E' = (a(a-c) : -b^2 : c(c-a) \quad \text{and} \quad F' = (a(a-b) : b(b-a) : -c^2).$$

Applying Lemma 1 to D' and $X = (0 : a+b-c : c+a-b)$, (likewise to E' and Y, and to F' and Z), we obtain the points

$$X' = \left(\frac{-a^2}{a(b+c)-(b-c)^2} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right),$$

$$Y' = \left(\frac{a}{b+c-a} : \frac{-b^2}{b(c+a)-(c-a)^2} : \frac{c}{a+b-c}\right),$$

$$Z' = \left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{-c^2}{c(a+b)-(a-b)^2}\right).$$

These are clearly the vertices of the circumcevian triangle of X_{57}. We summarize this in the following proposition.

Proposition 2. *If X' (respectively Y', Z') are the second intersections of $D'X$ (respectively $E'Y$, $F'Z$) and the circumcircle, then $X'Y'Z'$ is the circumcevian triangle of X_{57}.*

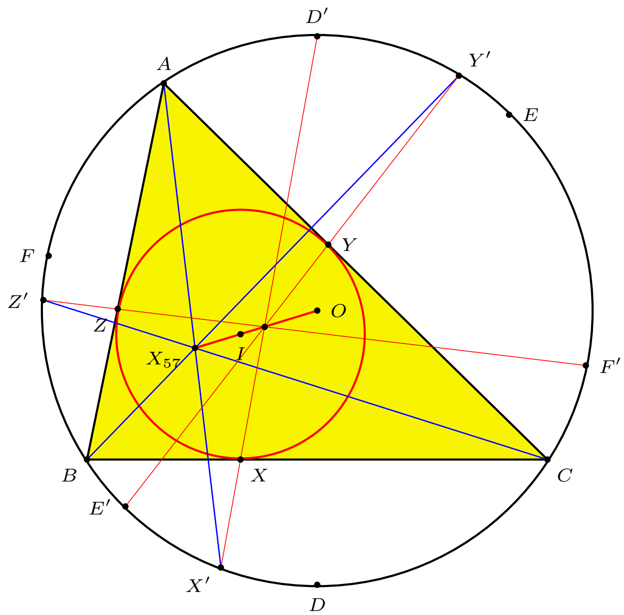

Figure 3.

Remark. The lines $D'X$, $E'Y$, $F'Z$ intersect at X_{55}, the internal center of similitude of the circumcircle and the incircle.

Proposition 3. *Let X'', Y'', Z'' be the second intersections of the circumcircle with the lines DX, EY, FZ respectively. The lines AX'', BY'', CZ'' bound the anticevian triangle of X_{57}.*

Proof. By Lemma 1, these are the points

$$X'' = \left(\frac{a^2}{s-a} : \frac{b(b-c)}{s-b} : \frac{c(c-b)}{s-c}\right),$$
$$Y'' = \left(\frac{a(a-c)}{s-a} : \frac{b^2}{s-b} : \frac{c(c-a)}{s-c}\right),$$
$$Z'' = \left(\frac{a(a-b)}{s-a} : \frac{b(b-a)}{s-b} : \frac{c^2}{s-c}\right).$$

The lines AX'', BY'', CZ'' have equations

$$\begin{array}{ccccccc} & & \frac{s-b}{b}y & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & & & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & + & \frac{s-b}{b}y & & & = & 0. \end{array}$$

They clearly bound the anticevian triangle of X_{57}. See Figure 4. □

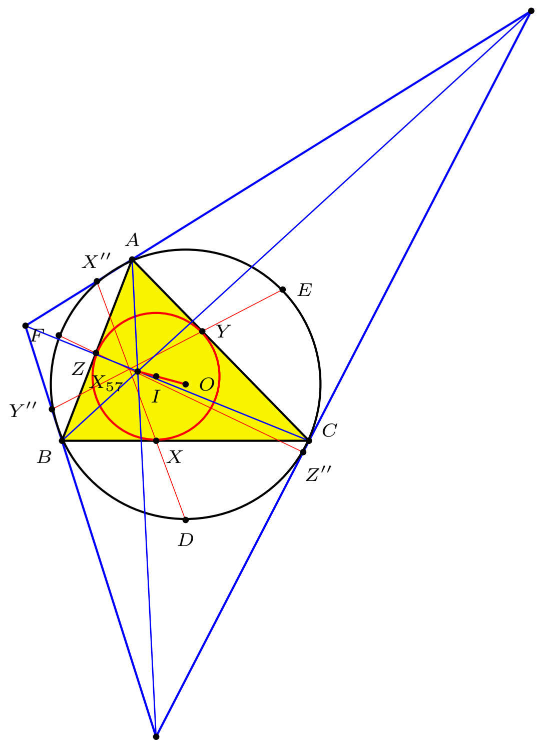

Figure 4.

Remark. The lines DX, EY, FZ intersect at X_{56}, the external center of similitude of the circumcircle and incircle.

Proposition 4. *X_{57} is the perspector of the triangle bounded by the polars of A, B, C with respect to the circle through the excenters.*

Proof. As is easily verified, the equation of the circumcircle of the excentral triangle is

$$a^2yz + b^2zx + c^2xy + (x + y + z)(bcx + cay + abz) = 0.$$

The polars are the lines

$$\begin{array}{rcrcrcl} \frac{x}{s} & + & \frac{y}{b} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{s} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{b} & + & \frac{z}{s} & = & 0. \end{array}$$

They bound a triangle with vertices

$$\left(-\frac{a(s^2-bc)}{s(s-b)(s-c)} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : -\frac{b(s^2-ca)}{s(s-c)(s-a)} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : \frac{s}{s-b} : -\frac{c(s^2-ab)}{s(s-a)(s-b)}\right).$$

This clearly has perspector X_{57}. □

Proposition 5. *X_{57} is the perspector of the reflections of the Gergonne point in the intouch triangle.*

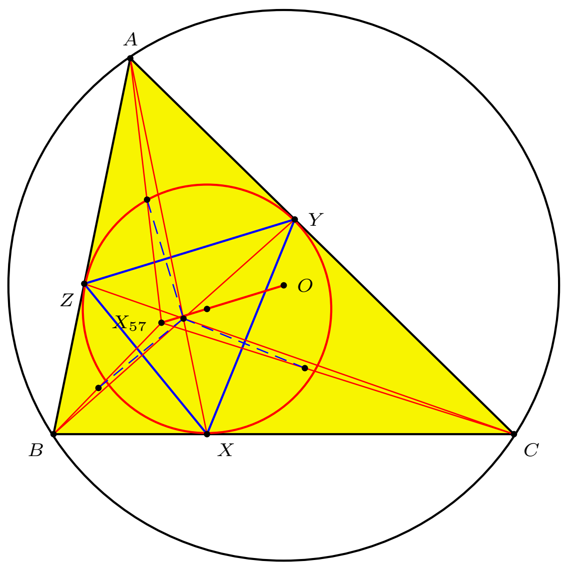

Figure 5.

More generally, the reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at

$$\left(u\left(-\frac{a^2}{u^2}+\frac{b^2}{v^2}+\frac{c^2}{w^2}+\frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right).$$

See [2]. For example, if P is the incenter, this perspector is the point

$$X_{35} = (a^2(b^2+bc+c^2-a^2) : b^2(c^2+ca+a^2-b^2) : c^2(a^2+ab+b^2-c^2))$$

which divides the segment OI in the ratio $OX_{35} : X_{35}I = R : 2r$.

Finally, we also mention from [5] that X_{57} is the orthocorrespondent of the incenter. This means that the trilinear polar of X_{57}, namely, the line

$$\frac{s-a}{a}x + \frac{s-b}{b}y + \frac{s-c}{c}z = 0$$

intersects the sidelines BC, CA, AB at X, Y, Z respectively such that $IX \perp IA$, $IY \perp IB$, and $IZ \perp IC$.

3. A locus of perspectors

As an extension of the result of [4], we consider, for a real number t, the triangle $X_tY_tZ_t$ with X_t, Y_t, Z_t dividing the segments XX_1, YY_1, ZZ_1 in the ratio

$$XX_t : X_tX_1 = YY_t : Y_tY_1 = ZZ_t : Z_tZ_1 = t : 1 - t.$$

Proposition 6. *The triangle $X_tY_tZ_t$ is perspective with ABC. The locus of the perspector is the Soddy line joining the incenter to the Gergonne point.*

Proof. We compute the coordinates of X_t, Y_t, Z_t. It is well known that $BX = s - b$, $XC = s - c$, *etc.*, so that, in absolute barycentric coordinates,

$$X = \frac{(s-c)B + (s-b)C}{a}, \qquad Y = \frac{(s-a)C + (s-c)A}{b}, \qquad Z = \frac{(s-b)A + (s-a)B}{c}.$$

Since the intouch triangle XYZ has (acute) angles $\frac{B+C}{2}$, $\frac{C+A}{2}$, and $\frac{A+B}{2}$ at X, Y, Z respectively, the pedal X_1 of X on YZ divides the segment in the ratio

$$YX_1 : X_1Z = \cot\frac{C+A}{2} : \cot\frac{A+B}{2} = \tan\frac{B}{2} : \tan\frac{C}{2} = s - c : s - b.$$

Similarly, Y_1 and Z_1 divide ZX and XY in the ratios

$$ZY_1 : Y_1X = s - a : s - c, \qquad XZ_1 : Z_1Y = s - b : s - a.$$

In absolute barycentric coordinates,

$$\begin{aligned} X_1 =& \frac{(s-b)Y + (s-c)Z}{a} \\ =& \frac{(b+c)(s-b)(s-c)A + b(s-c)(s-a)B + c(s-a)(s-b)C}{abc}. \end{aligned}$$

It follows that

$$\begin{aligned} X_t =& (1-t)X + tX_1 \\ =& \frac{t(b+c)(s-b)(s-c)A + b(s-c)(c - t(s-b))B + c(s-b)(b - t(s-c))C}{abc}. \end{aligned}$$

In homogeneous barycentric coordinates, this is

$$X_t = (t(b+c)(s-b)(s-c) : b(s-c)(c - t(s-b)) : c(s-b)(b - t(s-c))).$$

The line IX_t has equation

$$bc(b-c)(s-a)x + c(s-b)(ab - 2s(s-c)t)y - b(s-c)(ca - 2s(s-b)t)z = 0.$$

The line IX_t intersects BC at the point

$$\begin{aligned} X_t' =& (0 : b(s-c)(ca - 2s(s-b)t) : c(s-b)(ab - 2s(s-c)t) \\ =& \left(0 : \frac{b(ca - 2s(s-b)t)}{s-b} : \frac{c(ab - 2s(s-c)t)}{s-c}\right). \end{aligned}$$

Similarly, the lines IY_t and IZ_t intersect CA and AB respectively at

$$Y_t' = \left(\frac{a(bc-2s(s-a)t)}{s-a} : 0 : \frac{c(ab-2s(s-c)t)}{s-c}\right),$$
$$Z_t' = \left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : 0\right).$$

The triangle $X_t'Y_t'Z_t'$ is perspective with ABC at the point

$$\left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : \frac{c(ab-2s(s-c)t)}{s-c}\right).$$

As t varies, this perspector traverses a straight line. Since the perspector is the Gergonne point for $t=0$ and the incenter for $t=\infty$, this line is the Soddy line joining these two points. □

The Soddy line has equation

$$(b-c)(s-a)^2x + (c-a)(s-b)^2y + (a-b)(s-c)^2z = 0.$$

Here are some triangle centers on the Soddy line, with the corresponding values of t. The symbol r_a stands for the radius of the A-excircle.

t	perspector	first barycentric coordinate
1	X_{77}	$\frac{a(b^2+c^2-a^2)}{s-a}$
2	X_{1442}	$\frac{a(b^2+bc+c^2-a^2)}{s-a}$
$\frac{1}{2}$	X_{269}	$\frac{a}{(s-a)^2}$
$\frac{R}{s}$	X_{481}	$2r_a - a$
$\frac{-R}{s}$	X_{482}	$2r_a + a$
$\frac{2R}{s}$	X_{175}	$r_a - a$
$\frac{-2R}{s}$	X_{176}	$r_a + a$
$\frac{3R}{2s}$	X_{1372}	$4r_a - 3a$
$\frac{-3R}{2s}$	X_{1371}	$4r_a + 3a$
$\frac{R}{2s}$	X_{1374}	$4r_a - a$
$\frac{-R}{2s}$	X_{1373}	$4r_a + a$

The infinite point of the Soddy point is the point

$$X_{516} = (2a^3-(b+c)(a^2+(b-c)^2) : 2b^3-(c+a)(b^2+(c-a)^2) : 2c^3-(a+b)(c^2+(a-b)^2)).$$

It corresponds to $t = \frac{R(4R+r)}{s^2}$. The deLongchamps point X_{20} also lies on the Soddy line. It corresponds to $t = \frac{2R(2R+r)}{s^2}$.

4. Emelyanov's first problem

From the coordinates of X_t, we easily find the intersections

$$A_t = AX_t \cap BC, \quad B_t = BX_t \cap CA, \quad C_t = CX_t \cap AB.$$

These are

$$\begin{aligned}A_t =&(0 : b(s-c)(c-(s-b)t) : c(s-b)(b-(s-c)t),\\ B_t =&(a(s-c)(c-(s-a)t) : 0 : c(s-a)(a-(s-c)t),\\ C_t =&(a(s-b)(b-(s-a)t) : b(s-a)(a-(s-b)t) : 0).\end{aligned} \tag{2}$$

They are collinear if and only if

$$\begin{aligned}&(a-(s-b)t)(b-(s-c)t)(c-(s-a)t)\\ +&(a-(s-c)t)(b-(s-a)t)(c-(s-b)t)=0.\end{aligned} \tag{3}$$

Since this is a cubic equation in t, there are three values of t for which A_t, B_t, C_t are collinear. One of these is $t = 2$ according to [4]. The other two roots are given by

$$abc - abct + 2(s-a)(s-b)(s-c)t^2 = 0. \tag{4}$$

Since $abc = 4Rrs$ and $(s-a)(s-b)(s-c) = r^2 s$, where R and r are respectively the circumradius and inradius, this becomes

$$2R - 2Rt + rt^2 = 0. \tag{5}$$

From this,

$$t = \frac{R \pm \sqrt{R^2 - 2Rr}}{r} = \frac{R \pm d}{r},$$

where d is the distance between O and I.

We identify the lines corresponding to these two values of t.

Proposition 7. *Corresponding to the two roots of (4), the lines containing A_t, B_t, C_t are the tangents to the incircle perpendicular to the OI-line.*

Lemma 8. *Consider a triangle ABC with intouch triangle XYZ, and a line $\mathcal{L}$ intersecting the sides BC, CA, AB at A', B', C' respectively. The line $\mathcal{L}$ is tangent to the incircle if and only if one of the following conditions holds.*

(1) *The intersection $BB' \cap CC'$ lies on the line YZ.*
(2) *The intersection $CC' \cap AA'$ lies on the line ZX.*
(3) *The intersection $AA' \cap BB'$ lies on the line XY.*

Proof. Let $A'B'$ be a tangent to the incircle. By Brianchon's theorem applied to the circumscribed hexagon $AYB'A'XB$ it immediately follows that AA', YX and $B'B$ are concurrent.

Now suppose AA', YX and $B'B$ are concurrent. Consider the tangent through A' (different from BC) to the incircle. Let B'' be the intersection of this tangent with AC. It follows from the preceeding that AA', YX and $B''B$ are concurrent. Therefore B'' must coincide with B'. This means that $A'B'$ is a tangent to the incircle. □

5. Proof of Proposition 7

The lines BB_t and CC_t intersect at the point

$$A'' = \left(\frac{a}{s-a}(b-(s-a)t)(c-(s-a)t) \right. \\ : \frac{b}{s-b}(c-(s-a)t)(a-(s-b)t) \\ \left. : \frac{c}{s-c}(a-(s-c)t)(b-(s-a)t) \right).$$

This point lies on the line $YZ: \ -(s-a)x+(s-b)y+(s-c)z=0$ if and only if

$$\begin{aligned} &-a(b-(s-a)t)(c-(s-a)t) \\ &+b(c-(s-a)t)(a-(s-b)t) \\ &+c(a-(s-c)t)(b-(s-a)t)=0. \end{aligned}$$

This reduces to equation (4) above. By Lemma 8, these two lines are tangent to the incircle. We claim that these are the tangents perpendicular to the line OI. From the coordinates given in (2), the equation of the line B_tC_t is

$$\begin{aligned} &-\frac{(s-a)(a-(s-b)t)(a-(s-c)t)}{a}x \\ &+\frac{(s-b)(a-(s-c)t)(b-(s-a)t)}{b}y \\ &+\frac{(s-c)(a-(s-b)t)(c-(s-a)t)}{c}z=0. \end{aligned}$$

According to [6], lines perpendicular to OI have infinite point

$$X_{513}=(a(b-c):b(c-a):c(a-b)).$$

The line B_tC_t contains the infinite point X_{513} if and only if the same equation (4) holds. This shows that the two lines in question are indeed the tangents to the incircle perpendicular to the OI-line.

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